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Hexagonal Lattice Dome — Illustration of a Nontrivial Bifurcation Problem

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Abstract

The deformation of a hexagonal lattice dome under an external load is an example of a parameter dependent system which is equivariant under the symmetry group of a regular hexagon. In this paper the mixed symbolic-numerical algorithm SYMCON is applied to analyze its steady state solutions automatically showing their different symmetry and stability properties.

Contents

1. Introduction

We consider the hexagonal lattice dome which was introduced by HEALEY [9]. This is an example of a parameter dependent, equivariant system

$$\begin{array}{rclll} F(x,\lambda) & = & 0, & & F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, \\ F(\vartheta_t x,\lambda) & = & \vartheta_t F(x,\lambda) & & \forall\, t \in H\,, \end{array}$$

where ϑ is a linear representation of a group H. Symmetry considerations applied to analysis give theoretical results such as the equivariant branching lemma (GOLUBITSKY, STEWART, SCHAEFFER [7], [8], VANDERBAUWHEDE [21]), which shows the existence of emanating branches at symmetry breaking bifurcation points. For a numerical treatment of this problem class see DELLNITZ, WERNER [1], HEALEY [9], IKEDA, MUROTA, FUJII [13]. Several authors ([4], [10], [11], [12], [15], [20]) have used the block diagonal form of the transformed Jacobian which is a consequence of the theory of linear representations (SERRE [18], STIEFEL, FÄSSLER [19]). The blocks are important for the detection of bifurcation points and appear in the augmented system (WERNER [22]) which is used in SYMCON for the computation of bifurcation points. Based on this knowledge the algorithm SYMCON (see [4]) automates the symmetry analysis by means of Computer Algebra. The symmetry exploitation organized and implemented in REDUCE ([14]) is combined with an effective algorithm for the numerical pathfollowing with implicit reparametrization (DEUFLHARD, FIEDLER, KUNKEL [2]).

The aim of this paper is to show the results of SYMCON applied to the example of the hexagonal lattice dome. In contrast to [20], where only some bifurcation diagrams are given, we also show an overview of stable solutions.

In the second section the problem is clearly formulated while in the third section the solutions with different isotropy groups are viewed. In the last section the stability of solutions is discussed.

2. The Lattice Dome

In HEALEY [9] an example of a deformation of a hexagonal lattice dome is given. There are seven free nodes $I \in \{A, B, C, D, E, F, G\}$ with displacement vectors $x_I \in \mathbb{R}^3$ which form the unknowns $x = (x_A, \ldots, x_G) \in \mathbb{R}^{21}$ of the system. The coordinates are chosen depending on the nodes in a way that the radial and tangential displacement and the height displacement of the node are measured (Fig. 1). Note that the coordinate system is different from the one chosen in [20]. Rods are connecting these nodes and connect also with immovable nodes. The height of the center A is 3 and the other points $B-G$ have height 1.5. The rods $B-C, C-D, \ldots, F-G$ have length 9, the others $\sqrt{(9^2+1.5^2)}$.

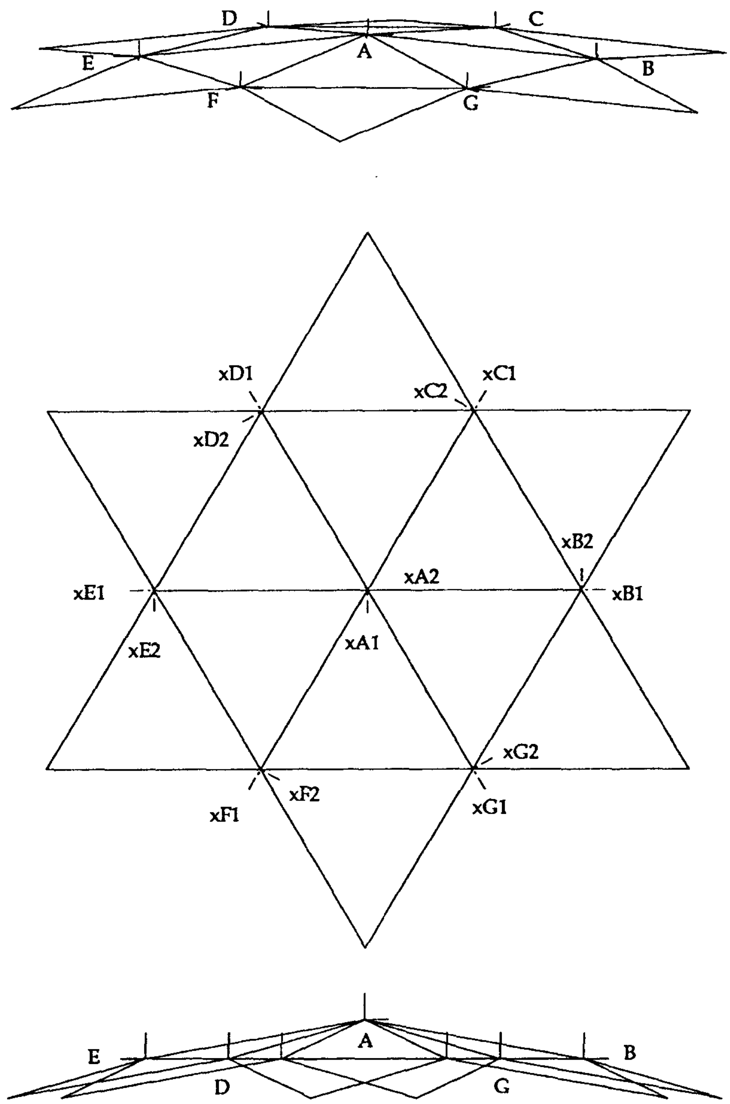


Figure 1: Hexagonal Lattice Dome

$$\vartheta_r = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & \cdots & & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & \vdots & & & \vdots \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & \cdots & 0 & 0 & I_{3,3} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & 0 & 0 & \cdots & I_{3,3} \\ 0 & \cdots & 0 & I_{3,3} & 0 & \cdots & 0 \end{pmatrix}$$

$$\vartheta_s = \begin{pmatrix} K_{3,3} & 0 & \cdots & & \cdots & & 0 \\ 0 & J_{3,3} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & 0 & \cdots & \cdots & 0 & J_{3,3} \\ & \vdots & \vdots & 0 & 0 & J_{3,3} & 0 \\ \vdots & \vdots & \vdots & 0 & J_{3,3} & 0 & \vdots \\ & \vdots & 0 & J_{3,3} & 0 & 0 & \vdots \\ 0 & 0 & J_{3,3} & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$I_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad K_{3,3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad J_{3,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 2: Two representation matrices ϑ_r and ϑ_s

Obviously the truss structure is D_6–invariant where D_6 denotes the dihedral group

$$\{id, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}.$$

In this paper the possible deformations of the dome are discussed if an external load (e.g. a weight) affects at each node $A - G$.

In engineering sciences usually this is modelled in the following way. The internal energy is given by $U(x) = \sum_{(I,J)} U_{I,J}$, where the sum is taken over all rods. The energy of one rod is

$$U_{I,J} := \frac{1}{8}EA\left(\frac{2 \cdot e_{IJ} \cdot (x_J - x_I)}{l_{IJ}} + \frac{\|x_J - x_I\|^2}{l_{IJ}^2}\right)^2,$$

where l_{IJ} is the undeformed length of the rod connecting the nodes I, J. The unit vector e_{IJ} is parallel to this rod pointing from I to J. The vectors x_I, x_J, and e_{IJ} are assumed to be taken in a global coordinate system. We have chosen the system in A and transformed the other vectors to this coordinate system. The constants E and A are Young's modulus and the cross sectional area, respectively.

The external load λ affecting at each node $A - G$ causes the potential energy $\lambda x^T e$ with $e = (e_3, \ldots, e_3) \in \mathbb{R}^{21}$ and $e_3 = (0, 0, 1)$. Finally by the principle of stationary potential energy the system of equations is given by

$$F(x, \lambda) = D_x U(x) + \lambda e = 0. \tag{1}$$

So the function $F : \mathbb{R}^{22} \to \mathbb{R}^{21}$ depends on the displacements x and the load parameter λ. Different choices of the parameter E and A lead to the same solution set up to rescaling of λ. We have chosen $EA = 10000$.

Because the dome and the energy function are obviously D_6–invariant, F is *D_6–equivariant*, i.e.

$$\vartheta_t F(x, \lambda) = F(\vartheta_t x, \lambda), \quad \forall t \in D_6,$$

where $\vartheta : D_6 \to \mathrm{GL}(\mathbb{R}^{21})$ is a real linear representation. For the theory of linear representations see [18], [19]. Two matrices of this representation are given in Fig. 2. Thus we end up with a parameter dependent equivariant system (1) which may be tackled by the symbolic-numerical algorithm SYMCON as described in [4]. In this particular case it is convenient to formulate the equations in REDUCE because the implemented equivariance check makes the correct implementation of the equations safer.

3. Isotropy Groups and Conjugate Solutions

Although the system (1) is D_6–equivariant, the solutions are not necessary D_6–invariant. The group $H = (D_6)_x$ giving the symmetry of a solution x

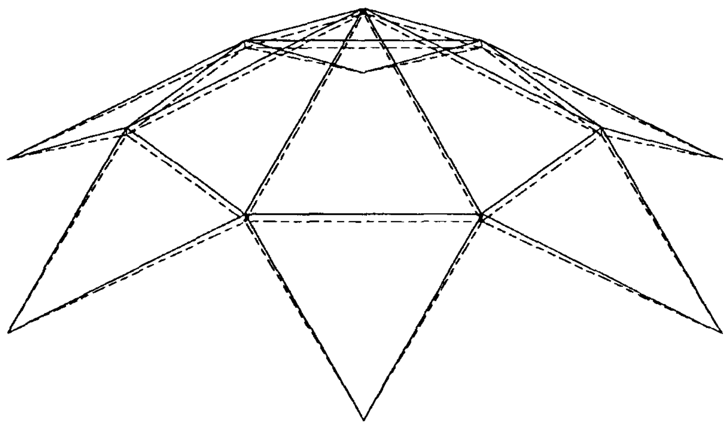


Figure 3: D_6–invariant stable solution

is called *isotropy group* of x (i.e. $\vartheta_t x = x \quad \forall\, t \in H$). Examples of solutions with different isotropy groups are given in Fig. 3 - Fig. 9. The pathfollowing of H–invariant solutions is done with a new implementation of ALCON (using implicit reparametrization and Broyden updates of the Jacobian) applied to the symmetry reduced system of (1) with respect to H. Note that some solutions are physically the same and are only distinguished by the viewpoint of an observer. By a rotation of the lattice dome they are identical. Then one speaks of *conjugate* solutions.

The dome has an additional symmetry, if the existence of a ground is omitted. Then it may be reflected with respect to a plane through the immovable nodes (see Fig. 10 and Fig. 11).

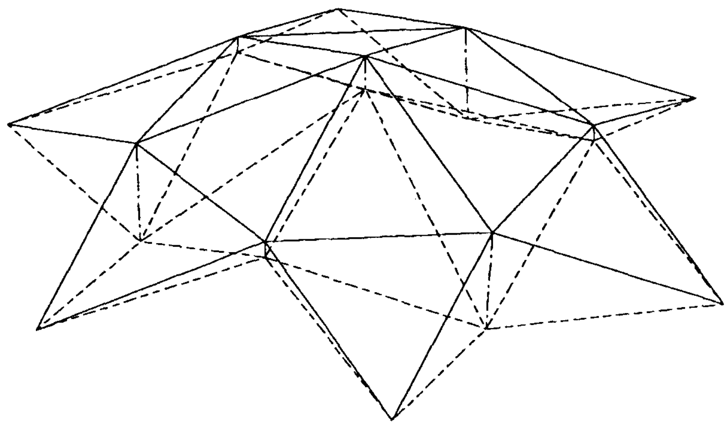


Figure 4: D_3–invariant stable solution

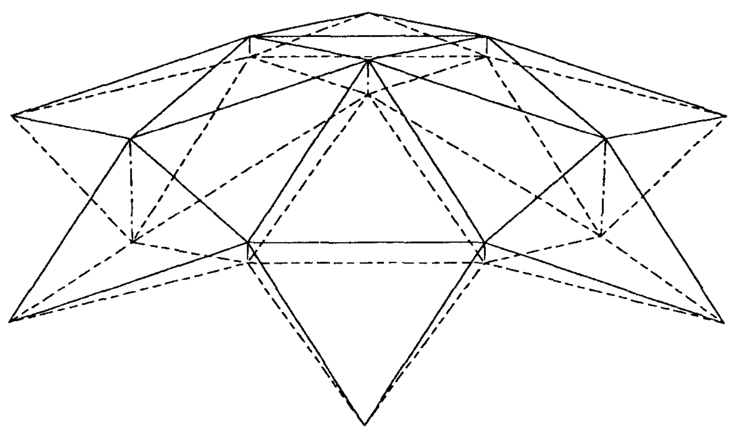


Figure 5: Stable solution with the isotropy of the Kleinian group

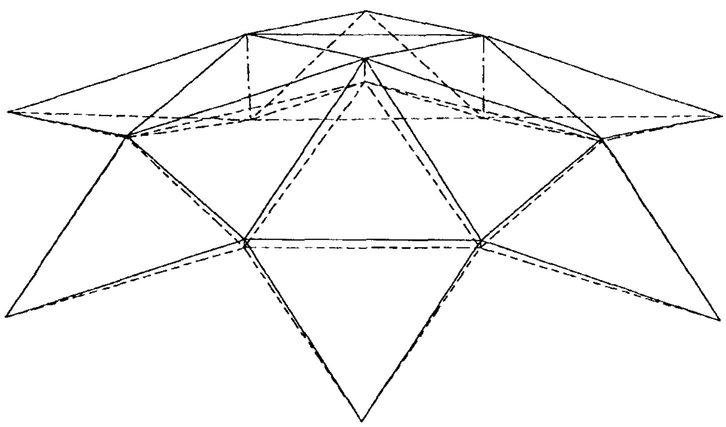


Figure 6: Z_2^1–invariant stable solution

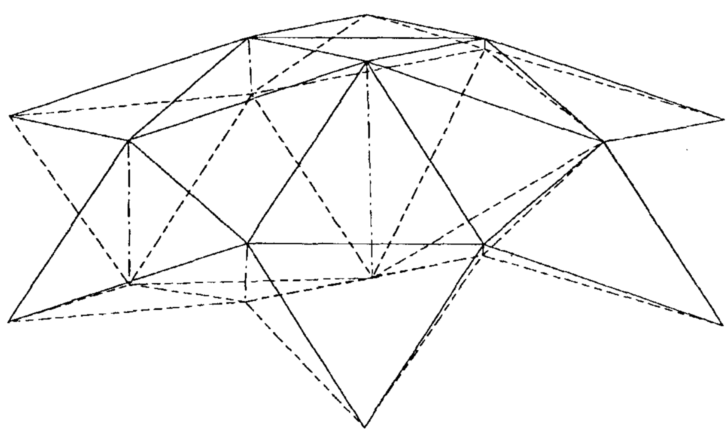


Figure 7: Z_2^4–invariant stable solution

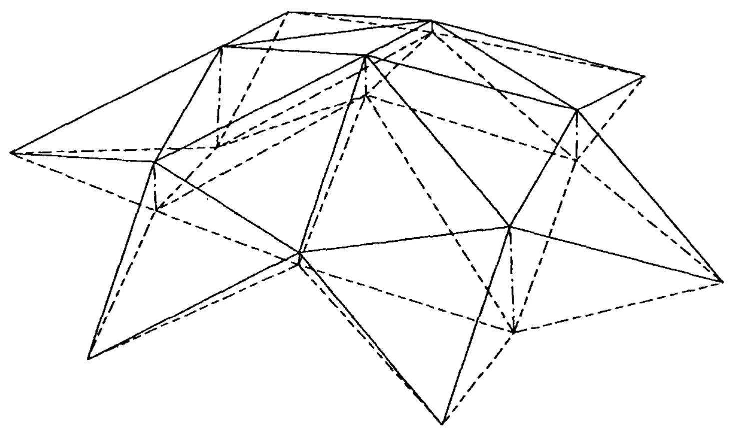


Figure 8: Unstable solution, invariant wrt a rotation of 180 degrees

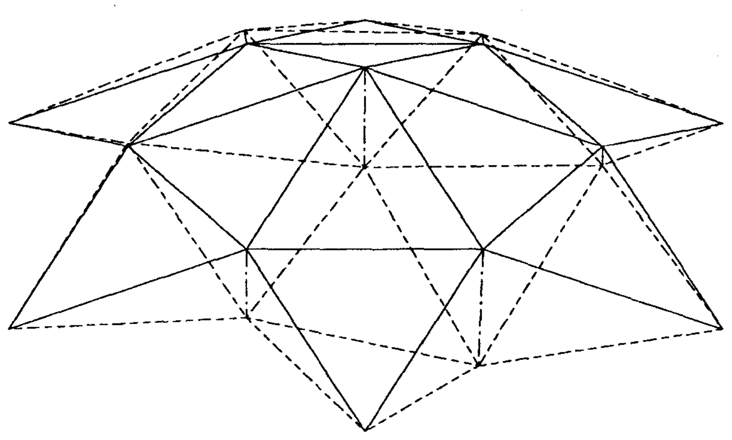


Figure 9: Unstable solution without symmetry

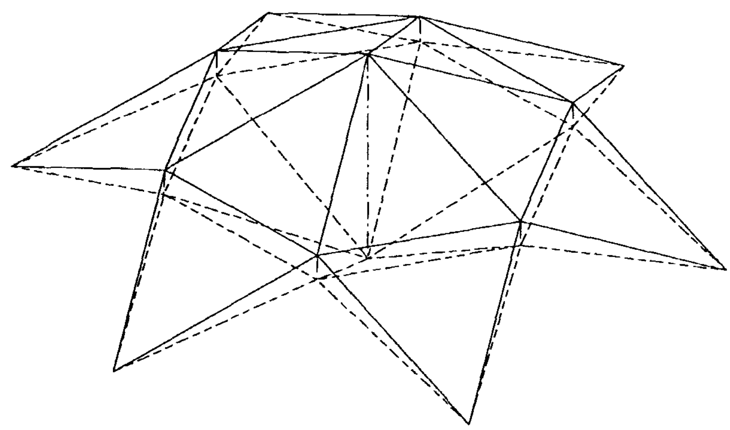


Figure 10: Stable D_6–invariant solution, symmetric to Fig. 11

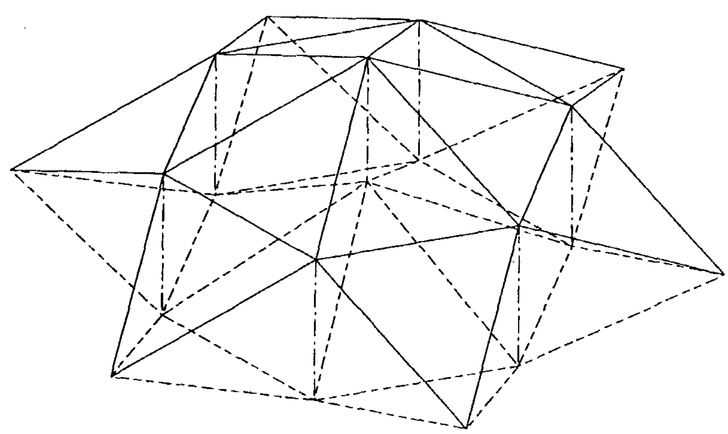


Figure 11: Stable D_6–invariant solution, symmetric to Fig. 10

4. Singular Points

Let H denote the isotropy group of a solution (x, λ). Then the Jacobian $D_xF(x, \lambda)$ has the symmetry of the isotropy group, i.e.

$$\vartheta_t D_x F = D_x F \vartheta_t \quad \forall\, t \in H\,.$$

Thus the transformed Jacobian $M^T D_x F M$ has block diagonal form for some transformation matrix M. These transformation matrices were computed in REDUCE using some projections from the theory of linear representations ([4], [18], [19]). The matrices are produced with GENTRAN ([6]) as C-code avoiding zero multiplication. So the additional work in [15] for this purpose is superfluous.

In this example the blocks have dimensions $3, 1, 2, 1, 4, 3$ for solutions with isotropy D_6 where the last two blocks of dimension 4 and 3 appear twice. For solutions with isotropy D_3 the blocks have dimension $5, 2, 7$.

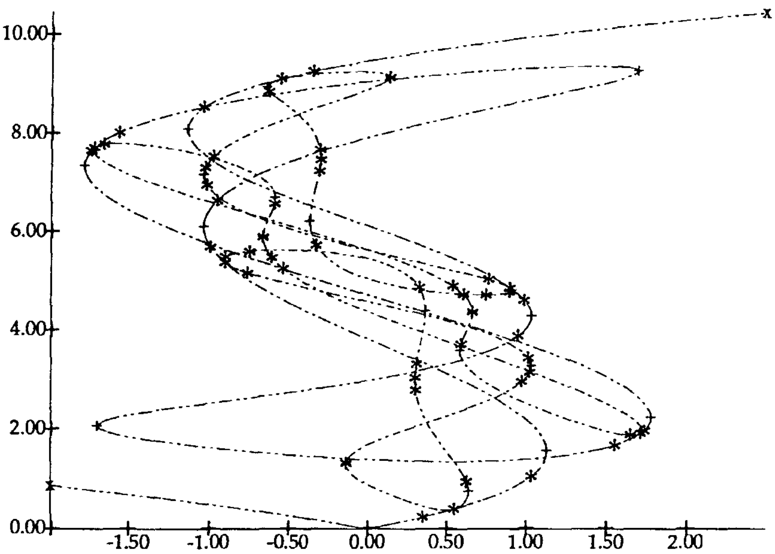
In turning points and symmetry breaking bifurcation points the Jacobian D_xF is singular. These singular points are detected during numerical pathfollowing by change of sign of the determinants of the blocks of the Jacobian. Using this criterion also multiple bifurcation points are found which is not possible without exploitation of symmetry although they appear generically in problems with symmetry. In [23] an alternative method for their detection is given which uses a bordering of the blocks.

The blocks are needed for three other tasks. Firstly they appear in the augmented systems for the bifurcation points (see [4] and [22]). Secondly their subcondition influences the automatic steplength control during pathfollowing (see [4]). Thirdly they are needed for the determination of stability (see Section 5).

At symmetry breaking bifurcation points branches with the isotropy of bifurcation subgroups (see [1]) bifurcate. See Fig. 13 and Table 1 for an example of a bifurcation scenario including a range of symmetry breaking bifurcation points of different types. The bifurcation subgroups may be arranged in a bifurcation graph. The graph for D_6 may be found in [1], [4], [13]. How the bifurcation graph is computed automatically for an arbitrary finite group will appear in [5]. It is only based on the data of irreducible representations of finite groups.

In SYMCON the organization of pathfollowing of emanating branches is done in REDUCE.

The turning points are computed with an iterative method based on Hermite interpolation (see [3]).



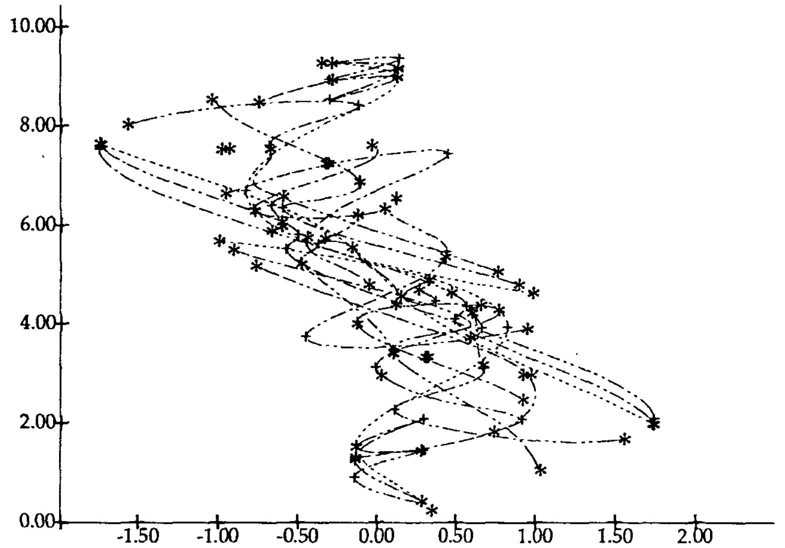


Figure 12: Solutions with the isotropy of the Kleinian group and D_6 (above) and D_3, Z_2^4 (bottom) ($||.||_2$ versus λ)

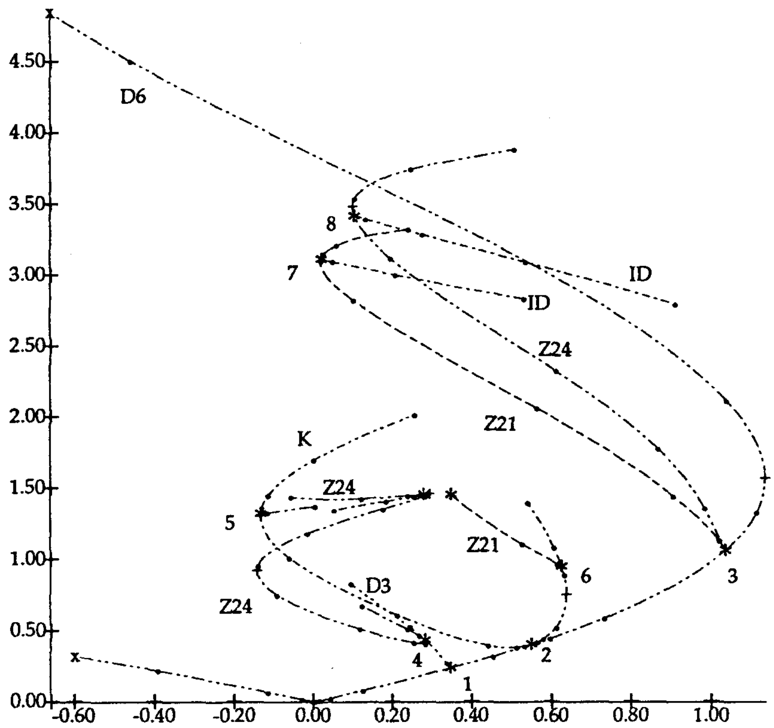


Figure 13: Nonconjugate solutions and different types of symmetry breaking bifurcation points (see Table 1)

nr	supergroup	isot. comp.	subgroups	type
1	D_6	3	D_3	symmetric
2	D_6	6	K_6^1, K_6^2, K_6^3	asymmetrical
3	D_6	5	$Z_2^0, Z_2^1, Z_2^2, Z_2^3, Z_2^4, Z_2^5$	symmetric
4	D_3	3	Z_2^0, Z_2^2, Z_2^4	asymmetrical
5	K_6^2	3	Z_2^4	symmetric
6	K_6^2	4	Z_2^1	symmetric
7	Z_2^1	2	Id	symmetric
8	Z_2^4	2	Id	symmetric

Table 1: Bifurcation points in Fig. 13

5. Stable Solutions

The variety of solutions with different isotropy groups is enormous as Fig. 12 shows. But there are only few stable solutions. A solution was accepted to be stable if the eigenvalues of the Jacobian at this point are all negative. Again the block diagonal structure is exploited. The transformation to this block diagonal form does not influence the signs and values of the eigenvalues.

Because $F(x, \lambda) = D_x U(x) + \lambda e$ has the potential U, the Jacobian is symmetric and all eigenvalues are real. The algorithm for the computation of eigenvalues and its implementation was taken from [16], [17].

The interpretation of the bifurcation diagrams Fig. 14, Fig. 15, and Fig. 16 is the following. Increasing the load parameter λ from 0 first D_6–invariant solutions are stable. At the symmetry breaking bifurcation point of a type which leads to D_3–invariant solutions the D_6-branch looses its stability. Further increase causes a jump to states which probably are invariant with respect to the Kleinian group.

In [13] Proposition 5 some statements on the stability of emanating branches depending on the supergroup and bifurcation subgroups are given. For D_6 this means the following. The solution invariant with respect to the Kleinian group emanating from a D_6-K symmetry breaking bifurcation point are unstable. The Z_2^i- invariant solutions ($i = 0, \ldots, 5$) emanating from a D_6-invariant bifurcation point turn all to the same direction and at most one orbit of conjugate solutions is stable. We didn't find any contradiction.

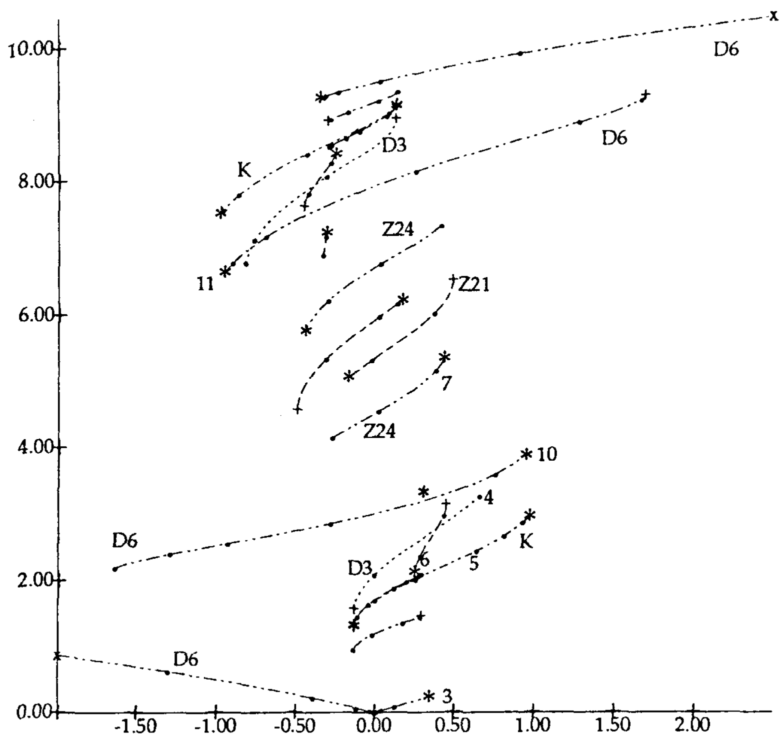


Figure 14: Stable solutions ($||.||_2$ versus λ), numbers refer to figures above

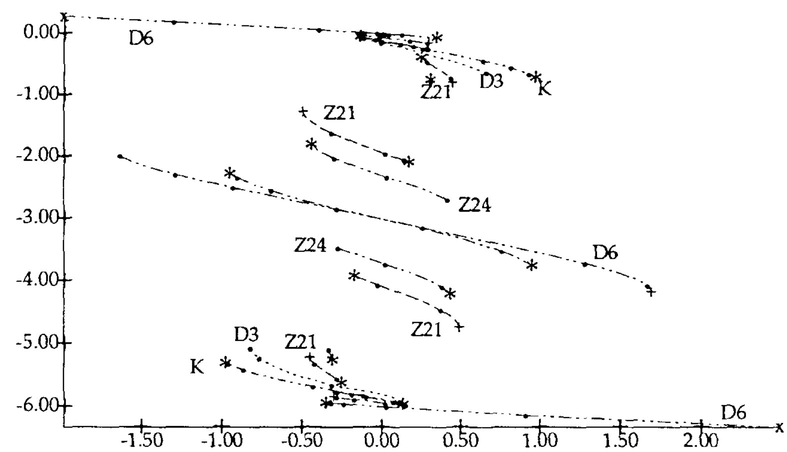


Figure 15: Stable solutions (x_{A3} versus λ)

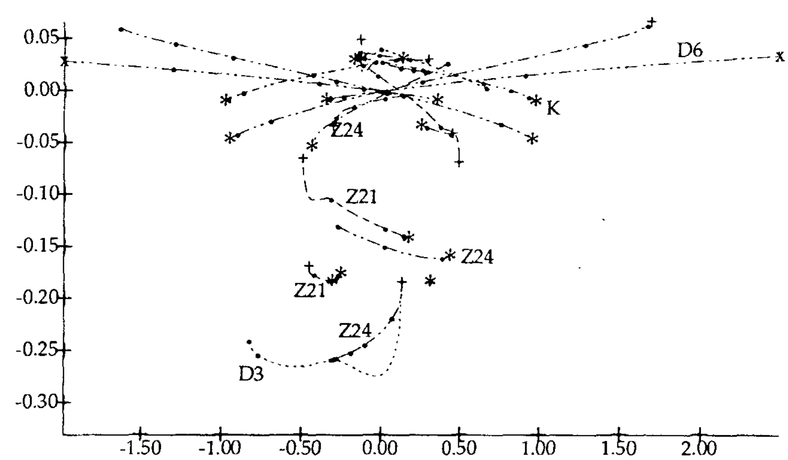


Figure 16: Stable solutions (x_{B1} versus λ)

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