Line Planning and Steiner Path Connectivity

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Overview

- Service Design
- Steiner Path Connectivity
- Line Planning
- Project Stadt+
Planning Problems in Public Transit

Service Design

Operational Planning

Operations Control

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Line Plan

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Line Planning and Steiner Path Connectivity

**Line Planning Problem**
Find a cost minimal set of lines and associated frequencies, s.t. a given set of travel demands can be transported in minimal time.

**Steiner Path Connectivity Problem**
Find a cost minimal set of paths that provide enough capacity to route a fastest multi-commodity flow.

**Features**
- Bicriteria problem (cost vs. quality)
- Passenger behavior (transfers)

Graphics: JavaView, F4
Line Planning and Steiner Path Connectivity

Line Planning Problem
- Find lines and frequencies to satisfy a given demand

Objectives
- Minimize operation costs
- Minimize travel time

Input
- Public transport network
- OD-matrix of travel demands
- Operation costs and travel times

Output
- Set of lines and frequencies
- Passenger flow

Steiner Path Connectivity Problem
- Find paths to connect a set of terminal nodes

Objectives
- Minimize path costs

Input
- Undirected graph
- Set of terminal nodes
- Path costs

Output
- Connecting set of paths
Steiner (Path) Connectivity Problem
(B., Karbstein., Pfetsch [2009])

\[
\begin{align*}
\text{min} & \quad \sum_{\ell \in \mathcal{L}} c_{\ell} x_{\ell} \\
\text{s.t.} & \quad \sum_{\ell \in \mathcal{L}_s(W)} x_{\ell} \geq 1 \quad \emptyset \neq W \cap T \neq V \\
& \quad x_{\ell} \in \{0, 1\}
\end{align*}
\]

Minimize cost
Connect all OD nodes
Complexity and Approximation

Proposition
1. The Steiner connectivity problem (SCP) is NP-hard even for T=V.
2. The SCP is solvable in polynomial time for |T| = r, r fixed (T set of OD nodes).
3. The SCP is not approximable in the general case.

Theorem
There exists a \((k + 1)\)-approximation algorithm for the SCP, where

\[ k = \min\{\text{max. path length}, \text{max. number of terminals per path}\}. \]

Proof: Primal dual algorithm similar to Goemans, Williamson [1995] for the Steiner forest problem.

Basic property of minimal solutions (w.r.t. inclusion):

Proposition (B, Karbstein [2014]): The average path-degree (nr. of paths incident to a node) of a terminal node is \( \leq k + 1 \).

For Steiner trees the node degree (nr. of edges) of non-terminal nodes is \( \geq 2 \); the analogue is not true for minimal solutions of the SCP.
(SCP\textsubscript{cut}) \quad \min \sum_{p \in \mathcal{P}} c_p x_p \quad \text{subject to}\ \begin{align*} \sum_{p \in \mathcal{P}_{\delta(W)}} x_p &\geq 1 \quad \forall \emptyset \neq W \cap T \subsetneq T \\ x_p &\in \{0, 1\} \quad \forall p \in \mathcal{P} \end{align*} 

where $\mathcal{P}_{\delta(W)} = \{p \in \mathcal{P} : \mathcal{P}_{\delta(W)} \cap p \neq \emptyset\}$ (Steiner path cuts) is the set of paths that cross a Steiner cut.
Steiner Path Cut

\[
W = \{d, f, g\} \quad \rightarrow \quad x_1 + x_2 + x_4 + x_6 \geq 1
\]

Weaker:
\[
x_1 + x_2 + 2x_4 + x_6 \geq 1
\]
Separation of Steiner Path Constraints

Construct Steiner connectivity digraph $D'$ for $\hat{x} \in \mathbb{R}_0^P$:

- Consider cuts between $s$ and $t$; $V' = \{s, t\} \cup \{v_p, w_p : p \in P\}$
- Insert arcs $a = (v_p, w_p), p \in P$ with capacity $\kappa_a = \hat{x}_p$
- Insert arcs $a = (s, v_p), (w_p, t), p \in P$ with capacity $\kappa_a = \hat{x}_p$
- Insert arcs $a = (w_{\tilde{p}}, v_p), p, \tilde{p} \in P, r \notin p, p \cap \tilde{p} \neq \emptyset$ with capacity $\kappa_a = \min\{\hat{x}_p, \hat{x}_{\tilde{p}}\}$
**Proposition**

There is a one-to-one correspondence between minimal directed \((s, t)\)-cuts in \(D'\) and minimal \((s, t)\)-Steiner path cuts in \(G\), and the capacities are equal.

**Proposition**

The separation problem for Steiner path cut constraints can be solved in polynomial time.
Let \( P = (V_1, V_2, \ldots, V_k) \), \( V_i \cap T \neq \emptyset \), \( i = 1, \ldots, k \), \( V = \bigcup_{i=1}^{k} V_i \).

The Steiner partition inequality is defined as

\[
\sum_{p \in \mathcal{P}} a_p \cdot x_p \geq k - 1
\]

where \( a_p := |\{i \in \{1, \ldots, k\}: V_i \cap p \neq \emptyset\}| - 1 \).

\( x_1 + x_2 + x_3 \geq 1 \)
\( x_1 + x_2 + x_4 \geq 1 \)
\( x_1 + x_3 + x_4 \geq 1 \)

\( x_2 = x_3 = x_4 = 0.5 \) satisfies all Steiner path cuts

\( 2x_1 + x_2 + x_3 + x_4 \geq 2 \) separates this solution
Let $\overline{P} := \{p \in P: a_p = 0\}$. Generalization of a result of Grötschel, Monma & Stoer [1990]:

**Proposition**

A Steiner partition inequality is facet defining if the following properties are satisfied.

1. $G[V_i]$ is connected by $\overline{P}$, $\forall i$
2. $G[V_i]$ contains no Steiner-path-bridge in $\overline{P}$, $\forall i$
3. Each path is incident with at most two node sets, i.e., $a_p \in \{0,1\}$
4. The shrunk graph (each node set a single node) is 2-node-path-connected

Only property 4 is necessary!
Directed Cut Formulation

Can use Steiner connectivity digraph to get equivalent directed formulation:

All "path"-arcs get the cost of the corresponding path. All other arcs get 0 cost.

→ Directed Steiner tree problem in $D'$
Directed Cut Formulation

\[(SCP_{arc}^r) \min \sum_{a \in A'} c'_a x_a \]

\[
\text{s.t. } \sum_{a \in \delta^-(W)} x_a \geq 1 \quad \forall W \subseteq V' \setminus \{r\}, \ W \cap T \neq \emptyset \\
\quad x_a \in \{0, 1\} \quad \forall a \in A'.
\]

**Theorem**

\[P_{LP}(SCP_{cut}) = P_{LP}(SCP_{arc}) \bigg|_P\]

Consequence: objective value of \((SCP_{arc}^r)\) is independent of \(r\).
Flow Balance Constraints

Flow balance constraints:

\[ \sum_{a \in \delta^+(v)} y_a \geq \sum_{a \in \delta^-(v)} y_a \quad \forall v \in V' \setminus T \]

\[ (SCP_{arc+}^r) \min \sum_{a \in A'} c'_a y_a \]

s.t. \[ \sum_{a \in \delta^-(W)} y_a \geq 1 \]

\[ y_{v_p w_p} \geq \sum_{a \in \delta^-(v_p)} y_a \quad \forall v_p \in V' (p \in \mathcal{P} : r \notin p) \]

\[ \sum_{a \in \delta^+(w_p)} y_a \geq y_{(v_p, w_p)} \quad \forall w_p \in V' (p \in \mathcal{P} : t \notin p \forall t \in T) \]

\[ y_a \in \{0, 1\} \quad \forall a \in A' \]

- Quality depends on root
Theorem

\[ P_{LP}(SCP_{arc}^+) \mid \mathcal{P} \] satisfies all Steiner partition inequalities.

\[ 2x_1 + x_2 + x_3 + x_4 = 2x_{1 \overline{1}} + x_{2 \overline{2}} + x_{3 \overline{3}} + x_{4 \overline{4}} \geq 2x_{a_1} + x_{a_2} + x_{a_3} + x_{\overline{1}4} + x_{\overline{2}4} + x_{\overline{3}4} \geq 1 + x_{a_1} + x_{a_3} + x_{\overline{1}4} + x_{\overline{2}4} \geq 2 + x_{\overline{1}4} \geq 2 \]
The Case \(|T| = 2\)

\[ (SCP_{cut}) \quad \min \sum_{p \in \mathcal{P}} c_p x_p \]

\[ \sum_{p \in \mathcal{P} \delta(W)} x_p \geq 1 \quad \forall \emptyset \neq W \cap T \subsetneq T \]

\[ x_p \in \{0, 1\} \quad \forall p \in \mathcal{P} \]

**Theorem**

\((SCP_{cut})\) is TDI for \(T = \{s, t\}\). In particular, it is integral.

**Totally dual integral (TDI):** For each integral \(c\) for which the optimum is finite, the dual has an integral optimal solution.

**Proof:** Primal-dual algorithm.
The Case $|T| = 2$

\[
\text{(SCP}_{\text{cut}}) \quad \min \sum_{p \in \mathcal{P}} c_p x_p
\]

\[
\sum_{p \in \mathcal{P}_{\delta(W)}} x_p \geq 1 \quad \forall \emptyset \neq W \cap T \subsetneq T
\]

\[
x_p \in \{0, 1\} \quad \forall p \in \mathcal{P}
\]

c $\equiv 1$: A Menger companion theorem follows:

**Theorem**

The minimum cardinality of an \((s, t)\)-connecting set is equal to the maximum number of path-disjoint \((s, t)\)-disconnecting sets.
Min-Max Results

- An \((s, t)\)-connected set of paths connects two nodes \(s\) and \(t\) \((\triangleq (s, t)\)-path).
- An \((s, t)\)-disconnecting set of paths breaks all \((s, t)\)-connected sets \((\triangleq (s, t)\)-cut).

**Theorem (Menger and companion theorem for path-connectivity)**

1. The minimum cardinality of an \((s, t)\)-disconnecting set is equal to the maximum number of path-disjoint \((s, t)\)-connected sets.
2. The minimum cardinality of an \((s, t)\)-connected set is equal to the maximum number of path-disjoint \((s, t)\)-disconnecting sets.

- 1. is hypergraph folklore, 2. is new.
- Generalizes class of blocking pairs of ideal incidence matrices of paths and cuts.

**Example (2):** Network with 5 paths

- \((s, t)\)-connected:
  1. green
  2. yellow
  3. red

- \((s, t)\)-disconnecting (path-disjoint):
  1. green
  2. yellow, blue, brown
  3. red
Relation to Line Planning

Observation for $c \equiv 1$:

- minimum number of paths connecting $s$ and $t$
- lower bound on number of transfers $- 1$ in line planning

Idea: use this lower bound in line planning

- associate with a passenger route $r$ the minimum number of transfers $k$
- $k$ depends on all possible lines
- include constraints to ensure direct connections, i.e., for $k_r = 0$
  - $y_{r,0}$ – variable for direct connection
  - $y_{r,1}$ – variable for connections with at least one transfer
Line Planning and Steiner Path Connectivity

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Features:
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Graphics: JavaView, F4
Line Planning Problem

- Find lines and frequencies to satisfy a given demand

Objectives

- Minimize operation costs
- Minimize travel time

Input

- Public transport network
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Output

- Set of lines and frequencies
- Passenger flow

Steiner Path Connectivity Problem

- Find paths to connect a set of terminal nodes

Objectives

- Minimize path costs

Input

- Undirected graph
- Set of terminal nodes
- Path costs

Output

- Connecting set of paths
Example

- Line capacities 50
- Demands
  \[a \to f = 50, a \to b = 50, d \to f = 20, d \to c = 80\]
- Solution
  lines \(l_3, l_4\) with frequency 2
  line \(l_5\) with frequency 1
Minimize transfers/transfer time
Scholl [2005], Schöbel & Scholl [2005], Schmidt [2012]
- **Advantage**: detailed treatment of transfers
- **Disadvantage**: *change-and-go-graph* on the basis of all possible lines; large scale model

Maximize travel quality
Nachtigall & Jerosch [2008]
- **Advantage**: utility for each path including all transfers
- **Disadvantage**: capacity constraint for each partial route and line; large scale model

Minimize pareto function of line cost and travel times
B., Grötschel & Pfetsch[2005], B., Neumann & Pfetsch [2008]
- **Advantage**: allows line pricing; computationally tractable
- **Disadvantage**: ignores transfers within same transportation mode
Basic Line Planning Model (B)  
(B., Grötschel & Pfetsch [2007])

- Line capacities 50
- Demands
  \[ a \rightarrow f = 50, \ a \rightarrow b = 50 \]
  \[ d \rightarrow f = 20, \ d \rightarrow c = 80 \]
- Solution
  - lines \( l_3, l_4 \) with frequency 2
  - line \( l_5 \) with frequency 1

Travel time on path is sum of travel times on edges

\[
\begin{align*}
p_1 &= (a, e, f), & \tau_{p_1} &= \tau_{ae} + \tau_{ef} \\
p_2 &= (a, e, f, b), & \tau_{p_2} &= \tau_{ae} + \tau_{ef} + \tau_{fb} \\
p_3 &= (d, g, f), & \tau_{p_3} &= \tau_{dg} + \tau_{gf} \\
p_4 &= (d, g, c), & \tau_{p_4} &= \tau_{dg} + \tau_{gc}
\end{align*}
\]

Note: direct connection are not distinguished from transfer connections, e.g., paths \( p_1 \) and \( p_3 \)
Basic Line Planning Model (B)  
(B., Grötschel & Pfetsch [2007])

\[
\begin{align*}
\min & \quad \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} + (1 - \lambda) \sum_{p \in \mathcal{P}} \tau_p y_p \\
\text{Subject to} & \quad \sum_{p \in \mathcal{P}_{st}} y_p = d_{st} \quad \forall s, t \in D \\
& \quad \sum_{p \ni a} y_p \leq \sum_{\ell \in \mathcal{L} \ni a} \sum_{f \in \mathcal{F}} k_{\ell,f} x_{\ell,f} \quad \forall a \in \mathcal{A} \\
& \quad \sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L} \\
& \quad y_p \geq 0, x_{\ell,f} \in \{0, 1\}
\end{align*}
\]

Variables: \( x_{\ell,f} = 1 \) if line \( \ell \in \mathcal{L} \) is chosen with frequency \( f \in \mathcal{F} \); \( x_{\ell,f} = 0 \) otherwise
\( y_p \geq 0 \) passengers on path \( p \in \mathcal{P} \)

Features
- Complete line pool
- Multi-criteria objective
- Integrated passenger routing

Disadvantage
- No transfers \( \rightarrow \) Direct Connection Model (Metric Inequalities)
Change & Go Model (CG)
(Schöbel & Scholl [200])

- Copy every node number of lines containing node times, i.e., nodes \((v, \ell) \forall v \in V, v \ni \ell \in \mathcal{L}\)
- Complete transfer graph for every node, i.e., edges \(((v, \ell), (v, \ell')) \forall \ell, \ell' \in \mathcal{L}\)

Travel time on path is sum of travel times on travel + transfer edges

\[
p_1 = (a, e, f), \quad \tau_{p_1} = \tau_{(a, \ell_3)}(e, \ell_3) + \tau_{(e, \ell_3)}(f, \ell_3)
\]
\[
p_2 = (a, e, f, b), \quad \tau_{p_2} = \tau_{(a, \ell_3)}(e, \ell_3) + \tau_{(e, \ell_3)}(f, \ell_3) + \tau_{(f, \ell_3)}(f, \ell_5) + \tau_{(f, \ell_5)}(b, \ell_5)
\]
\[
p_3 = (d, g, f), \quad \tau_{p_3} = \tau_{(d, \ell_4)}(g, \ell_4) + \tau_{(g, \ell_4)}(g, \ell_5) + \tau_{(g, \ell_5)}(f, \ell_5)
\]
\[
p_4 = (d, g, c), \quad \tau_{p_4} = \tau_{(d, \ell_4)}(g, \ell_4) + \tau_{(g, \ell_4)}(c, \ell_4)
\]

Note: all transfers are taken into account
Change & Go Model (CG)  
(Schöbel & Scholl [200])

\[
\begin{align*}
\min & \quad \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} + (1 - \lambda) \sum_{p \in \mathcal{P}} \tau_p y_p \\
\text{Minimize cost and travel time} \\
\sum_{p \in \mathcal{P}_{st}} y_p = d_{st} \quad & \forall s, t \in D \\
\text{Transport all demand} \\
\sum_{p \ni a} y_p \leq \sum_{\ell \in \mathcal{L} \ni a} \sum_{f \in \mathcal{F}} \kappa_{\ell,f} x_{\ell,f} \quad & \forall (a, \ell) \in \mathcal{A}_{\ell} \\
\text{Capacity constraints} \\
\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad & \forall \ell \in \mathcal{L} \\
\text{One frequency per line} \\
y_p \geq 0, \\
x_{\ell,f} \in \{0, 1\} \\
Variables: \ x_{\ell,f} = 1 \ if \ line \ \ell \in \mathcal{L} \ is \ chosen \ with \ frequency \ f \in \mathcal{F}; \ x_{\ell,f} = 0 \ otherwise \\
y_p \geq 0 \ passengers \ on \ path \ p \in \mathcal{P}
\end{align*}
\]

Features
- Complete line pool
- Multi-criteria objective
- Integrated passenger routing
- Correct handling of transfers

Disadvantage
- Very large graph
Direct Line Connection Model (DLC)
(B., Karbstein [2012])

- Idea: associate with a pax path either a direct connection line or a transfer penalty:
  - $y_{p,0}$ number of pax on $p$ traveling directly with $\ell$
  - $y_{p,1}$ number of pax on $p$ traveling with $\geq 1$ transfer

Associate transfer penalty $\sigma$ with non-direct connections

- $p_1 = (a, e, f)$, $y_{p_1,0}^\ell = 50$, $\tau_{p_1} = \tau_{ae} + \tau_{ef}$
- $p_2 = (a, e, f, b)$, $y_{p_2,1}^\ell = 50$, $\tau_{p_2} = \tau_{ae} + \tau_{ef} + \tau_{fb} + \sigma$
- $p_3 = (d, g, f)$, $y_{p_3,1}^\ell = 20$, $\tau_{p_3} = \tau_{dg} + \tau_{gf} + \sigma$
- $p_4 = (d, g, c)$, $y_{p_4,0}^\ell = 80$, $\tau_{p_4} = \tau_{dg} + \tau_{gc}$

Note: $y_{p_1,0}^\ell = y_{p_3,0}^\ell = 0$, since $\ell_1, \ell_7$ not in solution
Direct Line Connection Model (DLC)
(B., Karbstein [2012])

Variables:
\[ x_{\ell,f} \in \{0, 1\} \quad \text{choose line } \ell \text{ with frequency } f \]
\[ z_{p,0} \in \mathbb{R}_+ \quad \text{passenger flow on direct connection } (p, \ell) \]
\[ y_{p,1} \in \mathbb{R}_+ \quad \text{passenger flow on } p \text{ with at least one transfer} \]

Objective:
\[
\min \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} + (1 - \lambda) \left( \sum_{p \in \mathcal{P}^0} \sum_{\ell \in \mathcal{L} : p \in \mathcal{P}^0, \ell} \tau_{p,0} z_{p,0}^\ell + \sum_{p \in \mathcal{P}} \tau_{p,1} y_{p,1} \right)
\]

transport all passengers
\[
\sum_{\ell \in \mathcal{L}} \sum_{p \in \mathcal{P}^0, \ell} z_{p,0}^\ell + \sum_{p \in \mathcal{P}_{st}} y_{p,1} = d_{st} \quad \forall (s, t) \in D
\]

capacity constraints
\[
\sum_{\ell \in \mathcal{L}} \sum_{p \in \mathcal{P}^0, \ell : a \in p} z_{p,0}^\ell + \sum_{p \in \mathcal{P} : a \in p} y_{p,1} \leq \sum_{\ell \in \mathcal{L} : e(a) \in \ell} \sum_{f \in \mathcal{F}} k_{\ell,f} x_{\ell,f} \quad \forall a \in A
\]

direct line connection (capacity) constraints
\[
\sum_{p \in \mathcal{P}^0, \ell : a \in p} z_{p,0}^\ell \leq \sum_{f \in \mathcal{F}} k_{\ell,f} x_{\ell,f} \quad \forall \ell \in \mathcal{L}, e(a) \in \ell
\]

one frequency per line
\[
\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L}
\]
Relaxation I – Line Aggregation

Idea: Aggregate $y_{p,0} = \sum_{\ell \in \mathcal{L}} y_{p,0}^\ell$

Problem: Replace dlc constraints

$$
\sum_{p \in \mathcal{P}_0^0, \alpha \in p} z_{p,0}^\ell \leq \sum_{f \in \mathcal{F}} \kappa_{f,\ell} \times x_{\ell,f}
$$

Idea: find set of paths that have the same direct connection lines

- $\mathcal{L}_{st}^0(\alpha) = \text{set of dc lines for } (s, t) \text{ containing arc } \alpha$
- $[s, t]_\alpha = \{(u, v) : \mathcal{L}_{uv}^0(\alpha) = \mathcal{L}_{st}^0(\alpha)\}$
- $[s, t]_{\alpha}^\leq = \{(u, v) : \mathcal{L}_{uv}^0(\alpha) \subseteq \mathcal{L}_{st}^0(\alpha)\}$

$$(d, g) : y_{adfg,0} + y_{dgf,0} \leq \sum_{f \in \mathcal{F}} \kappa_{\ell_1,f} \times \ell_1,f + \sum_{f \in \mathcal{F}} \kappa_{\ell_7,f} \times \ell_7,f$$

$$(a, d) : y_{adfg,0} \leq \sum_{f \in \mathcal{F}} \kappa_{\ell_1,f} \times \ell_1,f$$
Direct Connection Model (DC)
(B., Karbstein [2012])

Here: same travel times as in DLC

\[ p_1 = (a, e, f), \quad y_{p_1,0} = 50, \quad \tau_{p_1} = \tau_{ae} + \tau_{ef} \]

\[ p_2 = (a, e, f, b), \quad y_{p_2,1} = 50, \quad \tau_{p_2} = \tau_{ae} + \tau_{ef} + \tau_{fb} + \sigma \]

\[ p_3 = (d, g, f), \quad y_{p_3,1} = 20, \quad \tau_{p_3} = \tau_{dg} + \tau_{gf} + \sigma \]

\[ p_4 = (d, g, c), \quad y_{p_4,0} = 80, \quad \tau_{p_4} = \tau_{dg} + \tau_{gc} \]
Direct Connection Model (DC)
(B., Karbstein [2012])

Variables:  \( x_{\ell,f} \in \{0, 1\} \)  
choose line \( \ell \) with frequency \( f \)

\( y_{p,0} \in \mathbb{R}_+ \)  
passenger flow on a direct connection path

\( y_{p,1} \in \mathbb{R}_+ \)  
passenger flow on a path with at least one transfer

Objective:

\[
\min \lambda \sum_{\ell \in \mathcal{L}} \sum_{f \in \mathcal{F}} c_{\ell,f} x_{\ell,f} + (1 - \lambda) \left( \sum_{p \in \mathcal{P}^0} \tau_{p,0} y_{p,0} + \sum_{p \in \mathcal{P}} \tau_{p,1} y_{p,1} \right)
\]

transport all passengers

\[
\sum_{p \in \mathcal{P}^0} y_{p,0} + \sum_{p \in \mathcal{P}} y_{p,1} = d_{st} \quad \forall (s, t) \in D
\]

capacity constraints

\[
\sum_{p \in \mathcal{P}^0 : a \in p} y_{p,0} + \sum_{p \in \mathcal{P} : a \in p} y_{p,1} \leq \sum_{\ell \in \mathcal{L}} \sum_{e(a) \in \ell} \sum_{f \in \mathcal{F}} \kappa_{\ell,f} x_{\ell,f} \quad \forall a \in A
\]

direct connection (capacity) constraints

\[
\sum_{(u,v) \in [s,t]_a} \sum_{p \in \mathcal{P}^0(a)} y_{p,0} \leq \sum_{\ell \in \mathcal{L}^0(a)} \sum_{f \in \mathcal{F}} \kappa_{\ell,f} x_{\ell,f} \quad \forall a \in A, [s,t]_a \in D(a)
\]

one frequency per line

\[
\sum_{f \in \mathcal{F}} x_{\ell,f} \leq 1 \quad \forall \ell \in \mathcal{L}
\]
Problem: How to identify direct connection paths, i.e., what is $\mathcal{P}^{0+}$?

Idea: For OD pair $(s, t)$ consider direct connection st-passenger routing digraph $G_{st}$ induced by all direct connection lines for $(s, t)$.

$\mathcal{P}_{st}^{0+} =$ set of all paths in $G_{st}$, $\mathcal{P}^{0+} = \bigcup_{s,t} \mathcal{P}_{st}^{0+}$
Problem: How to identify direct connection paths, i.e., what is $P^{0+}$?

Idea: For OD pair $(s, t)$ consider direct connection st-passenger routing digraph $G_{st}$ induced by all direct connection lines for $(s, t)$

$P^{0+}_{st} = \text{set of all paths in } G_{st}, P^{0+} = \bigcup_{s,t} P^{0+}_{st}$
Example

- **Line capacity** = 50
- **Demands**
  \[ c \rightarrow a = 50, \quad d \rightarrow b = 50 \]
  \[ c \rightarrow e = 50 \]
- **Path** \((c, d, e)\) is considered as a direct connection since it is a path in \(G_{ce}\)
- **Line capacity** = 50
- **Demands**
  \[ c \rightarrow a = 50, \quad d \rightarrow b = 50 \]
- **DC-constraints on** \((b, c)\)
  \[
y_{abc,0} \leq \sum_{f \in F} \kappa_{l_1,f} x_{l_1,f} + \kappa_{l_3,f} x_{l_3,f}
  \]
  \[
y_{bcd,0} \leq \sum_{f \in F} \kappa_{l_1,f} x_{l_1,f} + \kappa_{l_2,f} x_{l_2,f}
  \]
- \(y_{abc,0} = y_{bcd,0} = 50\), but either \((a, c)\) or \((b, d)\) must transfer
DC Model Discussion

- The direct line connection model (DLC) accounts exactly for the number of direct travelers in a system optimum. It is a first order approximation of model (CG).

- Model (DC) is a relaxation of the projection of model (DLC) onto the space of the direct connection path variables, i.e., (DC) approximates the number of direct travelers.

- The relaxation is due to
  - a larger set of direct connection paths \( P_{st}^0 \subseteq P_{st}^{0+} \),
  - the direct connection constraints, a small, combinatorial set of all projected direct line connection constraints.

- (DC) is algorithmically tractable:
  - passenger paths variables not dependent on lines;
  - pricing passenger path variables is a shortest path problem.

- Model (DC) can be seen as transfer improvement of model (B)
Proposition

1. Uncapacitated case (B., G., Pfetsch [2009]): The fundamental classes of Steiner cut and Steiner partition inequalities can be generalized to the Steiner connectivity problem and hence to the line planning problem.

2. Capacitated case (K. [recently]): The fundamental classes of band inequalities and Steiner partition band inequalities can be generalized to the line planning problem.

Example (2): Network with 4 lines

- Capacity \( \kappa_e = 10 \)
- Frequency \( F = \{1,2\} \)
- Steiner partition band inequality
  \[ 2x_{1,2} + x_{2,2} + x_{3,2} + x_{4,2} \geq 2 \]
- The yellow or two other lines with frequency 2 have to be chosen.

Public transport network partitioned into 3 components
Numbers give max. number of passengers leaving/entering component
Column Generation Approach

begin
solve line planning problem (IP)
solve LP relaxation
compute prices
compute pax paths
compute lines
all fixed?
no yes
stop?
no yes
fix lines
end
end
Test Instances

Dutch

China
Test Instances

Sioux Falls

Potsdam
### Test Instances

| problem       | $|D|$ | $|V_o|$ | $|V|$ | $|A|$ | $|L|$ | x-vars | dc-cons | cons |
|---------------|-----|-------|------|------|------|--------|---------|------|
| Dutch1        | 420 | 23    | 23   | 106  | 402  | 1 608  | 1 832   | 1 080 |
| Dutch2        | 420 | 23    | 23   | 106  | 2 679| 10 716 | 7 544   | 3 341 |
| Dutch3        | 420 | 23    | 23   | 106  | 7 302| 29 208 | 9 736   | 7 945 |
| China1        | 379 | 20    | 20   | 98   | 474  | 1 896  | 2 754   | 1 178 |
| China2        | 379 | 20    | 20   | 98   | 4 871| 19 484 | 8 162   | 5 457 |
| China3        | 379 | 20    | 20   | 98   | 19 355| 77 420 | 12 443  | 19 931 |
| SiouxFalls1   | 528 | 24    | 24   | 124  | 866  | 3 464  | 4 400   | 1 779 |
| SiouxFalls2   | 528 | 24    | 24   | 124  | 9 397| 37 588 | 16 844  | 10 197 |
| SiouxFalls3   | 528 | 24    | 24   | 124  | 15 365| 61 460 | 21 220  | 16 145 |
| Potsdam98a    | 7 734| 107   | 344  | 2 746| 207  | 7 776  | 3 538   | 9 970 |
| Potsdam98b    | 7 734| 107   | 344  | 2 746| 1 907| 7 572  | 60 902  | 11 991 |
| Potsdam98c    | 7 734| 107   | 344  | 2 746| 4 342| 17 313 | 76 640  | 14 366 |
| Potsdam2009   | 4 443| 236   | 851  | 5 542| 3 433| 14 140 | 30 780  | 12 006 |

- Frequencies: 3, 6, 9, 18 (\(\triangle\) cycle time of 60, 30, 20, 10 min in 3 h)
- Could not solve CG root LP for red instances in 10 h
## DC vs. B

<table>
<thead>
<tr>
<th>Problem</th>
<th>Time</th>
<th>Nodes</th>
<th>Gap</th>
<th>Time</th>
<th>Nodes</th>
<th>Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dutch1</td>
<td>15s</td>
<td>329</td>
<td>opt.</td>
<td>10h</td>
<td>5 940 327</td>
<td>0.03%</td>
</tr>
<tr>
<td>Dutch2</td>
<td>&lt;1h</td>
<td>11 532</td>
<td>opt.</td>
<td>10h</td>
<td>815 966</td>
<td>0.04%</td>
</tr>
<tr>
<td>Dutch3</td>
<td>10h</td>
<td>57 273</td>
<td>0.05%</td>
<td>10h</td>
<td>151 053</td>
<td>0.08%</td>
</tr>
<tr>
<td>China1</td>
<td>10h</td>
<td>814 964</td>
<td>0.32%</td>
<td>10h</td>
<td>3 754 582</td>
<td>0.11%</td>
</tr>
<tr>
<td>China2</td>
<td>10h</td>
<td>5 366</td>
<td>0.53%</td>
<td>10h</td>
<td>129 217</td>
<td>0.15%</td>
</tr>
<tr>
<td>China3</td>
<td>10h</td>
<td>997</td>
<td>0.47%</td>
<td>10h</td>
<td>37 519</td>
<td>0.18%</td>
</tr>
<tr>
<td>SiouxFalls1</td>
<td>10h</td>
<td>458 379</td>
<td>0.10%</td>
<td>&lt;1h</td>
<td>347 999</td>
<td>opt.</td>
</tr>
<tr>
<td>SiouxFalls2</td>
<td>10h</td>
<td>13 868</td>
<td>0.09%</td>
<td>10h</td>
<td>110 836</td>
<td>0.01%</td>
</tr>
<tr>
<td>SiouxFalls3</td>
<td>10h</td>
<td>3 230</td>
<td>0.10%</td>
<td>10h</td>
<td>44 713</td>
<td>0.00%</td>
</tr>
<tr>
<td>Potsdam98a</td>
<td>10h</td>
<td>7 357</td>
<td>0.09%</td>
<td>10h</td>
<td>6 266</td>
<td>0.12%</td>
</tr>
<tr>
<td>Potsdam98b</td>
<td>10h</td>
<td>62</td>
<td>0.28%</td>
<td>10h</td>
<td>2 491</td>
<td>0.26%</td>
</tr>
<tr>
<td>Potsdam98c</td>
<td>10h</td>
<td>10</td>
<td>0.27%</td>
<td>10h</td>
<td>661</td>
<td>0.25%</td>
</tr>
<tr>
<td>Potsdam2010</td>
<td>10h</td>
<td>2</td>
<td>0.81%</td>
<td>10h</td>
<td>2 123</td>
<td>0.41%</td>
</tr>
</tbody>
</table>

- Line costs proportional to line lengths plus fixed cost
- Objective weighing parameter $\lambda = 0.8$, transfer penalty $\sigma = 15$ min
Verifying Direct Travelers

<table>
<thead>
<tr>
<th>problem</th>
<th>travel time</th>
<th>cost</th>
<th>obj.</th>
<th>dir. trav. $^1$</th>
<th>dir. trav. $^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dutch1 (DC)</td>
<td>$1.279 \cdot 10^7$</td>
<td>68 900</td>
<td>2613305</td>
<td>179 496</td>
<td>179 496</td>
</tr>
<tr>
<td>Dutch1 (B)</td>
<td>$1.333 \cdot 10^7$</td>
<td>57 800</td>
<td>2711770</td>
<td>183 582</td>
<td>148 924</td>
</tr>
<tr>
<td>Dutch2 (DC)</td>
<td>$1.279 \cdot 10^7$</td>
<td>66 900</td>
<td>2612122</td>
<td>180 484</td>
<td>179 384</td>
</tr>
<tr>
<td>Dutch2 (B)</td>
<td>$1.319 \cdot 10^7$</td>
<td>57 500</td>
<td>2683071</td>
<td>183 582</td>
<td>156 251</td>
</tr>
<tr>
<td>Dutch3 (DC)</td>
<td>$1.279 \cdot 10^7$</td>
<td>66 900</td>
<td>2612122</td>
<td>180 484</td>
<td>179 384</td>
</tr>
<tr>
<td>Dutch3 (B)</td>
<td>$1.319 \cdot 10^7$</td>
<td>57 500</td>
<td>2683071</td>
<td>183 582</td>
<td>156 251</td>
</tr>
<tr>
<td>China1 (DC)</td>
<td>$1.259 \cdot 10^7$</td>
<td>267 937</td>
<td>2732445</td>
<td>749 736</td>
<td>716 040</td>
</tr>
<tr>
<td>China1 (B)</td>
<td>$1.559 \cdot 10^7$</td>
<td>233 268</td>
<td>3304432</td>
<td>759 950</td>
<td>509 526</td>
</tr>
<tr>
<td>China2 (DC)</td>
<td>$1.258 \cdot 10^7$</td>
<td>247 241</td>
<td>2714438</td>
<td>759 936</td>
<td>709 145</td>
</tr>
<tr>
<td>China2 (B)</td>
<td>$1.559 \cdot 10^7$</td>
<td>233 268</td>
<td>3304432</td>
<td>759 950</td>
<td>509 526</td>
</tr>
<tr>
<td>China3 (DC)</td>
<td>$1.245 \cdot 10^7$</td>
<td>244 361</td>
<td>2684860</td>
<td>759 950</td>
<td>714 728</td>
</tr>
<tr>
<td>China3 (B)</td>
<td>$1.559 \cdot 10^7$</td>
<td>233 268</td>
<td>3304432</td>
<td>759 950</td>
<td>509 526</td>
</tr>
</tbody>
</table>

$^1$ direct travelers computed by model

$^2$ direct travelers computed a posteriori at system optimum
## Verifying Direct Travelers

<table>
<thead>
<tr>
<th>problem</th>
<th>travel time</th>
<th>cost</th>
<th>obj.</th>
<th>dir. trav.</th>
<th>dir. trav.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SiouxFalls1 (DC)</td>
<td>$3.267 \cdot 10^6$</td>
<td>9 205</td>
<td>660675</td>
<td>360 600</td>
<td>358 888</td>
</tr>
<tr>
<td>SiouxFalls1 (B)</td>
<td>$3.633 \cdot 10^6$</td>
<td>8 295</td>
<td>733288</td>
<td>360 600</td>
<td>335 355</td>
</tr>
<tr>
<td>SiouxFalls2 (DC)</td>
<td>$3.392 \cdot 10^6$</td>
<td>5 787</td>
<td>682996</td>
<td>360 600</td>
<td>360 178</td>
</tr>
<tr>
<td>SiouxFalls2 (B)</td>
<td>$3.776 \cdot 10^6$</td>
<td>5 178</td>
<td>759365</td>
<td>360 600</td>
<td>326 625</td>
</tr>
<tr>
<td>SiouxFalls3 (DC)</td>
<td>$3.431 \cdot 10^6$</td>
<td>4 899</td>
<td>690200</td>
<td>360 600</td>
<td>355 068</td>
</tr>
<tr>
<td>SiouxFalls3 (B)</td>
<td>$3.695 \cdot 10^6$</td>
<td>4 283</td>
<td>742397</td>
<td>360 600</td>
<td>334 052</td>
</tr>
<tr>
<td>Potsdam98a (DC)</td>
<td>$5.076 \cdot 10^6$</td>
<td>27 044</td>
<td>1036865</td>
<td>70 513</td>
<td>71 075</td>
</tr>
<tr>
<td>Potsdam98a (B)</td>
<td>$5.102 \cdot 10^6$</td>
<td>29 018</td>
<td>1043617</td>
<td>83 702</td>
<td>68 900</td>
</tr>
<tr>
<td>Potsdam98b (DC)</td>
<td>$4.836 \cdot 10^6$</td>
<td>33 484</td>
<td>993938</td>
<td>78 745</td>
<td>79 511</td>
</tr>
<tr>
<td>Potsdam98b (B)</td>
<td>$4.970 \cdot 10^6$</td>
<td>28 302</td>
<td>1016610</td>
<td>84 879</td>
<td>73 983</td>
</tr>
<tr>
<td>Potsdam98c (DC)</td>
<td>$4.829 \cdot 10^6$</td>
<td>32 544</td>
<td>991772</td>
<td>79 694</td>
<td>79 576</td>
</tr>
<tr>
<td>Potsdam98c (B)</td>
<td>$4.952 \cdot 10^6$</td>
<td>29 320</td>
<td>1013779</td>
<td>84 979</td>
<td>74 356</td>
</tr>
<tr>
<td>Potsdam2010 (DC)</td>
<td>$1.032 \cdot 10^6$</td>
<td>9 314</td>
<td>213769</td>
<td>38 152</td>
<td>38 001</td>
</tr>
<tr>
<td>Potsdam2010 (B)</td>
<td>$1.073 \cdot 10^6$</td>
<td>8 734</td>
<td>221549</td>
<td>41 052</td>
<td>35 285</td>
</tr>
</tbody>
</table>
Verifying Direct Travelers

Deviation of direct travelers from a posteriori system optimum:

<table>
<thead>
<tr>
<th></th>
<th>DC</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>7%</td>
<td>49%</td>
</tr>
<tr>
<td>Min</td>
<td>0%</td>
<td>7.5%</td>
</tr>
<tr>
<td>Avg</td>
<td>1.9%</td>
<td>22.8%</td>
</tr>
</tbody>
</table>
Project Stadt+

Goal
- Rearrange line plan, minimize travel time, no cost increase

Data
- Public transport network including lengths and times of 2009
- Demand estimation (OD data) of 2007
- Vehicle capacity bus 125, tram 170

Planning Requirements
- Cycle time of (10,) 20, 30, 60 minutes; tram at least 20 minutes
- Minimum frequency 20, 60 (one line), 10, 30 (two lines) minutes
- Maximum travel time for each line ≤ 45 minutes
- Definition of endpoints for lines, important and forbidden stations
- No parallel traffic of bus and tram
- Fixed BVG and HVG lines

Objectives
- Cost per kilometer for bus and tram
- Penalty of 15 minutes for each transfer
Bus: ViP vs. ZIB

Buslinien

ViP
- 698
- 603
- 605
- 699

ZIB
- new13832
- new29524
- new28256
- new30369

Graphics: VISUM, ptv AG
Tram: ViP vs. ZIB

Line Planning and Steiner Path Connectivity | COeTL 2015
Comparing Cost and Travel Times

from 6:00 to 9:00 am

<table>
<thead>
<tr>
<th></th>
<th>ViP</th>
<th>ZIB</th>
<th>min cost</th>
<th>min time</th>
</tr>
</thead>
<tbody>
<tr>
<td>bus lines</td>
<td>16</td>
<td>16</td>
<td>17</td>
<td>120</td>
</tr>
<tr>
<td>tram lines</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>bus km</td>
<td>2392</td>
<td>2497</td>
<td>1644</td>
<td>10565</td>
</tr>
<tr>
<td>tram km</td>
<td>1440</td>
<td>1207</td>
<td>1054</td>
<td>3156</td>
</tr>
<tr>
<td>cost</td>
<td>8057</td>
<td>7717</td>
<td>5688(^1)</td>
<td>28091(^2)</td>
</tr>
<tr>
<td>time(^3) with transfer penalty(^4)</td>
<td>6.2484</td>
<td>6.1927</td>
<td>6.7344</td>
<td>6.0812</td>
</tr>
<tr>
<td>time(^3) without transfer penalty</td>
<td>5.1641</td>
<td>5.1550</td>
<td>5.2889</td>
<td>5.1422</td>
</tr>
</tbody>
</table>

\(^1\) optimal solution
\(^2\) 0.4% duality gap
\(^3\) in 10\(^7\) sec
\(^4\) transfer penalty 15 min
## Comparing Travel Times with VISUM

<table>
<thead>
<tr>
<th></th>
<th>ZIB</th>
<th>ViP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average travel time</td>
<td>36 min 3 s</td>
<td>36 min 39 s</td>
</tr>
<tr>
<td>Average carriage time</td>
<td>16 min 8 s</td>
<td>17 min 42 s</td>
</tr>
<tr>
<td>Average time in vehicle</td>
<td>13 min 8 s</td>
<td>14 min 36 s</td>
</tr>
<tr>
<td>Average transfer waiting time</td>
<td>1 in 30 s</td>
<td>1 min 29 s</td>
</tr>
<tr>
<td>Average walking time</td>
<td>1 min 38 s</td>
<td>1 min 37 s</td>
</tr>
<tr>
<td>Average perceived travel time</td>
<td>26 min</td>
<td>27 min 37 s</td>
</tr>
<tr>
<td>Total number transfers</td>
<td>10595</td>
<td>11141</td>
</tr>
<tr>
<td>0 transfers</td>
<td>37338</td>
<td>36851</td>
</tr>
<tr>
<td>1 transfer</td>
<td>10088</td>
<td>10503</td>
</tr>
<tr>
<td>2 transfers</td>
<td>243</td>
<td>306</td>
</tr>
<tr>
<td>&gt;2 transfers</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>
Comparing Transfers and Loads

Direct travelers by districts:
- **Green**: ZIB > ViP
- **Red**: ViP > ZIB

Load:
- **Red**: ZIB > ViP
- **Blue**: ViP > ZIB

Legende
Bezirke
Personenfahrten ÖV 0 Umstiege (Diff)
- <= -2
- <= 1
- > 1
Strecken
Belastung ÖV [Pers] (AP,Diff)
- <= 0
- > 0
Conclusion

- The ViP line plan achieves a good compromise between minimizing travel times and costs.
- The ZIB line plan achieve additional improvements by shortening the tram network.
- The guidelines (for means of transportation, minimum service frequencies) strongly influence the result.
- Even if the network is changed only slightly, passenger routes can differ strongly.
- The current practice of data acquisition, computation of statistics, and evaluation of results is unsatisfactory.
- Given suitable data and parameterization, optimization tools can compute line plans which are at least on a par with plans computed by experienced planners.
Thank you for your attention

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