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Global Decoupling of Coupled Symmetric Oscillators

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Preprint SC-88-3 (Januar 1988)

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GLOBAL DECOUPLING OF COUPLED SYMMETRIC OSCILLATORS

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We model a symmetric system of coupled oscillators as a graph with symmetry group Γ. *Each vertex of the graph represents an "oscillator" or a "cell" of reactants. The magnitude (concentration) of the reactants in the ith cell is represented by a vector* x^i. *The edges represent the coupling of the cells. The cells are assumed to evolve by identical reaction-diffusion equation which depends on the sum of the reactants in the nearest neighbors. Thus the dynamics of the system is described by a nonlinear differential system*

$$(*) \qquad \dot{x}^i = f(x^i, \sum_{j \in \mathcal{N}_i} x^j),$$

where the sum ranges over the set $\mathcal{N}_i$ *of neighbors of cell i. If f also has a symmetry (e.g., oddness), there are geometric conditions on the graph such that the nonlinear system (*) decouples globally into a product flow on certain sums of isotropy subspaces. Thus we may detect higher-dimensional tori of solutions of (*) which are not amenable to other types of analysis. We present a number of examples, such as bipartite graphs, complete graphs, the square, the octahedron, and a 6-dimensional cube.*

∘ 1. Introduction

A graph of coupled oscillators can be described as follows. Each vertex i represents a physical, chemical or biological system (or *cell*), and these cells are coupled along the edges of the graph. In many applications, many of the cells are identical to each other. For example, one could consider a ring of identical cells (a *Turing ring*). Or perhaps the odd cells are identical, as are the even cells, in a ring with an even total number of cells. For references on networks of such cells (plexuses), see Alexander [1986], Alexander and Auchmuty [1986], Fiedler [1987], Golubitsky and Stewart [1986], Othmer [1985], Othmer and Scriven [1971], Schreiber et al. [1986], Smale [1974], Swift [1986], Turing [1952], and several papers in Othmer [1986]. It is our goal to draw conclusions about the dynamics of the coupled system, knowing the dynamics of the individual cells and the graph structure of the coupling.

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* Partially supported by the National Science Foundation.

† Partially supported by Deutsche Forschungsgemeinschaft, SFB 123 and a Konrad-Zuse Fellowship.

Consider the example of a Turing ring with four cells. The dynamics can be represented by the system

$$\dot{x}^i = \phi(x^i) + (x^{i+1} - 2x^i + x^{i-1}). \tag{1.1}$$

Here $x^i \in \mathbb{R}^N$, $\phi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, and $i = 1, 2, 3, 4 \pmod 4$. This system can be represented graphically by a square (Fig. 1.1), with "discretized" diffusion coupling along the edges. If ϕ is odd $(\phi(-\xi) = -\phi(\xi))$, then (1.1) decouples. Indeed let $X = \mathbb{R}^{4N} = \{(x^1, x^2, x^3, x^4)\}$ and consider the subspaces

$$X_1 = \{x \in X : x^1 = -x^3, x^2 = x^4 = 0\},$$
$$X_2 = \{x \in X : x^1 = x^3 = 0, x^2 = -x^4\}.$$

Then X_1 and X_2 are obviously invariant under the flow of (1.1); moreover (and this is the important extra structure), so is the direct sum $X_1 \oplus X_2$. Thus the flow on $\mathbb{R}^{2N} = X_1 \oplus X_2$ is a product flow governed by the decoupled systems

$$\dot{x}^1 = \phi(x^1) - 2x^1, \qquad x^3 = -x^1, \tag{1.2a}$$
$$\dot{x}^2 = \phi(x^2) - 2x^2, \qquad x^4 = -x^2. \tag{1.2b}$$

That is, since $x_1 = -x_3$, the coupling effects of these cells on x_2 and x_4 cancel each other. Cells 2 and 4 do not "see" cells 1 and 3, and *vice-versa*. The two sets of cells are completely decoupled. For example a periodic solution $x^1(t)$ of (1.2a) and a periodic solution $x^2(t)$ of (1.2b) give rise to an invariant 2-dimensional torus of (1.1) with a product flow on it. This phenomenon was observed in Alexander and Auchmuty [1986], but it seemed rather mysterious and was not explained as a decoupling phenomenon. Indeed the present paper is an attempt to demystify this example. For a bifurcation analysis of the square with a general reaction term, see Swift [1986].

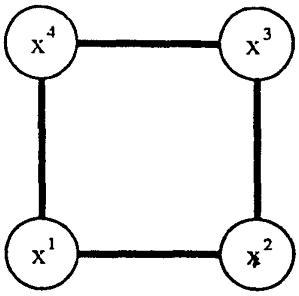

Fig. 1.1. A Turing ring of four coupled cells

More generally, we consider the following system:

$$\dot{x} = F(x), \qquad x \in X = \mathbb{R}^{nN}. \tag{1.3}$$

Here $x = (x^1, \ldots, x^n)$, with each $x^i \in \mathbb{R}^N$, where $i \in I = \{1, \ldots, n\}$ indexes the set of vertices of a graph $\mathcal{G}$. More specifically, each x^i, $i \in I$, is governed by the system

$$\dot{x}^i = f(x^i, \sum_{j \in I} A^i_j x^j), \tag{1.4}$$

where A^i_j are constant coupling matrices (of size $N \times N$). It is implicit that A^i_j is the zero matrix unless vertices i and j are connected by an edge. It is possible that $A^i_j \neq A^j_i$; that is, the coupling need not be symmetric. The ***symmetry*** Γ of the graph $\mathcal{G}$ is the group of vertex permutations $\gamma \in S_n$, the symmetric group on n elements, which satisfy

$$A^{\gamma(i)}_{\gamma(j)} = A^i_j$$

for all $i, j \in I$. We also consider a symmetry within each cell. We assume:

(1.5a) *$f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ commutes with the continuous linear action of a topological group G:*

$$f(g\xi, g\eta) = gf(\xi, \eta)$$

for all $g \in G$, $\xi, \eta \in \mathbb{R}^N$,

(1.5b)
$$gA^i_j = A^i_j g$$

for all $g \in G$ and all $i, j \in I$.

For example, for the Turing ring above A^i_j is the identity matrix if $j \equiv i \pm 1 \pmod 4$, and the zero matrix otherwise. Moveover $\Gamma = D_4$ is the symmetry of the square and, since f is odd, $G = Z/2Z = \langle g \rangle$ acts by $g\xi = -\xi$.

In fact, $G = Z/2Z$ acting as above due to the oddness of f is common. Another possibility is that each vertex of the (macro-)graph $\mathcal{G}$ might itself consist of some small (micro-)graph with symmetry G and suitable coupling to the neighboring (macro-)vertices. In other words, we have a lexicographic composition of graphs. Carrying this idea slightly further, we could assume each cell has spatial extent so that each x^i is a function of time and space; also (1.4) would be replaced by a partial differential system. The group G could arise from a spatial symmetry in the cells.

Assumptions (1.5) imply that the system (1.3) is equivariant (covariant) with respect to the action ρ of the direct product $\Gamma \times G$ on X given by

$$(\rho(\gamma, g)x)^i = gx^{\gamma^{-1}(i)} \tag{1.6}$$

for all $\gamma \in \Gamma$, $g \in G$, $i \in I$. Equivariance of system (1.3) means that

$$F(\rho(\gamma, g)x) = \rho(\gamma, g)F(x), \tag{1.7}$$

for all $\gamma \in \Gamma$, $g \in G$, $x \in X$.

Let Σ be any subgroup of $\Gamma \times G$. By (1.7), the linear subspace

$$X^\Sigma = \{x \in X : \sigma \cdot x = x \text{ for all } \sigma \in \Sigma\} \tag{1.8}$$

of Σ-fixed vectors x is invariant under the flow of (1.3). Consider two subgroups Σ^1, Σ^2 and associated fixed-point subspaces X^{Σ^1}, X^{Σ^2}. In Theorem 2.1, we present a simple condition which assures that not only X^{Σ^1}, X^{Σ^2}, but also their direct sum $X^{\Sigma^1} \oplus X^{\Sigma^2}$, are invariant under the flow, and the flow on $X^{\Sigma^1} \oplus X^{\Sigma^2}$ is given by the direct product of the respective flows on X^{Σ^1}, X^{Σ^2}. In other words, the flow on $X^{\Sigma^1} \oplus X^{\Sigma^2}$ decouples into a flow on X^{Σ^1} and a flow on X^{Σ^2}.

For the Turing ring above, the decoupling (1.2) can be described easily: if γ_1 is the permutation $(1\,3)$ and $\gamma_2 = (2\,4)$, then $\Sigma_j = \langle(\gamma_j, g)\rangle$, $j = 1, 2$, where $g(\xi) = -\xi$.

We indicate some abstract results in section 2. In section 3 we discuss some simple examples. In this paper we do not attempt complete lists of decoupling effects in these examples. Instead, we sketch some particular phenomena in octahedral, complete, and complete bipartite graphs, including a 6-dimensional hypercube.

As a warning, we emphasize that the whole setting of coupled oscillators with symmetry is infinitely degenerate in the following sense. For generic $\Gamma \times G$-equivariant systems (1.3), the global decoupling phenomena discussed here are not to be expected. For example, Swift has studied the normal form for generic Hopf bifurcation with D_4 symmetry. Additional equivariance with respect to $\Gamma \times G = D_4 \times Z/2Z$ will not, in itself, force the complete decoupling which is seen with the Turing ring. Hopf bifurcation in the Turing ring is accompanied by primary bifurcation of a 2-torus with a product flow on it (Alexander and Auchmuty [1986]). Remarkably, primary bifurcation of a 2-torus still occurs with certain structurally stable D_4-equivariant Hopf bifurcations (Swift [1986]). The flow on these tori is not a periodic product flow. It is the fact that (1.4) has a special form that causes the degeneracy; viz. the x_j, $j \neq i$ enter (1.4) only via the sum $\sum_{j\in I} A^i_j x^j$. At present, any of the examples studied in section 3 is beyond reach of a normal-form analysis in the generic $\Gamma \times G$-equivariant setting. Beyond their significance for experimental setups as referred to in Othmer [1986], Schreiber et al. [1986], we hope that coupled oscillators can serve as a model and starting point for future perturbations and variations.

Acknowledgements. The octahedral example was pointed out to us by Albrecht Brandis. The authors have also profited immensely from cheerful and spirited criticism by Marty Golubitsky.

• 2. Decoupling

In this section we indicate some abstract results on decoupling. Theorem 2.1 contains the basic idea. As corollaries, we consider decoupling into a k-fold product flow, and the special case of a free G-action. Throughout we assume that equivariance assumptions (1.5) hold.

Let Σ be a subgroup of $\Gamma \times G$, with associated fixed-space X^Σ, cf. (1.8). Denote

$$I(\Sigma) = \{i \in I : x^i = 0 \text{ for all } x \in X^\Sigma\}. \tag{2.1}$$

It is not excluded that $I(\Sigma) = \emptyset$ or, if $X^\Sigma = \{0\}$, that $I(\Sigma) = I = \{1, \ldots, n\}$.

2.1 Theorem. *Let Σ_1 and Σ_2 be subgroups of $\Gamma \times G$ such that*

$$I(\Sigma_1) \cup I(\Sigma_2) = I. \tag{2.2}$$

Then the sum

$$X_{12} = X^{\Sigma_1} \oplus X^{\Sigma_2}$$

is direct, i.e. $X^{\Sigma_1} \cap X^{\Sigma_2} = \{0\}$. If $x \in X_{12}$ is written $x = x_1 + x_2 \in X^{\Sigma_1} \oplus X^{\Sigma_2}$, the following hold:

(2.3a) *X_{12} is invariant for the flow of $\dot{x} = F(x)$,*

(2.3b) *$F(x_1 + x_2) = F(x_1) + F(x_2)$ for all $x_1 \in X^{\Sigma_1}$, $x_2 \in X^{\Sigma_2}$,*

(2.3c) *the flow on X_{12} is a product flow.*

Proof. We show that $X^{\Sigma_1} \cap X^{\Sigma_2} = \{0\}$, and that (2.3.b) holds. The other claims are then obvious, because F maps X^{Σ_j} into itself, for $j = 1, 2$. Let $x \in X^{\Sigma_1} \cap X^{\Sigma_2}$, and pick any $i \in I$. Then either $i \in I(\Sigma_1)$, which implies $x^i = 0$ because $x \in X^{\Sigma_1}$, or else $i \in I(\Sigma_2)$, which analogously implies $x^i = 0$. Thus $x = 0$. To show (2.3.b), it is sufficient to check each component of F. That is, we show

$$F^i(x_1 + x_2) = F^i(x_1) + F^i(x_2) \tag{2.3.b^i}$$

for all $i \in I(\Sigma_1)$. The case $i \in I(\Sigma_2)$ is analogous. Recall that the linear map $A: X \to X$ given by $(Ax)^i = \sum_{j\in I} A^i_j x^j$ is $(\Gamma \times G)$-equivariant. Therefore A maps each of the spaces X^{Σ_1}, X^{Σ_2} into itself. To check (2.3.b)^i for $i \in I(\Sigma_1)$, we observe that

$$\begin{aligned} F^i(x_1 + x_2) &= f(x_1^i + x_2^i, (Ax_1)^i + (Ax_2)^i) \\ &= f(x_2^i, (Ax_2)^i), \\ F^i(x_1) &= 0, \\ F^i(x_2) &= f(x_2^i, (Ax_2)^i), \end{aligned}$$

because $x_1^i = 0$, $(Ax_1)^i = 0$, and (since $F(x_1) \in X^{\Sigma_1}$) also $F^i(x_1) = 0$. This proves (2.3.b) and the theorem.

2.2 Corollary. *Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ be subgroups of $\Gamma \times G$ such that for any $1 \le j_1 < j_2 \le k$,*

$$I(\Sigma_{j_1}) \cup I(\Sigma_{j_2}) = I. \tag{2.4}$$

Then F decomposes into a product flow on the direct sum

$$X^{\Sigma_1} \oplus X^{\Sigma_2} \oplus \cdots \oplus X^{\Sigma_k}$$

via

$$F(x_1 + x_2 + \cdots + x_k) = F(x_1) + F(x_2) + \cdots + F(x_k).$$

Proof. Induction on $k \ge 2$.

Definition (2.1) of $I(\Sigma)$ is somewhat inconvenient for applications. To arrive at a more geometric definition, let

$$G_i = \{g \in G : \text{there exists } (\gamma, g) \in \Sigma \text{ with } \gamma(i) = i\}, \tag{2.5}$$

for any $i \in I$. Note that G_i is a subgroup of G.

2.3 Proposition.

$$I(\Sigma) = \{i \in I : (\mathbb{R}^N)^{G_i} = \{0\}\}. \tag{2.6}$$

Proof. We prove first that $i \in I(\Sigma)$ implies $(\mathbb{R}^N)^{G_i} = \{0\}$. Let $\xi \in (\mathbb{R}^N)^{G_i}$. We claim that there exists an $x \in X^\Sigma$ with $x^i = \xi$. But then $\xi = x^i = 0$, because $i \in I(\Sigma)$. It remains to prove the claim. Define $x^j = g\xi$ if there exists $(\gamma, g) \in \Sigma$ with $\gamma(i) = j$, and $x^j = 0$ otherwise. Note that $x = (x^1, \ldots, x^n)$ is well-defined because $\xi \in (\mathbb{R}^N)^{G_i}$; obviously $x^i = \xi$. Moreover the orbit $\Sigma \cdot x = \{x\}$ by definition of x and hence $x \in X^\Sigma$, which proves the claim.

Conversely, we prove that $(\mathbb{R}^N)^{G_i} = \{0\}$ implies $i \in I(\Sigma)$. Indeed $x \in X^\Sigma$ implies $x^i \in (\mathbb{R}^N)^{G_i}$; hence $x^i = 0$. Therefore $i \in I(\Sigma)$, and the proof is complete.

An important special case is the following:

(2.7) *G acts freely on $\mathbb{R}^N$, i.e. $g\xi = \xi$ if and only if g is the identity or $\xi = 0$.*

Let $p_1: \Gamma\times G \to \Gamma$ and $p_2: \Gamma\times G \to G$ denote projection onto the first and second factors, respectively. Let Σ be a subgroup of $\Gamma \times G$ with $X^\Sigma \neq \{0\}$, and define $\Delta = p_1\Sigma$. Then $p_1: \Sigma \to \Delta$ is an isomorphism by assumption (2.7), since $x \in X^\Sigma \setminus \{0\}$ and $\rho(1, g)x = x$ imply $g = 1$ so that $\ker p_1 = (1, 1)$. Therefore there exists a homomorphism $s = p_2p_1^{-1}: \Delta \to G$ such that Σ becomes a twisted copy of Δ:

(2.8) $$\Sigma = \Delta^s = \{(\gamma, s(\gamma)) : \gamma \in \Delta\}.$$

This construction is reminiscent of one in Golubitsky and Stewart [1985].

Given $\gamma \in \Gamma$, denote

$$\mathrm{fix}(\gamma) = \{i \in I : \gamma(i) = i\}.$$

In the above setting, Proposition 2.3 immediately yields the following.

2.4 Corollary. *If G acts freely on $\mathbb{R}^N$, then*

$$I(\Sigma) = \bigcup_{\gamma\in\Delta\setminus\ker(s)} \mathrm{fix}(\gamma).$$

Decoupling requires several subgroups and maps, usually demarked with subscripts. For example, for the 4-vertex Turing ring of section 1, $\Delta_1 = \langle(1\,3)\rangle \cong Z/2Z$, $\Delta_2 = \langle(2\,4)\rangle \cong Z/2Z$ and for $i = 1, 2$, the map s_i takes $(1\,3)$, respectively $(2\,4)$, to the nontrivial element of $G \cong Z/2Z$. The group G acts freely and by Corollary 2.4, $I(\Sigma_1) = \{2, 4\}$ and $I(\Sigma_2) = \{1, 3\}$. We consider more interesting examples in section 3.

• 3. Examples

Throughout this section we put A^i_j the identity matrix if there is an edge of $\mathcal{G}$ between i and j and the zero matrix otherwise. We discuss the octahedral graph, complete graphs, complete bipartite graphs, and a special example in the 6-dimensional cube, viz. the 4-octahedron. We encounter a surprisingly rich variety of decoupling phenomena.

3.1 The octahedron (Fig. 3.1)

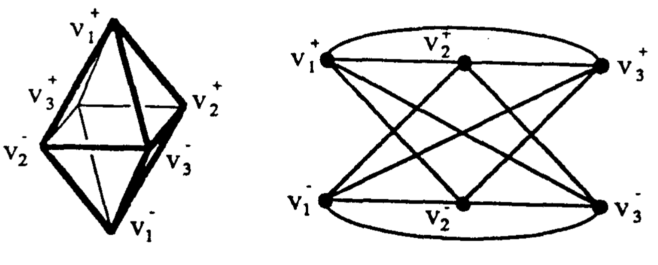

Figure 3.1. The octahedron

The octahedron has six vertices, which, deviating from the notation $I = \{1,\dots,6\}$ suggested by sections 1 and 2, we denote $v_j^\pm$ for $j = 1,2,3$. The vertices v_j^+ and v_j^- are diametrically opposite each other. Let $\Delta_j \cong Z/2Z$ be generated by interchanging only v_j^+ and v_j^-. Let $G \cong Z/2Z$ with the nontrivial element g operating by $g(\xi) = -\xi$. Let $s_j\colon \Delta_j \to G$ denote the canonical isomorphism and define $\Sigma_j = \Delta_j^{s_j}$, $j = 1,2,3$, as in (2.8). Since G acts freely on $I\!R^N$, the groups Σ_j satisfy the assumption of corollary 2.2 with $k = 3$. Indeed $I(\Sigma_j) = \{v_i^\pm : i \neq j\}$. In the resulting solution of (1.3), the opposite vertices oscillate as the negatives of each other and their coupling effects on any other vertex cancel.

Alternatively, we could let $\Delta_2 \cong Z/4Z$ rotate the vertices v_2^+, v_3^+, v_2^-, v_3^- cyclically, and $G \cong \Delta_4$ act freely on $I\!R^N$, N even. Then Theorem 2.1 applies and the discrete rotating wave dynamics in the (v_2, v_3)-plane decouple from the top and bottom vertices.

Note that decoupling depends not only on the abstract groups Γ and G, but also on the representation of $\Gamma \subset S_n$. The cube, as the dual polyhedron of the octahedron, also has the octahedral group as its group of symmetries, but the action of Γ on the vertices is different and any decoupling involves different subgroups.

Generalizations to the hyperoctahedral graph (k-dimensional octahedron), with vertices $v_j^\pm$, $1 \leq j \leq k$ and edges joining any two vertices except v_j^+ and v_j^-, are immediate. See Fig. 3.2. The 4-vertex Turing ring is the 2-dimensional octahedron.

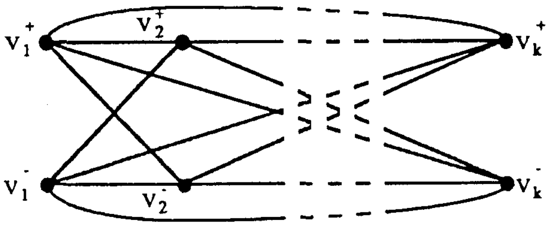

Figure 3.2. The k-octahedron

3.2 Complete bipartite graphs (Fig. 3.3)

Let the vertex set I decompose as a disjoint union $I_1 \cup I_2$, with $p_j = |I_j| \geq 2$. The edges of the graph $\mathcal{G}$ join any vertex in I_1 to any vertex in I_2, but no two vertices in I_1, or I_2, are joined to each other. These graphs, denoted K_{p_1,p_2}, are called *complete bipartite graphs*. See Fig. 3.3.

The automorphism group Γ of K_{p_1,p_2} contains $S_{p_1} \times S_{p_2}$, the direct product of the symmetric groups. Let $\gamma_1 = (\gamma_1, 1)$, $\gamma_2 = (1, \gamma_2)$ be any two nontrivial elements of Γ of orders q_1, q_2, respectively. Let G act freely and assume G contains the cyclic groups Z/q_1Z, Z/q_2Z with 1-1 homomorphisms $s_j\colon \langle\gamma_j\rangle \to G$, $j = 1,2$. Define $\Sigma_j = \Delta_j^{s_j}$ as before and apply Theorem 2.1 and Corollary 2.4. Obviously $I(\Sigma_j) = \mathrm{fix}(\gamma_j) \supset I \backslash I_j$, $j = 1,2$, i.e., assumption (2.2) holds. Thus decoupling occurs. In the resulting solution of (1.3), the p_1 vertices on I_1 are synchronized to have a fixed "phase"-relation defined by the actions of γ_1 and Z/q_1Z. This synchronization occurs although the subgraph on these vertices has no edges. An analogous statement holds for the p_2 vertices of I_2.

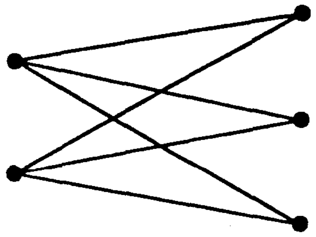

Figure 3.3. The complete bipartite graph $K_{2,3}$

In light of Corollary 2.4, our approach generalizes directly to complete k-partite graphs $K_{p_1,\dots,p_k}$, for which the vertex set $I = I_1 \cup \cdots \cup I_k$ is a disjoint union and edges join precisely those vertices in I_i, I_j, $i \neq j$. The k-octahedron discussed in section 3.1 is the complete k-partite graph with $p_1 = \cdots = p_k = 2$. For another example, see section 3.5.

3.3 Complete graphs (Fig. 3.4)

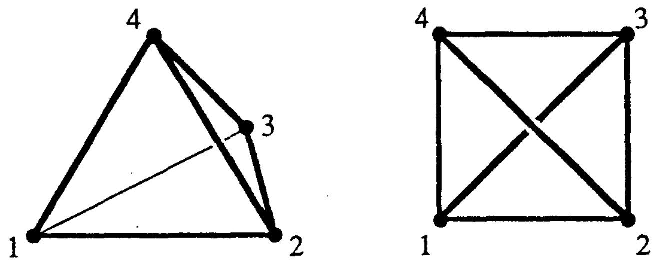

Figure 3.4. The complete graph K_4

Let K_n denote the complete graph on n vertices; i.e. all $\binom{n}{2}$ possible edges are present. Of course, $\Gamma = S_n$. Let $n \geq 4$, and let $\gamma_j \in S_n \backslash \{1\}$, $j = 1, 2$ be such that $\text{fix}(\gamma_1) \cup \text{fix}(\gamma_2) = I$, i.e. γ_1 and γ_2 move different vertices. As in section 3.2, we consider the case that G acts freely and that there exist 1-1 homomorphisms $s_j \colon \langle \gamma_j \rangle \to G$ which allow us to put $\Sigma_j = \Delta_j^{s_j}$ and to apply Theorem 2.1 and Corollary 2.4 In the same spirit, Corollary 2.2 applies with $j = 1, \dots, k$. Note that, as in the previous example, K_n can decouple in many genuinely different ways even if G is quite small.

3.4 Infinite lattices

The simplest infinite lattice is the integers with edges between each integer and its predecessor and successor. The group Γ is the group Z of translations. If $G = \langle g \rangle \cong Z/2Z$ operates by $g: \xi \mapsto -\xi$, there is a solution to (1.3) (ignoring any technical problems with the fact that (1.3) is an infinite system) in which cells of distance 2 oscillate as negatives of each other and the sets of even and odd cells decouple from each other. The lattice is a covering of the 4-vertex Turing ring (and indeed any Turing ring) and this decoupling is a "lift" of the decoupling considered in Section 1. There is a general principle here. Suppose Σ is a subgroup of the group Γ of a graph $\mathcal{G}$ which operates freely on $\mathcal{G}$. Let $\mathcal{G}_0 = \mathcal{G}/\Sigma$ be the quotient graph. Then any decoupling of oscillators on $\mathcal{G}_0$ induces a "lifted" decoupling on $\mathcal{G}$. The resulting solutions live in the subspace $X^\Sigma \subset X$. We use this principle in the next example.

It is clear that more interesting modes of decoupled oscillation can occur for higher-dimensional lattices. In particular, it would be interesting to investigate 3-dimensional crystalline lattices and ramifications for modelling molecular oscillations in crystals.

3.5 A hypercube

The k-cube $\mathcal{G} = K_2^k$ is given by the vertex set

$$I = \{v = (w_j)_{1 \le j \le k} : w_j \in \{\pm 1\}\}$$

with edges between those v which differ in precisely one component w_j. Below we consider the 6-cube. Differently from the previous examples we define a single subgroup Σ of $\Gamma \times G$ first and restrict our attention to the associated fix-space X^Σ. Within X^Σ, the dynamics can then be identified with that on a 4-octahedron $K_{2,2,2,2}$ of coupled oscillators. Finally, we specify a decoupling effect for a free action of $G = Z/6Z \cong Z/2Z \times Z/3Z$.

Define $\Sigma = \Delta^s$ with $\Delta \cong$ the 8-element group $Z/2Z \times Z/2Z \times Z/2Z$ generated by the three involutions $\alpha_{1,2,4}$, $\alpha_{1,3,5}$, $\alpha_{2,3,6}$, where α_{j_1,j_2,j_3} maps the coordinates $w_{j_i} \mapsto -w_{j_i}$ and leaves the other three coordinates fixed. Geometrically these involutions are reflections in 3-dimensional coordinate planes: α_{j_1,j_2,j_3} is the reflection in the coordinate plane $w_{j_1} = w_{j_2} = w_{j_3} = 0$. For any group G, let the map $s: \Delta \to G$ be the trivial map. This defines Σ. Each orbit of Δ in I contains eight elements, precisely one of which has the form $(w_1, w_2, w_3, +1, +1, +1)$. Within X^Σ, the eight vertices in any one such orbit oscillate synchronously since s is trivial on Δ. Consider the factor graph $\mathcal{G}_0 = \mathcal{G}/\Delta$. This graph has eight vertices $[w_1, w_2, w_3] = \{(w_1, w_2, w_3, +1, +1, +1)\}$. The reader can verify that two vertices $[w_1, w_2, w_3]$ and $[w_1', w_2', w_3']$ are *not* connected by an edge if and only if $w_1 = -w_1'$, $w_2 = -w_2'$, $w_3 = -w_3'$. This identifies $\mathcal{G}_0$ as a 4-octahedron of coupled oscillators with vertices $v_j^\pm$, $1 \le j \le 4$ as in 3.1. The nonlinearity of the system on $\mathcal{G}_0$ is again f. Note that although each edge of $\mathcal{G}_0$ is covered by 8 edges of $\mathcal{G}$, the coupling matrices on $\mathcal{G}_0$ are the identity matrices. Moreover, flows on the 4-octahedron are naturally identified with flows in X^Σ on the 6-cube. In particular, any decoupling in the 4-octahedron lifts to a decoupling in the 6-cube.

For example, let $G \cong Z/2Z = \langle -1 \rangle$. If Γ^0 is the group of $\mathcal{G}_0$, let $\Delta_j^0 \cong Z/2Z$ be generated by $v_j^+ \mapsto v_j^-$ for $1 \le j \le 4$, and let $s_j^0: \Delta_j^0 \to G$ be the nontrivial map. From Corollary 2.4 and 2.2 with $k = 4$, the $\Sigma_j^0 = (\Delta_j^0)^{s_j}$ decouple the flow on $\mathcal{G}_0$ (or in X^Σ). In the resulting oscillation, the vertices in any orbit of Δ oscillate synchronously and any vertex v and $\alpha_{1,2,3} \cdot v$ oscillate as negatives of each other.

Now let $G \cong Z/6Z$. Let Δ and $\mathcal{G}_0$ be as above. Let Δ_1^0 be as above and $s_1^0\colon \Delta_1^0 \to G$ be the nontrivial map. Let $\Delta_2 \cong Z/3Z$ be generated by the 3-cycle $v_2^\iota \mapsto v_3^\iota \mapsto v_4^\iota \mapsto v_2^\iota$ for $\iota = +$ or $-$, and define $s_2\colon \Delta_2^0 \to G$ to be an isomorphism onto $Z/3Z \subset G$. As usual, $\Sigma_2^0 = (\Delta_2^0)^{s_2}$. By Corollary 2.4

$$I(\Sigma_1^0) = \{v_j^\iota : \iota = \pm, 2 \le j \le 4\},$$
$$I(\Sigma_2^0) = \{v_1^+, v_1^-\}.$$

The flow decouples by Theorem 2.1. In the resulting oscillation on $\mathcal{G}_0$, the two vertices v_1^+, v_1^- oscillate as negatives of each other. The remaining six vertices consist of two triples $\{v_2^\iota, v_3^\iota, v_4^\iota\}$, for $\iota = \pm$. Within each triple of vertices, the cell states are related by the action of $Z/3Z \subset G$.

These examples, as well as many others, demonstrate that a fixed graph of coupled symmetric oscillators can split into independent substructures in various genuinely different ways. This suggests the architecture of a system of coupled symmetric oscillators is a structure with a rich and flexible dynamics.

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