Network Design and Operation (WS 2015)

Excercise Sheet 3
Submission: Mo, 09. November 2015, tutorial session

Exercise 1. 10 Points

Weiszfeld’s algorithm iterates the operator $T$ defined as

$$p_{j+1} := T(p_j) := \frac{1}{\sum_{i=1}^{m} \frac{1}{\|p_j - v_i\|^2}} \sum_{i=1}^{m} \frac{v_i}{\|p_j - v_i\|^2}, \quad j = 1, 2, \ldots$$

to determine the median $p^* = \arg\min_{\mathbb{R}^2} \sum_{i=1}^{m} \|p - v_i\|^2$ of a given set of points $V = \{v_i\}_{i=1}^{m}$ in $\mathbb{R}^2$ w.r.t. Euclidean distances. Consider $V = \{(-1,3),(0,0),(9,9),(10,0)\}$ and compute the median $p^*$ numerically. Try $p_0 = (8,-1)$ and two other starting points of your choice. Plot the resulting iterates.

Exercise 2. 10 Points

Consider a triangle $\Delta v_1v_2p$ in $\mathbb{R}^2$ and let $c = (v_1 + v_2)/2$ be the median of $v_1$ and $v_2$. Prove that $\|p - c\|_2 \leq (\|p - v_1\|_2 + \|p - v_2\|_2)/2$.

Exercise 3. 10 Points

Prove that the Fermat or Torricelli point in a triangle with an obtuse angle ($\geq 2\pi/3$) is the vertex at this obtuse angle.

Exercise 4. 10 Points

Prove that the median (w.r.t. Euclidean distance) of four points in the plane is

a) the intersection of the two diagonals, if the points form a convex quadrilateral.

b) the point in the triangle, if three points form a triangle containing the fourth.

Hint: Use the Weiszfeld conditions. Choose favorable coordinates in b).

Exercise 5. Tutorial Session

Consider the $k$-median problem $k/N/\cdot/\ell_1/\sum$ w.r.t. Manhattan distances on the $11 \times 11$-grid graph $N$ generated by the points $\{0, \ldots, 10\}^2$ for a set of points $V = \{(1,8),(2,0),(4,10),(5,4),(7,7),(9,2),(10,5)\}$. Solve the following problems:

a) $1/N/\cdot/\ell_1/\sum$

b) $1/N/R = [0,8] \times [5,5]/\ell_1/\sum$

c) $1/N/R = [0,8] \times [5,6]/\ell_1/\sum$

d) $1/N/R = [0,8] \times [5,6] \cup [8,9] \times [4,7]/\ell_1/\sum$

e) $2/N/\cdot/\ell_1/\sum$

f) $2/N/R = [0,8] \times [5,5]/\ell_1/\sum$

g) $2/N/R = [0,8] \times [5,6]/\ell_1/\sum$

h) $2/N/R = [0,8] \times [5,6] \cup [8,9] \times [4,7]/\ell_1/\sum$

Use integer programming where necessary, and plot all solutions. Which scenarios suffer most from the restrictions?
Figure 1: 1-median $\ell_1$-problem.