we consider the symmetric TSP on a graph $G = (V, E)$ with edge weights $w_{ij} = 6$

**Def. 6.1** (k-opt move): let $T \subseteq E$ be a tour in $G$, $k \in \mathbb{N}$,

$X = \{x_{1}, \ldots, x_{k}\} \subseteq T$ set of cut-edges
$Y = \{y_{1}, \ldots, y_{k}\} \subseteq \mathbb{E} \setminus T$ set of cut-edges
$X \rightarrow Y$ improving $k$-opt move
$g(x, y) = \|x\| - \|y\|$ length of $k$-opt move $X \rightarrow Y$

**Lemma 6.2** (Proposed by Jun & Kwonig et al. [1972]): let $S_0 \ldots S_n \in \mathbb{R}$
be a sequence of n real numbers s.t. $\sum_{i=0}^{n} S_i > 0$. Then, there is an index $k \in \{0, \ldots, n-1\}$ such that the partial sums

$\sum_{i=0}^{k} S_i \leq 0 \quad j = 0, \ldots, n-1$ 

**Proof:** let $k$ be the largest index s.t. $S_0 + \ldots + S_{k-1}$ is minimal. Then

$g_x + \ldots + g_j = (g_0 + \ldots + g_j) - (g_0 + \ldots + g_{k-1}) > 0 \quad k \leq j \leq n-1$

$g_x + \ldots + g_j + g_{k} + \ldots + g_i \geq g_y + g_{x} + g_{y} + \ldots + g_{i} > 0$ for $0 \leq j < k$

**Problem 6.2** (Douglas [1973, Problem 22]): along a speed bane dune are some
gas stations. The total amount of gasoline available in dune is
capped to what our car (which has a very large tank) needs for
going around the dune. Prove that there is a gas-station such that
if we start there with an empty tank, we should be able to go
around the dune without running out of gasoline.
Def. 6.4 (Sequential k-opt move): A k-opt move $X \rightarrow Y$ is a sequential exchange $k$-opt.

$$X = \{x_1, \ldots, x_k\} = \{t_{2i-1}, t_{2i}\}_{i=1}^k$$

$$Y = \{y_1, \ldots, y_k\} = \{t_{2i}, t_{2i+1}\}_{i=1}^k$$

$$t_{2i} = t_{2i+1}$$

Ex. 6.5 (Sequential 4-opt move):

Def. 6.6: A sequential exchange $k$-opt $X \rightarrow Y$ is

1) disjoint: $\Delta X \cap Y = \emptyset$

2) improving: $\Rightarrow g_i = g_i(X_i, Y_i) = \frac{1}{2} x_{i-1} + y_{i+1} > 0$ for $i = 1, \ldots, k$

3) feasible: $\Rightarrow [t_{2i-1}, t_{2i}] \rightarrow [t_{2i}, t_{2i+1}]$ is an i-opt move, $i = 2, \ldots, k$

4) restricted: $\Rightarrow t_{2i+1} \in N_i(t_{2i})$ in some (small) neighborhood of $t_{2i}$

5) sequential: $\Rightarrow X \rightarrow Y$ is disjoint, improving, feasible, depending on the level $i$

Obs. 6.7: i) There is a 1-1 correspondence between sequential $k$-opt moves and negative alternating cycles of length $2k$ in $A$.

ii) Consider $X_i \rightarrow Y_i$, $i = 2, \ldots, k$. If

$$X_i = \{t_{2i-1}, t_{2i}, t_{2i+1}, t_{2i+2}\} \cup \{t_{2i-2}, t_{2i-1}\}$$

$$Y_i = \{t_{2i}, t_{2i+1}, t_{2i+2}, t_{2i+3}\} \cup \{t_{2i-1}, t_{2i-2}\}$$

is an i-opt move, then a 1/2 move for $t_{2i}$ and hence for $X_i$, i.e., the degrees of freedom in a sequential k-opt move are the choices of $x_i, y_i, \ldots, y_{i+1}$, while $x_{2i}, t_{2i}$ are uniquely determined.

iv) every improving 2- and 3-opt move is sequential. ($\Rightarrow E(i)$)

v) none of the 4-opt moves described are not sequential, e.g., the
So-called double-bridge move (Huang, Price & Teller 1992) \( \rightarrow 6x \)

\[ x_i, y_i \rightarrow y_i \] produces an \( h \)-tree \( i = 1, \ldots, h - 1 \)

This move leads to a data structure in which

\[ x_i, y_i \rightarrow y_i \] is appended (Toussaint & Hoggard 1997).

vi) Popular reductions are

a) Nearest neighbors (Liu & Shannon [1973]):

\[ N_i(u) = N_2(u) - \text{argmin} \] \( i \geq 3 \)

b) Nearest neighbors (Heldermann [2006]):

\[ N_i(u) = \text{argmin} \] \( i \geq 3 \)

vi) Let \( \Lambda \in V \), \( \Gamma \) be an minimal \( h \)-tree, and for \( \Lambda \in E \) let

\[ T(\Lambda) \text{ be a minimal } h \text{-tree containing edge } \Lambda \text{. Heldermann [2006]} \]

\[ \alpha(\Lambda) = \text{cost} \] \( \Lambda \text{ in } T \text{ and } T \text{ in } G \text{, and distance cycle } (\rightarrow 6x) \text{ in } O(n^3) \text{ time}. \]

\[ \alpha(\Lambda) = 0 \text{ if } \Lambda \in T \text{ (6x)} \]

\[ \alpha(\Lambda) \text{ can be computed by adding } \Lambda \text{ to } T \text{ and deleting the } \]

\[ \alpha(\Lambda) \text{ can be computed in } O(n^2) \text{ time (and } \Lambda \text{ in } O(n) \text{ space).} \]
Lemma 6.3 \( f(v_i) \) can be computed in \( O(n) \) time for any fixed \( v_i \).

Proof (Sketch):

\( f(v_2) \) can be computed from \( f(v_1) \). \( \Box \)

Alg 6.3 - Steiner-Tree-Cover (Shiloach & Tarjan [1976])

Input: \( G = (V, E), \quad \alpha = \infty \quad \text{and} \quad \beta = \infty \).

Output: \( T \) is a \( (T, \alpha, \beta) \)-tree cover.

1. \( i = 1, \quad i \neq 1 \), \( G^* = G, \quad \alpha = \frac{1}{i}, \quad \beta = \frac{1}{i} \).

2. \( i < |V| \)

3. \( t_i \in V \quad \text{s.t.} \quad \gamma_i = t_i \quad \text{and} \quad \gamma_i = \min_{v \in V} \gamma_i(v) > 0 \).

4. \( i = i + 1 \)

5. \( t_i \in V \quad \text{s.t.} \quad \gamma_{i-1} = t_i \quad \text{and} \quad \gamma_i = \min_{v \in V} \gamma_i(v) > 0 \).

6. If no such \( t_i \) exists and \( i > 3 \) then

\( i = i - 1 \), goto 3.
5. If $i = 2$ goto 1 and try the alternative $t_2$ ("eliminate sep").
6. If $i = 1$ goto 1 and try $h$ not yet used.
7. Output $T^*$.

Ex 6.10 (via - Maximum - Heuristic, undirected TSP, "e2002[2002]"): 

Diagram of a network with labeled nodes and edges. The diagram shows a path after 1 or 2 ILU-Scalings. The optimal tour is indicated with a line through the network.