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Combinatorial Packing Problems

ZIB-Report 03-03 (March 2003)

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March 06, 2003

Abstract. This article investigates a certain class of combinatorial packing problems and some polyhedral relations between such problems and the set packing problem.

MSC 2000. 90C27
Key words. Packing problems, polyhedral combinatorics.

1 Introduction

Packing constraints are one of the most common problem characteristics in combinatorial optimization. They come up in problems of vehicle and crew scheduling, VLSI and network design, and frequency assignment, see [39, 16] for surveys. The pure form is the *set packing* or *stable set problem* (SPP) in a graph $G = (V, E)$ with node weights w; it asks for a maximum weight set of mutually non-adjacent nodes. This problem has been studied extensively, and deep structural and algorithmic results have been achieved in areas such as anti-blocking theory, the theory of perfect graphs, perfect and balanced matrix theory, and semidefinite programming, see [7, 20, 35, 8] for surveys. There is, in particular, a substantial structural and algorithmic knowledge of the set packing polytope, with many classes of strong and polynomial time separable inequalities such as odd hole, odd antihole, orthonormal representation constraints and other classes [36, 34, 45, 38, 20].

Several research directions try to translate some of these results to broader settings. A first line investigates generalizations of set packing such as node packing in hypergraphs [42], independence systems [34, 37, 13, 26], transitive packing, [30, 31, 32, 41], and mixed integer packing [2, 3]. This work aims at a unified polyhedral theory. A second direction is the theory of matrix cuts [27], which generalizes the semidefinite separation machinery that had been developed for the solution of the stable set problem in perfect graphs [20] to arbitrary 0/1 programs. A third approach is the construction of discrete set

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packing relaxations [9, 10], see also [40]. This technique allows to transfer set packing inequalities and separation algorithms to other combinatorial problems.

Our aim in this paper is to continue in this general direction. We consider a class of combinatorial optimization problems of packing type where a Dantzig-Wolfe decomposition gives rise to a canonical, yet exponential, set packing formulation, namely, the formulation that one would use in a column generation approach. This alternative formulation allows, at least in principle, to understand combinatorial packing problems completely in terms of set packing theory. We show that such Dantzig-Wolfe set packing formulations of combinatorial packing problems have structural properties that relate them to the original formulation and make them interesting sources of cutting planes.

The article consists of two parts. In Section 2 we introduce the concept of combinatorial packing. We give two examples of such problems, namely, on packings of two stable sets in bipartite graphs and independent sets in any number of matroids, which are naturally integral. Dantzig-Wolfe set packing formulations of combinatorial packing problems are discussed in Section 3. It is shown that such formulations give rise to cutting planes and that the intersection graphs associated with Dantzig-Wolfe formulations of combinatorial 2-packing problems are perfect.

2 Combinatorial Packing

We introduce in this section the notion of combinatorial packing. This concept subsumes a variety of combinatorial optimization problems, among them the Steiner tree packing problem, the multicommodity flow problem with unit capacities, the multiple knapsack problem, and the coloring problem. It will turn out that for some problems of this type, namely, the 2-coloring problem in bipartite graphs and the matroid packing problem, the integrality of the individual subproblems carries over to the packing composition.

Consider a family of some number k of combinatorial optimization problems

$$(\mathrm{IP}^i) \qquad \max\ {c^i}^{\mathrm{T}} x^i, \quad M^i x^i \leq b^i,\ 0 \leq x^i \leq \mathbb{1},\ x^i \in \mathbb{Z}^E, \qquad i = 1, \ldots, k \quad (1)$$

on the same ground set E. These are the *individual problems.* Associated with each of them is an *individual polytope* $P^I_{\mathrm{IP}^i} = \mathrm{conv}\{x^i \in \{0,1\}^E \mid M^i x^i \leq b^i\}$ and its fractional relaxation $P_{\mathrm{IP}^i} = \{0 \leq x^i \leq \mathbb{1} \mid M^i x^i \leq b^i\}$. An individual problem with the property $P^I_{\mathrm{IP}^i} = P_{\mathrm{IP}^i}$ is called *integral.*

A *packing* is a collection of individual solutions $x^1, \ldots, x^k$ of $\mathrm{IP}^1,\ldots,\mathrm{IP}^k$, respectively, such that each element of the ground set is contained in at most one solution. The problem to find a maximum weight packing is the *combinatorial packing problem* (CPP) associated with the individual problems IP^i, $i = 1, \ldots, k$. A CPP with k individual problems is a *(combinatorial) k-packing* *problem*. The IP formulation of a CPP reads

$$\begin{array}{lrll} \text{(CPP)} & \max \sum_{i=1}^{k} {c^i}^{\mathrm{T}} x^i & & \\ \text{(i)} & M^i x^i \le b^i, & i = 1, \ldots, k & \\ \text{(ii)} & x^i \ge 0, & i = 1, \ldots, k & \\ \text{(iii)} & \sum_{i=1}^{k} x^i \le \mathbb{1} & & \\ \text{(iv)} & x^i \in \mathbb{Z}^E, & i = 1, \ldots, k. & \end{array} \tag{2}$$

We call CPP (iii) the *packing constraints*. It will be convenient to use the notation $x^{\mathrm{T}} = ({x^1}^{\mathrm{T}}, \ldots, {x^k}^{\mathrm{T}})$ and $c^{\mathrm{T}} = ({c^1}^{\mathrm{T}}, \ldots, {c^k}^{\mathrm{T}})$. Likewise, we shall view the ground set of a combinatorial k-packing problem as a disjoint union $\dot\bigcup E^i = E^1 \,\dot\cup\, \ldots \,\dot\cup\, E^k$ of copies of the ground sets of the individual problems, where E^i is the copy of the ground set of problem IP^i. Associated with the CPP are finally the *combinatorial packing polytope* and its fractional relaxation

$$\begin{array}{l} P^I_{\mathrm{CPP}} = \mathrm{conv}\{x \in \{0,1\}^{\dot\cup E^i} \mid \sum_{i=1}^{k} x^i \le \mathbb{1},\ M^i x^i \le b^i,\ i = 1, \ldots, k\}. \\ P_{\mathrm{CPP}} = \{x \in [0,1]^{\dot\cup E^i} \mid \sum_{i=1}^{k} x^i \le \mathbb{1},\ M^i x^i \le b^i,\ i = 1, \ldots, k\}. \end{array} \tag{3}$$

A CPP is *integral* if $P^I_{\mathrm{CPP}} = P_{\mathrm{CPP}}$. If all individual problems as well as CPP itself are integral, we say that CPP is *naturally integral*.

2.1 Examples of combinatorial packing problems

The Multicommodity Flow Problem with Unit Capacities involves a *supply digraph* $D_S = (V, A_S)$ and a *demand digraph* $D_D = (V, A_D)$, both on the same node set V. We denote an arc from a node s to a node t in these digraphs by st. There are non-negative weights $w \in \mathbb{Q}_+^{A_S}$ on the arcs A_S of the supply digraph. A *multiflow* is a collection of pairwise arc disjoint directed st-paths in D_S, one for each arc $st \in A_D$ of the demand digraph. The *multicommodity flow problem with unit capacities* (MCFP) asks for a multiflow of minimum weight [1, 16, 12].

The MCFP is a combinatorial path packing problem. The individual problems are shortest path problems, one for each demand arc $st \in A_D$:

$$\begin{array}{ll} \min\ w^{\mathrm{T}} x^{st} & \\ \quad x^{st}(\delta^+(v)) - x^{st}(\delta^-(v)) = e_s - e_t, & \forall v \in V \\ 0 \le x^{st} \le \mathbb{1} & \\ \quad x^{st} \in \mathbb{Z}^{A_S}. & \end{array} \tag{4}$$

Combining the shortest path problems in a CPP adds the packing constraints $\sum_{st \in A_S} x^{st} \le \mathbb{1}$ that model the edge disjointness of the paths.

The Steiner Tree Packing Problem involves a graph $G = (V, E)$, some number k of sets of *terminal* nodes $T^1, \ldots, T^k \subseteq V$, and non-negative edge weights $w^1, \ldots, w^k \in \mathbb{Q}_+^E$. The *Steiner tree packing problem* (PST) is to find a collection of Steiner trees $S^1, \ldots, S^k$ spanning the terminals $T^1, \ldots, T^k$, respectively, such that no two Steiner trees have an edge in common [29, 23, 21, 22, 24]. Note that terminal sets of two nodes will be joined by paths such that the PST subsumes the MCFP.

The PST is a combinatorial packing problem. The individual problems, one for each terminal set T^i, $i = 1, \ldots, k$, are Steiner tree problems

$$\begin{aligned} \min\ & {w^i}^{\mathrm{T}} x^i \\ & x^i(\delta(W)) \geq 1, \qquad \forall W \subseteq V: \ W \cap T^i \neq \emptyset \neq (V \setminus W) \cap T^i \\ 0 \leq\ & x^i \leq \mathbb{1} \\ & x^i \in \mathbb{Z}^E. \end{aligned} \tag{5}$$

Combining the problems in a CPP forces the Steiner trees to be edge disjoint.

The Generalized Assignment Problem deals with a set of *jobs* J to be processed by a set of *machines* I with capacities α^i. There are resource demands a^i_j and profits w^i_j for the assignment of job j to machine i. The *generalized assignment problem* (GAP) is to find a maximum profit assignment of jobs to machines [28, 18]. The special case where the resource demands and availabilities do not depend on the machines, i.e., when $a^i = a^k$ and $\alpha^i = \alpha^k$ for all $i, k \in I$, is known as the *multiple knapsack problem* (MKP) [28, 14, 15].

The GAP models combinatorial packings of job-machine assignments. There is an individual knapsack problem for each of the machines $i \in I$

$$\max\ {w^i}^{\mathrm{T}} x^i, \quad {a^i}^{\mathrm{T}} x^i \leq \alpha^i,\ 0 \leq x^i \leq \mathbb{1},\ x^i \in \mathbb{Z}^J. \tag{6}$$

The packing constraints forbid assignments of jobs to more than one machine.

The k-Coloring Problem involves a graph $G = (V, E)$ with node weights $w \in \mathbb{Q}^V_+$ and some number $k \in \mathbb{N}$ of colors. The *k-coloring problem* (k-COL) asks for a collection of k mutually disjoint stable sets (color classes) of maximum weight [44].

A combinatorial packing formulation of the k-coloring problem is based on k individual stable set problems

$$\max\ w^{\mathrm{T}} x^i, \quad x^i_u + x^i_v \leq 1 \quad \forall uv \in E,\ 0 \leq x^i \leq \mathbb{1},\ x^i \in \mathbb{Z}^V, \tag{7}$$

one for each color $1 \leq i \leq k$. The packing constraints $\sum_{i=1}^k x^i \leq \mathbb{1}$ guarantee that each node can take at most one color.

We finish our list of examples here and remark that, in the same way, graph decomposition problems, constrained path packing problems that arise, e.g., in vehicle routing and duty scheduling, and a variety of other problems are also combinatorial packing problems.

2.2 Natural integrality

The example of the multicommodity flow problem shows that combinatorial packing problems can be hard even if all of the individual subproblems are easy and, in particular, even if complete descriptions of the individual polyhedra are explicitly known. There are, however, cases where the integrality of the individual problems carries over to the entire combinatorial packing problem. We give now two examples of combinatorial packing problems that have this natural integrality property.

The Bipartite 2-Coloring Problem (BIP-2-COL) is the special case of the 2-coloring problem where $G = (V, E)$ is a bipartite graph G. The individual problems are two set packing problems in this graph G. Their IP formulations can be stated as

$$\max\ w^{\mathrm{T}} x^i, \quad A x^i \leq \mathbb{1}, x^i \geq 0,\ x^i \in \mathbb{Z}^V, \qquad (i = 1, 2) \tag{8}$$

where $A = A(G)$ denotes the edge-node incidence matrix of G. It is well known (see, e.g., [35, III.1., Corollary 2.9]) that the edge-node incidence matrices of bipartite graphs are totally unimodular. Hence, the individual coloring problems are integral.

The IP formulation of the entire bipartite 2-coloring problem reads

$$\begin{array}{rll} \text{(BIP-2-COL)} & \max\ w^{\mathrm{T}} x^1 + w^{\mathrm{T}} x^2 & \\ \text{(i)} & x_u^1 + x_v^1 \leq 1 & \forall uv \in E \\ \text{(ii)} & x_u^2 + x_v^2 \leq 1 & \forall uv \in E \\ \text{(iii)} & x_v^1 + x_v^2 \leq 1 & \forall v \in V \\ \text{(iv)} & x_v^1, x_v^2 \geq 0 & \forall v \in V \\ \text{(v)} & x_v^1, x_v^2 \in \{0,1\}^V & \forall v \in V. \end{array} \tag{9}$$

2.1 Proposition *The bipartite 2-coloring problem is naturally integral.*

Proof (empty). We show that the constraint matrix of the bipartite 2-coloring problem is totally unimodular. This is easily done be noting that BIP-2-COL can again be seen as a set packing problem in a larger bipartite graph H. Using the convention to view the ground set of a combinatorial packing problem as a disjoint union of the ground sets of the individual problems, this graph H has as its node set the ground set $V^1 \mathbin{\dot\cup} V^2$ of the bipartite 2-coloring problem, where V^1 is a copy of the node set of the first individual coloring problem, and V^2 a copy of the second node set. For every constraint BIP-2-COL (i), there is an edge u^1v^1 between the first copies u^1 and v^1 of nodes u and v; this edge is a copy of the respective edge uv in the first individual problem. Analogously, there is an edge u^2v^2 between the second copies u^2 and v^2 of nodes u and v for every constraint BIP-2-COL (ii); this edge is a copy of the respective edge uv in the second individual problem. The graph H contains thus two disjoint copies G^1 and G^2 of G, one on the nodes V^1, the other one the nodes V^2. The only additional edges between these copies come from the constraints BIP-2-COL (iii). There is an edge v^1v^2 that joins the two copies of each original node for every packing constraint.

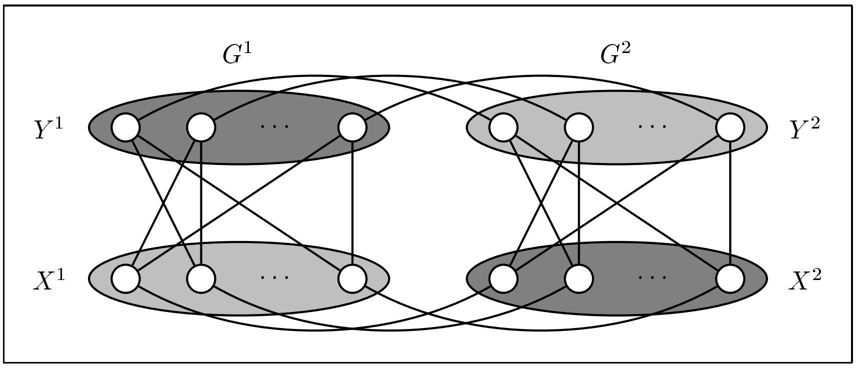

Figure 1: Bipartite 2-Coloring.

Let $X \cup Y$ be a bipartition of the nodes of G. The nodes of H can be partitioned into corresponding copies X^1, Y^1, X^2, and Y^2. Edges run between X^1 and Y^1 (first copy G^1 of G), X^2 and Y^2 (second copy G^2 of G), X^1 and X^2 (packing constraints on the copies of X), and Y^1 and Y^2 (packing constraints on the copies of Y), see Figure 1. It follows that $(X^1 \cup Y^2) \cup (X^2 \cup Y^1)$ is a bipartition of H. $\square$

The Matroid Packing Problem involves some number k of not necessarily identical matroids on the same ground set E with not necessarily identical non-negative weights $w^1, \ldots, w^k \in \mathbb{Q}_+^E$. The *matroid packing problem* (MPP) is to find a maximum weight collection of independent sets, one from each matroid, such that no two independent sets intersect on a common element.

The matroid packing problem can be stated as the following integer program:

$$\begin{array}{lrll}
\text{(MPP)} & \max \sum {w^i}^{\mathrm{T}} x^i & & \\
\text{(i)} & x^i(F) \le r^i(F) & \forall F \subseteq E, & i = 1, \ldots, k \\
\text{(ii)} & x^i \ge 0 & & i = 1, \ldots, k \\
\text{(iii)} & \sum x^i \le \mathbb{1} & & \\
\text{(iv)} & x^i \in \{0,1\}^E & & i = 1, \ldots, k.
\end{array} \tag{10}$$

Here, r^i denotes the rank function of matroid i. It is known (see, e.g., [35, Theorem 3.5]) that the individual matroid problems are integral.

2.2 Proposition *The matroid packing problem is naturally integer.*

Proof (empty). The reason for the natural integrality of the matroid packing problem is that this problem can be reinterpreted as a matroid intersection problem involving two matroids. Both of these matroids have $E^1 \cupdot \ldots \cupdot E^k$ as their ground set. The first matroid is simply the disjoint union of the k individual matroids. The second matroid is also a disjoint union of k matroids, namely, the $|E|$ uniform matroids that are induced by the packing constraints MPP (iii). Consider the packing constraint $\sum_{i=1}^{k} x_e^i \leq 1$ for element e. The matroid that is associated with this constraint has as its ground set the set $\{e^1, \ldots, e^k\}$ of copies of the element e. The non-trivial independent sets of this matroid are precisely the one-element sets $\{e^1\}, \ldots, \{e^k\}$. The disjoint union of these $|E|$ uniform matroids forms the second matroid.

By definition, MPP (i) and (ii) are a complete polyhedral description for the first matroid. Trivially, MPP (iii) and (ii) are also a complete polyhedral description of the second matroid. It is, however, well known (see, e.g., [35, III.3., Theorem 5.9]) that the union of two such systems is a complete description of the polytope that is associated with the intersection of two matroids. $\square$

Having seen two examples of naturally integral CPPs, a "converse" question that comes up is whether the integrality of the individual problems is a necessary condition for the natural integrality of a CPP. This is true if the individual problems are down monotone. The following example shows, however, that this is not true in general.

2.3 Example Consider the combinatorial 2-packing problem

$$
\begin{array}{rcrcrcrl}
\max x1 & + & x2 & + & x3 & + & x4 & \\
x1 & + & x2 & & & & & \geq 1 \\
2x1 & + & 2x2 & & & & & \leq 3 \\
 & & & & x3 & + & x4 & \geq 1 \\
 & & & & 2x3 & + & 2x4 & \leq 3 \\
 & & & & \multicolumn{3}{r}{x1,\ x2,\ x3,\ x4} & \geq 0 \\
x1 & & & + & x3 & & & \leq 1 \\
 & & x2 & & & + & x4 & \leq 1 \\
 & & & & \multicolumn{3}{r}{x1,\ x2,\ x3,\ x4} & \in \mathbb{Z}.
\end{array}
\tag{11}
$$

The individual problems produce the polytopes $P_{\mathrm{IP}^i} = \operatorname{conv}\begin{pmatrix} 0 & \frac{1}{2} & 1 & 1 \\ 1 & 1 & 0 & \frac{1}{2} \end{pmatrix}$, $i = 1, 2$, which have fractional vertices. The entire CPP is, however, integral; its associated polytope is $P_{\mathrm{CPP}} = \operatorname{conv}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} = P^I_{\mathrm{CPP}}$.

3 Dantzig-Wolfe Set Packing Formulations

Combinatorial packing problems give rise to a natural alternative set packing formulation via Dantzig-Wolfe decomposition. This connection creates a possibility to study combinatorial packing problems in terms of set packing theory. We show in this section that such Dantzig-Wolfe set packing formulations have interesting structural properties that make them potentially useful sources of cutting planes for combinatorial packing problems.

Consider a combinatorial packing problem (2). Let $M^i \in \{0,1\}^{E\times\mathfrak{V}^i}$ be a matrix whose columns are the incidence vectors of the 0/1 solutions of the individual problem IP^i, $i=1,\ldots,k$. Let us identify the index $\mathfrak{v} \in \mathfrak{V}^i$ of such a column $M^i_{\cdot\mathfrak{v}}$ with the set associated with that column, i.e., we view $\mathfrak{v}$ as a subset of the ground set E^i whose incidence vector is $M^i_{\cdot\mathfrak{v}}$ (i.e., $\chi^{\mathfrak{v}} = M^i_{\cdot\mathfrak{v}}$).

A Dantzig-Wolfe decomposition subject to the substitutions

$$x^i = M^i\lambda^i, \quad \mathbb{1}^{\mathrm{T}}\lambda^i = 1,\ \lambda^i \geq 0,\ \lambda^i \in \{0,1\}^{\mathfrak{V}^i}, \qquad i=1,\ldots,k, \tag{12}$$

transforms (2) into the form

$$\begin{array}{rrll} \text{(XPP)} & \max \sum_{i=1}^k {w^i}^{\mathrm{T}} M^i\lambda^i & & \\ \text{(i)} & \mathbb{1}^{\mathrm{T}}\lambda^i = 1, & i=1,\ldots,k & \\ \text{(ii)} & \sum_{i=1}^k M^i\lambda^i \leq \mathbb{1} & & \\ \text{(iii)} & \lambda^i \geq 0, & i=1,\ldots,k & \\ \text{(iv)} & \lambda^i \in \{0,1\}^{\mathfrak{V}^i}, & i=1,\ldots,k. & \end{array} \tag{13}$$

We call XPP the *Dantzig-Wolfe formulation* associated with CPP. Constraints XPP (i) are the *convexity constraints*, and XPP (ii) are the *packing constraints.* Using the notation $\lambda^{\mathrm{T}} = ({\lambda^1}^{\mathrm{T}},\ldots,{\lambda^k}^{\mathrm{T}})$, $M=(M^1,\ldots,M^k)$, $C=\mathrm{diag}(\mathbb{1}^{\mathrm{T}})$, $w^{\mathrm{T}} = ({w^1}^{\mathrm{T}},\ldots,{w^k}^{\mathrm{T}})$, and $\mathfrak{V} = \mathfrak{V}^1 \cup \ldots \cup \mathfrak{V}^k$, XPP becomes

$$\text{(XPP)} \quad \max\ w^{\mathrm{T}}M\lambda, \quad C\lambda = \mathbb{1},\ M\lambda \leq \mathbb{1},\ \lambda \geq 0,\ \lambda \in \{0,1\}^{\mathfrak{V}}. \tag{14}$$

XPP is closely related to the set packing problem

$$\text{(SPP)} \quad \max\ w^{\mathrm{T}}M\lambda, \quad C\lambda \leq \mathbb{1},\ M\lambda \leq \mathbb{1},\ \lambda \geq 0,\ \lambda \in \{0,1\}^{\mathfrak{V}}. \tag{15}$$

In fact, XPP arises from SPP by forcing the relaxed convexity constraints $C\lambda \leq \mathbb{1}$ to equality. This is, however, not an essential change. XPP can, e.g., be transformed into the form SPP by adding a suitably large constant $M\cdot\mathbb{1}$ to the objective. As a (modified) packing problem, XPP can be restated in graph theoretical language in terms of the intersection graph $\mathfrak{G} = (\mathfrak{V},\mathfrak{E})$ that is associated with the constraint matrix $A = \binom{C}{M}$. This graph $\mathfrak{G}$ has a node $\mathfrak{v} \in \mathfrak{V} = \mathfrak{V}^1 \cup \ldots \cup \mathfrak{V}^k$ for each individual 0/1 solution. There is an edge $\mathfrak{u}\mathfrak{v}$ for any two individual solutions $\mathfrak{u}$ and $\mathfrak{v}$ that can not simultaneously be contained in a packing. This is the case when either $\mathfrak{u}$ and $\mathfrak{v}$ are both solutions of the same individual problem such that the columns $A_{\cdot\mathfrak{u}}$ and $A_{\cdot\mathfrak{v}}$ intersect on a convexity row, or when $\mathfrak{u}$ and $\mathfrak{v}$ contain both the same element $e \in E$, i.e., $A_{\cdot\mathfrak{u}}$ and $A_{\cdot\mathfrak{v}}$ intersect on the packing row associated with the element e. In terms of $\mathfrak{G}$, XPP is the problem to find a maximum weight packing in $\mathfrak{G}$ such that each "convexity clique" is covered exactly once. This connection to set packing has polyhedral consequences. Consider the polytopes

$$\begin{aligned} P^I_{\mathrm{XPP}} &= \{\lambda \in \{0,1\}^{\mathfrak{V}} : C\lambda = \mathbb{1}, M\lambda \leq \mathbb{1}\} \\ P^I_{\mathrm{SPP}} &= \{\lambda \in \{0,1\}^{\mathfrak{V}} : A\lambda \leq \mathbb{1}\} \end{aligned} \tag{16}$$

associated with XPP and SPP and their respective fractional relaxations P_{XPP} and P_{SPP}. The polytope P^I_{SPP} is the set packing polytope associated with $\mathfrak{G}$ and P^I_{XPP} is a face of P^I_{SPP}. The combinatorial packing polytope can be obtained from P^I_{XPP} by projection.

3.1 Proposition *P^I_{CPP} is the projection of the "extended set packing polytope"*

$$\{x \mid x = \operatorname{diag}(M^i)\lambda,\ \lambda \in P^I_{\text{XPP}}\} \tag{17}$$

on the space of the x-variables.

Proposition 3.1 states that all facets of the combinatorial packing polytope are projections of set packing inequalities in some high dimensional space. This means that it is, at least in principle, possible to study combinatorial packing problems in terms of set packing theory. We remark that such a study is necessary because a Dantzig-Wolfe formulation *per se* does only contain information on the individual problems, but not on packings. Namely, Proposition 3.1 implies the following relationship between CPP and XPP (see, e.g., [43, Section 2.3] for essentially the same result):

3.2 Corollary *Let CPP be a combinatorial packing problem with integral individual problems and let XPP be its Dantzig-Wolfe formulation. Then:*

The value of the LP relaxation of CPP is equal to the value of the LP relaxation of XPP.

For combinatorial packing problems with integral individual problems such as the multicommodity flow problem, one can therefore not gain much from just restating the problem in column generation form.

The natural way to exploit Proposition 3.1 algorithmically is by using *lift-and-project* techniques [4, 5]. Suppose we want to check some point $\overline{x}$ for membership in P^I_{CPP}. Suppose also for the moment that we have a complete description $D\lambda \leq d$ of P^I_{XPP} at hand. Then, by the Farkas lemma,

$$\begin{aligned} & \overline{x} \notin P^I_{\text{CPP}} \\ \iff & \{\lambda \mid D\lambda \leq d, \operatorname{diag}(M^i)\lambda = \overline{x}\} = \emptyset \\ \iff & \exists a, b : a^{\mathrm{T}}D + b^{\mathrm{T}}\operatorname{diag}(M^i) \geq 0,\ a \geq 0,\ a^{\mathrm{T}}d + b^{\mathrm{T}}\overline{x} < 0. \end{aligned} \tag{18}$$

However, as $0 \leq a^{\mathrm{T}}D\lambda + b^{\mathrm{T}}\operatorname{diag}(M^i)\lambda \leq a^{\mathrm{T}}d + b^{\mathrm{T}}x$ is valid for any $x \in P^I_{\text{CPP}}$, the inequality

$$a^{\mathrm{T}}d + b^{\mathrm{T}}x \geq 0 \tag{19}$$

is a valid inequality for P^I_{CPP} that is violated by $\overline{x}$; such a cut can be determined by solving an appropriate LP (involving an additional normalization constraint to bound the recession cone).

Ignoring the technical difficulty of this projection process for the moment, the success of the procedure clearly depends on the quality of the description $D\lambda \leq d$ for P^I_{XPP}. Knowledge of a *complete* description of P^I_{XPP} is surely an elusive goal in general. There are, however, significant cases where such a complete description is, in some sense, in fact available.

3.3 Proposition *The intersection graph associated with the Dantzig-Wolfe formulation of a combinatorial 2-packing problem is perfect.*

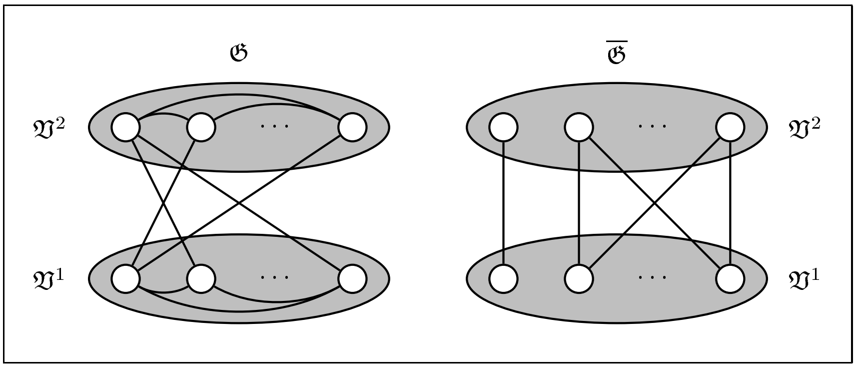

Figure 2: Intersection Graph of a Combinatorial 2-Packing Problem.

Proof (empty). We show that $\mathfrak{G}$ is the complement of a bipartite graph. The nodes of $\mathfrak{G}$ consist of the two sets $\mathfrak{V} = \mathfrak{V}^1 \cup \mathfrak{V}^2$ that correspond to the solutions of the first and the second individual problem, respectively. As there can be only one solution of each individual problem in a packing, the nodes of $\mathfrak{V}^1$ and $\mathfrak{V}^2$ form two cliques in $\mathfrak{G}$. These cliques are joined by the remaining edges, connecting solutions that have elements from the ground set in common, see Figure 2. In the complement graph $\overline{\mathfrak{G}}$, the sets $\mathfrak{V}^1$ and $\mathfrak{V}^2$ form two stable sets. Therefore, they induce a bipartition in $\overline{\mathfrak{G}}$, and hence $\overline{\mathfrak{G}}$ is perfect. □

Proposition 3.3 shows that all facets of combinatorial 2-packing polytopes are projections of *clique inequalities* [36]. The clique inequalities are subsumed by the larger class of *orthonormal representation constraints* which can be separated in polynomial time [20]. Proposition 3.3 suggests that such separation techniques, combined with lift-and-project methods, are potentially useful tools for the solution of combinatorial 2-packing problems. We remark that such techniques can, however, not lead to polynomial time algorithms for general combinatorial 2-packing problems, because this class contains $\mathcal{NP}$-hard problems such as the 2-commodity flow problem with unit capacities [17, Problem ND38].

A practical use of lift-and-project cutting planes from Dantzig-Wolfe formulations can not be that one builds up a larger and larger description of P^I_{XPP} in the exponential space $\mathbb{R}^{\mathfrak{V}}$, adding more and more cutting planes and columns. Doing so would be equivalent to a combined column generation and cutting plane approach to combinatorial packing problems with its well-known difficulties. Instead, we propose to accumulate cutting planes only in the compact original space $\mathbb{R}^{\cup E^i}$, and to use the Dantzig-Wolfe formulation solely as a separation tool.

The straightforward way to do that is as follows. Suppose we are given a point $\overline{x}$ to be tested for membership in P^I_{CPP}. The first step is to express $\overline{x}$ as a convex combination of individual solutions in the form $\overline{x}^i = M^i\lambda^i$, $i = 1, \ldots, k$. By Caratheodory's theorem, this can be done in such a way that the resulting multipliers λ^i have at most $|E| + 1$ nonzero components each. We then set up a subproblem of XPP that consists of the columns that appear in these convex combinations, apply whatever separation algorithms we have at hand, and add the resulting cuts. Projecting back, we have to be careful that our cut is dual feasible for the global XPP, i.e., we potentially have to lift a number of additional variables (this can happen because there may be more than one way to express $\overline{x}$ as a convex combination of 0/1 solutions). When this process results in a violated cutting plane for P^I_{CPP}, we add it to our current description of P^I_{CPP}, resolve, and iterate. The procedure that we have just sketched is admittedly expensive, but it points into a direction of a possible future algorithmic use of structural results such as Proposition 3.3.

We close this paper with an example that is supposed to avoid a possible misunderstanding. Proposition 3.3 does *not* make a statement that would relate perfection of the constraint matrix A of a Dantzig-Wolfe formulation or its intersection graph $\mathfrak{G}$ with natural integrality of the original formulation. The obstacle that prevents us from establishing such a connection is that the LP relaxation of a Dantzig-Wolfe formulation can have fractional vertices that correspond to integral packings.

3.4 Example Consider the following combinatorial packing problem with two uniform matroids of rank 2.

$$\begin{array}{lcccccccccccl}
\max & x_1^1 & + & x_2^1 & + & x_3^1 & + & x_1^2 & + & x_2^2 & + & x_3^2 & \\
& x_1^1 & + & x_2^1 & + & x_3^1 & & & & & & & \leq 2 \\
& & & & & & & x_1^2 & + & x_2^2 & + & x_3^2 & \leq 2 \\
& x_1^1 & & & & & + & x_1^2 & & & & & \leq 1 \\
& & & x_2^1 & & & & & + & x_2^2 & & & \leq 1 \\
& & & & & x_3^1 & & & & & + & x_3^2 & \leq 1 \\
& x_1^1 & & x_2^1, & & x_3^1, & & x_1^2, & & x_2^2, & & x_3^2 & \geq 0 \\
& x_1^1 & & x_2^1, & & x_3^1, & & x_1^2, & & x_2^2, & & x_3^2 & \in \mathbb{Z}.
\end{array} \tag{20}$$

By Proposition 2.2, the problem is naturally integral. The Dantzig-Wolfe formulation is

$$\begin{array}{l}
\max\, (0,1,1,1,2,2,2)\lambda^1 + (0,1,1,1,2,2,2)\lambda^2 \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & \\
& & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
& 1 & & & 1 & 1 & & & 1 & & & 1 & 1 & \\
& & 1 & & 1 & & 1 & & & 1 & & 1 & & 1 \\
& & & 1 & & 1 & 1 & & & & 1 & & 1 & 1
\end{pmatrix}
\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}
\begin{array}{l} = 1 \\ = 1 \\ \leq 1 \\ \leq 1 \\ \leq 1 \end{array} \\
\hspace{12em} \begin{array}{ll} \lambda^1, \lambda^2 & \geq 0 \\ \lambda^1, \lambda^2 & \in \mathbb{Z}^7. \end{array}
\end{array} \tag{21}$$

The constraint matrix A of this formulation is not perfect. The perfect clique matrix associated with the intersection graph of the Dantzig-Wolfe formulation is

$$
\left(\begin{array}{cccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & & & 1 & 1 & & & 1 & & & 1 & 1 & \\
 & & 1 & & 1 & & 1 & & & 1 & & 1 & & 1 \\
 & & & 1 & & 1 & 1 & & & & 1 & & 1 & 1 \\
\hline
 & & & & & & 1 & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & & 1 & & & 1 & & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & & & & 1 & 1 & 1 & 1 \\
 & & & & 1 & & & & 1 & 1 & & 1 & 1 & 1 \\
 & & & & 1 & & 1 & & & 1 & & 1 & 1 & 1 \\
 & & & & 1 & 1 & & & 1 & & & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 & & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 & & & & & & 1 & 1 \\
 & & 1 & & 1 & 1 & 1 & & & & & 1 & & 1 \\
 & & 1 & 1 & 1 & 1 & 1 & & & & & & & 1 \\
 & 1 & & & 1 & 1 & 1 & & & & & 1 & 1 & \\
 & 1 & & 1 & 1 & 1 & 1 & & & & & & 1 & \\
 & 1 & 1 & & 1 & 1 & 1 & & & & & 1 & &
\end{array}\right). \tag{22}
$$

This matrix adds 13 missing cliques to A. The clique in the last row contains the highlighted columns of A.

Similarly, one can verify that the 3-packing problem associated with three uniform matroids of rank 2 has an imperfect intersection graph.

Acknowledgement. I would like to thank an anonymous referee for helpful comments and suggestions, among them the idea to investigate the question behind Example 2.3. I would also like to thank Robert Weismantel and Alexander Martin for helpful comments.

References

[1] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows*, volume 1 of *Handbooks in OR and Mgmt. Sci.*, chapter IV, pages 211–369. Elsevier Sci. B.V., Amsterdam, 1989.

[2] A. Atamturk, G. L. Nemhauser, and M. W. P. Savelsbergh. The mixed vertex packing problem. TLI Tech. Rep. 98-05*, Georgia Inst. of Tech., 1998.

*Avail. at URL `http://akula.isye.gatech.edu/~atamturk/`

[3] A. Atamturk, G. L. Nemhauser, and M. W. P. Savelsbergh. Conflict graphs in solving integer programming problems. TLI Tech. Rep. 98-03†, Georgia Inst. of Tech., 1998.

[4] E. Balas. Disjunctive programming. *Annals of Disc. Math.*, 5:3–51, 1979.

[5] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Prog.*, 58:295–324, 1993.

[6] E. Balas and J. Clausen, editors. *Integer Programming and Combinatorial Optimization*, Proc. of the 4th Int. IPCO Conf., 1995.

[7] E. Balas and M.W. Padberg. Set partitioning: A survey. *SIAM Rev.*, 18:710–760, 1976.

[8] R. Borndörfer. *Aspects of Set Packing, Partitioning, and Covering.* Berichte aus der Mathematik. Shaker, Aachen, 1998. PhD thesis, Tech. Univ. Berlin‡.

[9] R. Borndörfer and R. Weismantel. Set packing relaxations of some integer programs. *Math. Prog.*, 88:425–450, 2000.

[10] R. Borndörfer and R. Weismantel. Discrete relaxations of combinatorial programs. *Disc. Appl. Math.*, 112(1–3):11–26, 2001.

[11] W. H. Cunningham, S. T. McCormick, and M. Queyranne, editors. *Integer Programming and Combinatorial Optimization*, Proc. of the 5th Int. IPCO Conf., Vancouver, British Columbia, Canada, 1996.

[12] M. Deza and M. Laurent. *Geometry of Cuts and Metrics.* Springer Verlag, Berlin, 1997.

[13] R. Euler, M. Jünger, and G. Reinelt. Generalizations of cliques, odd cycles and anticycles and their relation to independence system polyhedra. *Math. of OR*, 12(3):451–462, August 1987.

[14] C.E. Ferreira. *On Combinatorial Optimization Problems Arising in Computer System Design.* PhD thesis§, Tech. Univ. Berlin, 1994.

[15] C.E. Ferreira, A. Martin, and R. Weismantel. Solving multiple knapsack problems by cutting planes. *SIAM J. on Opt.*, 6:858 – 877, 1996.

[16] A. Frank. *Packing Paths, Circuits, and Cuts – A Survey*, pages 47–100. In Korte et al. [25], 1990.

[17] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman and Company, New York, 1979.

† Avail. at URL `http://akula.isye.gatech.edu/~atamturk/`
‡ Avail. at URL `http://www.zib.de/ZIBbib/Publications/`
§ Avail. at URL `http://www.zib.de/ZIBbib/Publications/`

[18] E.S. Gottlieb and M.R. Rao. The generalized assignment problem: Valid inequalities and facets. *Math. Prog.*, 46:31–52, 1990.

[19] R. Graham, M. Grötschel, and L. Lovász, editors. *Handbook of Combinatorics*. Elsevier Sci. B.V., Amsterdam, 1995.

[20] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, Berlin, 1988.

[21] M. Grötschel, A. Martin, and R. Weismantel. Packing steiner trees: A cutting plane algorithm and computational results. *Math. Prog.*, 72:125 – 145, 1996.

[22] M. Grötschel, A. Martin, and R. Weismantel. Packing steiner trees: further facets. *European Journal on Combinatorics*, 17:39 – 52, 1996.

[23] M. Grötschel, A. Martin, and R. Weismantel. Packing steiner trees: Polyhedral investigations. *Math. Prog.*, 72:101 – 123, 1996.

[24] M. Grötschel, A. Martin, and R. Weismantel. Packing steiner trees: separation algorithms. *SIAM Journal on Discrete Mathematics*, 9:233 – 257, 1996.

[25] B. Korte, L. Lovász, H.J. Prömel, and A. Schrijver, editors. *Paths, Flows, and VLSI-Layout*. Springer Verlag, Berlin, 1990.

[26] M. Laurent. A generalization of antiwebs to independence systems and their canonical facets. *Math. Prog.*, 45:97–108, 1989.

[27] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM J. on Opt.*, 1:166–190, 1991.

[28] S. Martello and P. Toth. *Knapsack Problems*. John Wiley & Sons Ltd, Chichester, 1990.

[29] A. Martin. *Packen von Steinerbäumen: Polyedrische Studien und Anwendung*. PhD thesis, Technische Universität Berlin, 1992.

[30] R. Müller. On the partial order polytope of a digraph. *Math. Prog.*, 73(1):31–49, 1996.

[31] R. Müller and A.S. Schulz. The interval order polytope of a digraph. In [6], pages 50–64, 1995.

[32] R. Müller and A.S. Schulz. Transitive packing. In [11], pages 430–444, 1996.

[33] George L. Nemhauser, A. H. G. Rinnooy Kan, and Michael J. Todd, editors. *Optimization*, volume 1 of *Handbooks in OR and Management Science*. Elsevier Sci. B.V., Amsterdam, 1989.

[34] G.L. Nemhauser and L.E. Trotter. Properties of vertex packing and independence system polyhedra. *Math. Prog.*, 6:48–61, 1973.

[35] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization.* John Wiley & Sons, Inc., 1988.

[36] M.W. Padberg. On the facial structure of set packing polyhedra. *Math. Prog.*, 5:199–215, 1973.

[37] M.W. Padberg. A note on zero-one programming. *Op. Res.*, 23(4):833–837, 1975.

[38] M.W. Padberg. On the complexity of set packing polyhedra. *Annals of Disc. Math.*, 1:421–434, 1977.

[39] M.W. Padberg. Covering, packing, and knapsack problems. *Annals of Disc. Math.*, 4:265–287, 1979.

[40] M.W. Padberg and T.-Y. Sung. An analytical comparison of different formulations of the travelling salesman problem. *Math. Prog.*, 52(2):315–357, 1991.

[41] A.S. Schulz. *Polytopes and Scheduling.* PhD thesis¶, Tech. Univ. Berlin, 1996.

[42] Yasuki Sekiguchi. A note on node packing polytopes on hypergraphs. *OR Letters*, 2(5):243–247, 1983.

[43] M. Sol. *Column Generation Techniques for Pickup and Delivery Problems.* PhD thesis, Tech. Univ. Eindhoven, 1994.

[44] B. Toft. *Colouring, Stable Sets and Perfect Graphs*, chapter 4, pages 233–288. In Graham et al. [19], 1995.

[45] L.E. Trotter. A class of facet producing graphs for vertex packing polyhedra. *Disc. Math.*, 12:373–388, 1975.

¶Avail. at URL `ftp://ftp.math.tu-berlin.de/pub/Preprints/combi/`