Network Design and Operation (WS 2015)

Excercise Sheet 6
Submission: Mo, 30. November 2015, tutorial session

Exercise 1. 8+2 Points

Let \( w^1 \geq \cdots \geq w^s > w^{s+1} = 0 \) be integers and denote by \( H(n) := \sum_{i=1}^{n} \frac{1}{i} \) the \( n \)-th harmonic number, \( n \in \mathbb{N} \). Prove that

\[
\sum_{i=1}^{s} \frac{(w^i - w^{i+1})}{w^i} \leq \sum_{i=1}^{s} \left( H(w^i) - H(w^{i+1}) \right) \leq H(w^1).
\]

Exercise 2. 4+4+2 Points

Consider a discrete stop location problem on a line, i.e., in a network \( N = (S, E) \) of collinear nodes. Prove:

a) The covering matrix \( A_{cov}^r \) associated with a discrete stop location problem on a line has the consecutive ones property, i.e., the columns and rows of \( A_{cov}^r \) can be permuted in such a way that \( a_{ri} = 1 = a_{rk} \) implies \( a_{rj} = 1, i \leq j \leq k \), for all \( r \in V, i, k \in S \).

b) A 0/1 matrix with the consecutive ones property is totally unimodular.

c) The set covering problem associated with a discrete stop location problem on a line can be solved by linear programming.

Exercise 3. 10 Points

Consider the continuous stop location problem in a network \( N = (S, E) \) with demand points \( V \) and covering radius \( r \):

\[
(CSL) \quad p = |U|/|N|/cov_r(U) = V/\ell_2/p.
\]

Prove: If (CSL) is feasible, the optimum is finite.

Exercise 4. 2+6+2 Points

Consider the continuous stop location problem in a network \( N = (S, E) \) with demand points \( V \) and covering radius \( r \)

\[
(CSL) \quad p = |U|/|N|/cov_r(U) = V/\ell_2/p
\]

and denote

\[ S_r := S \cup \{ x \in N : \exists v \in V : ||x - v||_2 = r \}. \]

Prove that (CSL) can be reduced to (DSL):
a) $S_r$ is finite.
b) Consider an edge $e = uv \in E$ and let $S_e^r := S_r \cap e = \{s_1, s_2, \ldots, s_k\}$. Order $\{s_1, \ldots, s_k\}$ as $u = s_1 <_e \cdots <_e s_n = v$ with respect to the natural order along the edge $e = uv$, starting from $u$. Let $s$ be a point on $uv$ between $s_i$ and $s_{i+1}$, i.e., $s_i <_e s <_e s_{i+1}$. Then $\text{cov}_r(s_i) \supseteq \text{cov}_r(s)$ and $\text{cov}_r(s_{i+1}) \supseteq \text{cov}_r(s)$.
c) (CSL) has an optimal solution $S^* \subseteq S_r$.

**Exercise 5. Tutorial Session**

Consider the bus stop location problem on the left of Fig. ?? with unit costs, potential stops $S = \{s_1, \ldots, s_7\}$, demand points $V = \{v_1, \ldots, v_5\}$ and covering radius $r$ with associated covering matrix

$$A_r^{\text{cov}} = \begin{pmatrix}
  s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 0 & 1 & 0 \\
  p_1 & p_2 & p_3 & p_4 & p_5
\end{pmatrix}$$

(cf. Schöbel [2003]). Solve this problem using set covering techniques, illustrating your reasoning in the figure.

![Figure 1: Bus stop location problem.](image)

**Exercise 6. Tutorial Session**

Peter wants to buy a new computer but has little money. To save costs, he decides to purchase individual parts that he plans to assemble himself. In a tedious internet search, he identifies five suppliers that sell the parts that he wants at the following prices (in Euros):
<table>
<thead>
<tr>
<th>supplier</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>main board</td>
<td>100</td>
<td>120</td>
<td>105</td>
<td>120</td>
<td>110</td>
</tr>
<tr>
<td>CPU</td>
<td>190</td>
<td>200</td>
<td>205</td>
<td>190</td>
<td>210</td>
</tr>
<tr>
<td>RAM</td>
<td>65</td>
<td>50</td>
<td>45</td>
<td>50</td>
<td>65</td>
</tr>
<tr>
<td>hard disk</td>
<td>105</td>
<td>80</td>
<td>85</td>
<td>100</td>
<td>105</td>
</tr>
<tr>
<td>frame</td>
<td>45</td>
<td>45</td>
<td>50</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

All suppliers charge (in Peter’s opinion) ridiculously high delivery costs independent of the size of the order, namely, suppliers 1, 2, and 3 charge 20 Euros, supplier 4 charges 25 Euros, and supplier 5 charges 10 Euros. Clenching his teeth, Peter recalls his math class. After all, a computer is also a kind of infrastructure. At least, he can ... yeah, what can he do?

a) Develop an optimization model for Peter’s computer acquisition problem.
b) What kind of model is this?
c) Solve the model. What is the optimal solution?
d) Solve the linear programming relaxation of the model. What happens?
e) Fix the suppliers and resolve the linear programming relaxation of the model. What happens now?
f) Peter is seldom at home, and his neighbors don’t want to store more than one parcel for him. What should he do?