Stable Set and Other Techniques for Frequency Assignment Problems

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Abstract. In mobile telephone systems stationary antennas broadcast and receive data from and to handies inside a geographical region of service around an antenna called a cell. Use of the same or adjacent frequencies in nearby cells results in interference. The frequency assignment problem is to find an assignment of frequencies to the antennas such that the resulting total interference is minimized.
We give two integer programming formulations for this problem. The first model is a stable set-type formulation, the second is called the orientation model. We discuss some of the properties of these formulations and use them to develop a number of heuristics for the frequency assignment problem. The methods are in use at the German mobile telephone system provider e plus. Our computational results for frequency assignment problems arising there show that significant improvements in network quality can be achieved.

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1 Frequency Assignment in Mobile Telephone Systems

Mobile telephone systems use stationary antennas to receive and broadcast data from and to the mobile stations (handies). The low powered radio signals used can be received only inside a (small) geographical region around the sending device. The region of service around an antenna is called a *cell*. Depending on an estimated demand, one or more radio frequencies are assigned to each antenna to transmit phone calls in its cell. Each assigned frequency provides a communication channel for a number of phone calls in a cell; such a communication channel is called a *carrier*.
Each mobile telephone system provider has only a limited number of frequencies at his disposal; these are spaced out evenly along the electromagnetic spectrum such that each frequency is separated by a constant distance from the following one. However, if the same or adjacent frequencies are used to establish carriers in nearby cells, *co-* or *adjacent channel interference* causes loss of transmission quality or breakdown of connections. This leads to stipulations that frequencies used in nearby pairs of cells have to be separated by a minimum distance, called *channel separation*, while smaller interferences might be unavoidable, but should at least be minimum. The *frequency assignment problem* is to assign to each carrier an available frequency such that minimum channel separations are respected and the resulting interference is minimized. The point in this formulation is that it asks for use of the complete available spectrum of frequencies to minimize interference. This concept is relatively new, an early reference is Plehn [Plehn]. Previous approaches attacked the problem by minimizing the number of used frequencies, see Box [1978] or Gamst [1988].
In mathematical terms, the frequency assignment problem can be stated as follows. Let

$$G = (V, E)$$

be a graph,

$$d \in \mathbb{N}_0^E, \quad \text{and} \quad \overline{p}, \overline{\overline{p}} \in [0,1]^E$$

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be vectors with the property

$$\overline{\overline{p}} \leq \overline{p},$$

and

$$C = \{0, 1, \ldots, \zeta\} \subseteq \mathbb{N}_0$$

be the set of zero and the first ζ integers. Each node $i \in V$ corresponds to a carrier, each edge $ij \in E$ has an associated minimum channel separation d_{ij}, an adjacent channel interference $\overline{\overline{p}}_{ij}$ and a co-channel interference $\overline{p}_{ij}$ (some of them possibly zero), and C is the set of available frequencies.

The frequency assignment problem can now be stated as follows. Introducing variables

$$y_i \in C \quad \forall i \in V$$

for the frequency assigned to carrier i, the frequency assignment problem becomes:

$$\begin{array}{rcll} \min & \sum\limits_{|y_i - y_j| = 0} \overline{p}_{ij} + \sum\limits_{|y_i - y_j| = 1} \overline{\overline{p}}_{ij} & & \\ |y_i - y_j| & \geq & d_{ij} & \forall ij \in E \\ y_i & \leq & \zeta & \forall i \in C \\ y_i & \geq & 0 & \forall i \in C \\ y_i & \in & Z & \forall i \in C. \end{array} \qquad \text{(FAP)}$$

This formulation is non-linear because of the channel-separation constraints and the objective function.

2 The Stable-Set Model

There is a natural connection between frequency assignment and graph coloring that was first noticed by Metzger [1970]: If the frequencies are interpreted as colors and all channel separations are 1, the frequency assignment problem asks for an assignment of colors to the nodes of a graph such that all nodes of the same color are mutually non-adjacent or, in other words, the nodes of the same color form a stable set. An adaptation of this idea to the interference case yields the first integer programming model, called the *stable set model*. For a similar approach see Aardal & van Hoesel [1995].

We use the following variables.

$$\begin{array}{rcll} y_i^f & = & \begin{cases} 1, & \text{if frequency } f \text{ is assigned to carrier } i \\ 0, & \text{else} \end{cases} & \forall i \in V, f \in C \\ \overline{z}_{ij} & = & \begin{cases} 1, & \text{if } \exists f : y_i^f = y_j^f = 1 \\ 0, & \text{else} \end{cases} & \forall ij \in E \\ \overline{\overline{z}}_{ij} & = & \begin{cases} 1, & \text{if } \exists f : y_i^f = y_j^{f+1} = 1 \\ 0, & \text{else} \end{cases} & \forall ij \in E \end{array}$$

The y-variables model frequency assignment. Note that there is one variable y_i^f for each possible assignment of a frequency f to a carrier i. The z-variables model co- and adjacent-channel interference: $\overline{z}_{ij}$ or $\overline{\overline{z}}_{ij}$ will be set to one if use of the same or adjacent frequencies for carriers i and j leads to interference. With these variables, the frequency assignment problem can be stated as the following integer linear program.

$$\begin{array}{llcll} \min & \overline{p}^T \overline{z} + \overline{\overline{p}}^T \overline{\overline{z}} & & & \\ & y_i^f + y_j^g & \leq & 1 & \forall ij \in E, f, g \in C : |f - g| < d_{ij} \quad (1) \\ & \sum_{f \in C} y_i^f & = & 1 & \forall i \in V \quad (2) \\ & y_i^f + y_j^f - \overline{z}_{ij} & \leq & 1 & \forall ij \in E, f \in C \quad (3) \\ & y_i^f + y_j^{f+1} - \overline{\overline{z}}_{ij} & \leq & 1 & \forall ij \in E, f, f+1 \in C \quad (4) \\ & y_i^f + y_j^{f-1} - \overline{\overline{z}}_{ij} & \leq & 1 & \forall ij \in E, f, f-1 \in C \quad (5) \\ & y_i^f & \in & \{0, 1\} & \forall i \in V, f \in C \quad (6) \\ & \overline{z}_{ij} & \in & \{0, 1\} & \forall ij \in E \quad (7) \\ & \overline{\overline{z}}_{ij} & \in & \{0, 1\} & \forall ij \in E \quad (8) \end{array} \qquad \text{(SIP)}$$

Constraints (1) models minimum channel separations by forbidding all pairs of conflicting assignments. (2) guarantees that each carrier gets a frequency assigned, (3) takes care of co-channel interference, (4) and (5) of adjacent channel interference by forcing the corresponding z-variables to 1.

3 The Orientation Model

In this section, we will give an alternative formulation for the frequency assignment problem that is based on different ideas. We introduce binary decision variables

$$\Delta_{ij} \in \{0,1\} \quad \forall ij \in E$$

with the meaning

$$\Delta_{ij} = \begin{cases} 1 & \text{if } y_j - y_i \geq 0 \\ 0 & \text{if } y_j - y_i \leq 0. \end{cases}$$

Let us assume here that $i < j$ for an edge $ij \in E$. Then, Δ_{ij} is 1 if the edge ij is oriented 'upward' from the lower channel y_i to the higher (or equal) channel y_j and 0 if the edge ij is oriented 'downward', that is, channel y_i is larger than or equal to channel y_j; if y_i and y_i are equal, Δ_{ij} can be both zero or one.
We can now state the frequency assignment problem as another integer linear program.

$$\begin{array}{llllll}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & & \\
& y_j - y_i & \geq & d_{ij}\Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in E & (1) \\
& -y_j + y_i & \geq & d_{ij}(1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in E & (2) \\
& y_j - y_i + \overline{z}_{ij} & \geq & \Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in E & (3) \\
& -y_j + y_i + \overline{z}_{ij} & \geq & (1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in E & (4) \\
& y_j - y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2\Delta_{ij} - \zeta(1-\Delta_{ij}) & \forall ij \in E & (5) \\
& -y_j + y_i + 2\overline{z}_{ij} + \overline{\overline{z}}_{ij} & \geq & 2(1-\Delta_{ij}) - \zeta\Delta_{ij} & \forall ij \in E & (6) \\
& y_i & \leq & \zeta & \forall i \in V & (7) \\
& \Delta_{ij} & \leq & 1 & \forall ij \in E & (8) \\
& \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \leq & 1 & \forall ij \in E & (9) \\
& y_i & \geq & 0 & \forall i \in V & (10) \\
& \Delta_{ij}, \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \geq & 0 & \forall ij \in E & (11) \\
& y_i & \in & \mathbb{Z} & \forall i \in V & (12) \\
& \Delta_{ij}, \overline{z}_{ij}, \overline{\overline{z}}_{ij} & \in & \mathbb{Z} & \forall ij \in E & (13)
\end{array} \qquad \text{(TIP)}$$

Conditions (7), (10), and (12) state that the channel assignment variables have to be chosen from the available spectrum. The objective sums up total interference; note that the coefficients are all non-negative. Constraints (1) and (2) model minimum channel separation. If $\Delta_{ij} = 1$, (2) is redundant, but (1) guarantees that the larger frequency y_j has a minimum distance of d_{ij} from the smaller one y_i, for $\Delta_{ij} = 0$, (1) is redundant but (2) active. Inequalities (3) and (4) deal with co-channel interference in a very similar way, (5) and (6) are for adjacent channel interference.
We can write the model in a more compact form using matrix notation.

$$\begin{array}{llll}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & \\
& Ay + Bz & \geq & d - D\Delta \\
& -y & \geq & -\zeta\mathbb{1} \\
& \Delta & \leq & \mathbb{1} \\
& y, \Delta, z & \geq & 0 \\
& y, \Delta, z & \in & \mathbb{Z}^{V \times E \times E \times E}
\end{array} \qquad \text{(TIP)}$$

Considering the feasible Δs, this formulation has relations to the linear ordering problem, because feasible orientations direct the edges of the underlying graph G in such a way that (in a sense) an acyclic digraph results, where all directed paths are of restricted length.

If the orientation Δ is fixed, the remaining problem reads

$$
\begin{array}{lrcl}
\min & p^{\mathrm{T}} z & & \\
 & Ay + Bz & \geq & d - D\Delta \\
 & -y & \geq & -\zeta \mathbb{1} \\
 & y, z & \geq & 0.
\end{array}
\qquad (\mathrm{TIP}(\Delta))
$$

It has a special structure because the matrix A is a network matrix and B is almost an identity matrix. An interesting case arises when each edge with non-zero co-channel interference has zero adjacent-channel interference and each edge with adjacent channel interference has a minimum channel separation of at least one, such that co-channel interference cannot occur. If these conditions guarantee that on all edges only either co- or adjacent channel interference is possible, the frequency assignment problem is called *simple*. In this, case, TIP(Δ) will be integral for any feasible Δ.

1 Theorem *If the frequency assignment problem TIP is simple, the problems TIP(Δ) will have all integral solutions for any feasible orientation Δ.*

In fact, in this case TIP(Δ) will be the dual of a min-cost flow problem. To see this, consider a simple TIP(Δ) in more detail. Let $\overline{E}$ be the set of edges with non-zero co-channel interference, $\overline{\overline{E}}$ the set of edges with non-zero adjacent-channel interference; further, let $\overline{A}$ and $\overline{\overline{A}}$ be the set of arcs resulting from orienting the edges in $\overline{E}$ and $\overline{\overline{E}}$ according to Δ. Then, after eliminating redundant constraints, the problem reads

$$
\begin{array}{llcll}
\min & \overline{p}^{\mathrm{T}}\overline{z} + \overline{\overline{p}}^{\mathrm{T}}\overline{\overline{z}} & & & \\
(x_{ij}) & y_j - y_i & \geq & d_{ij} & \forall (i,j) \in A \\
(\overline{x}_{ij}) & y_j - y_i + \overline{z}_{ij} & \geq & 1 & \forall (i,j) \in \overline{A} \\
(\overline{\overline{x}}_{ij}) & y_j - y_i \qquad + \overline{\overline{z}}_{ij} & \geq & 2 & \forall (i,j) \in \overline{\overline{A}} \\
(x_{ij}) & -y_j + y_i & \geq & d_{ij} & \forall (j,i) \in A \\
(\overline{x}_{ij}) & -y_j + y_i + \overline{z}_{ij} & \geq & 1 & \forall (j,i) \in \overline{A} \\
(\overline{\overline{x}}_{ij}) & -y_j + y_i \qquad + \overline{\overline{z}}_{ij} & \geq & 2 & \forall (j,i) \in \overline{\overline{A}} \\
(x_{tj}) & -y_j & \geq & -\zeta & \forall j \in V \\
 & y, z & \geq & 0 &
\end{array}
$$

Introducing the dual variables on the left of the inequalities, the dual looks as follows.

$$
\begin{array}{lrcll}
\max & d^{\mathrm{T}}x + \mathbb{1}^{\mathrm{T}}\overline{x} + 2\mathbb{1}^{\mathrm{T}}\overline{\overline{x}} - \zeta\mathbb{1}^{\mathrm{T}}x_t & & & \\
 & \left(x + \overline{x} + \overline{\overline{x}}\right)\left(\delta^-(j)\right) - \left(x + \overline{x} + \overline{\overline{x}}\right)\left(\delta^+(j)\right) - x_{tj} & \leq & 0 & \forall j \in V \quad (1) \\
 & \overline{x}_{ij} & \leq & \overline{p}_{ij} & \forall ij \in \overline{E} \quad (2) \\
 & \overline{\overline{x}}_{ij} & \leq & \overline{\overline{p}}_{ij} & \forall ij \in \overline{\overline{E}} \quad (3) \\
 & x, x_t & \geq & 0 & \qquad\quad (4)
\end{array}
\qquad (\mathrm{MCF})
$$

($\delta^-(j)$ denotes the arcs entering node j, $\delta^+(j)$ the arcs leaving j). If we interpret x and x_t as flow variables, MCF becomes a min-cost flow problem.
In general, TIP will not be simple. However, if $\overline{\overline{p}} \leq \frac{1}{2}\overline{p}$ holds, an easy transformation can be applied to yield a simple problem. In our applications at eplus, this conditions holds for practically all edges.

2 Theorem *Any frequency assignment problem TIP with the property*

$$\overline{\overline{p}} \leq \frac{1}{2}\overline{p}$$

can be polynomially transformed into an equivalent simple problem.

4 Frequency Assignment Heuristics

The two integer programming formulations of the last two sections inspire the development of several primal heuristics. All of these methods can be applied iteratively in combination with each other.

4.1 Heuristics Based on the Stable-Set Model

Elimination. Elimination is a "dual" heuristic. The idea is to iteratively eliminate possibilities to assign frequencies to carriers instead of choosing an assignment. In each step, each carrier has a set of still eligible frequencies. To each such frequency, a penalty is associated according to some interference based cost criterion. The assignment that is worst in this respect is eliminated, the penalties are updated, and the process is repeated.

Iterated 1-Opt. This is an improvement heuristics. Iteratively, a carrier is picked. If reassignment of its frequency yields an improvement, this is done until no longer possible or some other stopping criterion is met. There is also a randomized version of this heuristic.

DSATUR. DSATUR is an adaptation of an algorithm of Brélaz [1979] for graph coloring to the frequency assignment problem. The heuristic iteratively assigns frequencies to carriers. In each step, a partial assignment has already been computed and then minimum channel separations restrict the choice of frequencies for carriers that have not yet a frequency assigned. A carrier with the least number of possibilities is picked and —based on some interference related criterion— an assignment is made.

4.2 Heuristics Based on the Orientation Model

T-Coloring. T-Coloring is an opening heuristic. It neglects interference costs and only tries to find a feasible assignment that will induce a corresponding orientation. The algorithm is an adaptation of an algorithm by Costa [1993].

Min-Cost-Flow. Min-Cost-Flow is an improvement heuristic. Given a frequency assignment produced by some other heuristic, the corresponding orientation is computed. Fixing this orientation and (heuristically) transforming the problem into a simple frequency assignment problem, a min-cost flow problem results that is solved.

4.3 Computation

The methods of the previous two subsections were developped for the German telephone system provider eplus and are in use there. The frequency assignment problems arising there involve up to 2000 cells with 1, 2, 3 or even more carriers per cells and interference relations extend over long distances. Most of the heuristics produce feasible solutions in a couple of minutes. In our tests, Iterated 1-Opt and DSATUR performed best, Min-Cost-Flow usually yields small further improvements in interference.

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