Exercise 1

Proposition 1. The vector $x^0$ is a vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0_n\}$ if and only if the vector $(x^0, y^0) \in \mathbb{R}^{n+m}$ such that:

\[
\begin{pmatrix}
  x^0 \\
  y^0
\end{pmatrix} = \begin{pmatrix}
  x^0 \\
  b - Ax^0
\end{pmatrix}
\]

is a vertex of the polyhedron $P^{STD} = \{(x^0, y^0) \in \mathbb{R}^{n+m} : Ax + Iy = b, x \geq 0_n, y \geq 0_m\}$.

Prove that if $(x^0, y^0)$ is a vertex of $P^{STD}$ then $x^0$ is a vertex of $P$.

Exercise 2

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and assume that $A$ has full column rank. Additionally, denote by $I = \{1, \ldots, m\}$ the set of rows of $A$. Prove that $x^* \in P$ is a vertex of $P$ if and only if there exists a set $B \subseteq I$ such that $|B| = n$, $A_B$ is invertible and $A_Bx^* = b_B$ (here, we denote by $A_B$ and $b_B$ the rows of $A$ and the components of $b$ indexed by $B$, respectively).

Exercise 3

Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix. Prove that $A$ is totally unimodular if and only if the polyhedron

$P = \{x \in \mathbb{R}^n : \beta \leq Ax \leq b, \ell \leq x \leq u\}$

is integral for all vectors $\beta, b, \ell, u \in \mathbb{R}^n$.

Exercise 4

Let $A$ be an $m \times n$ binary matrix (i.e, $A \in \{0, 1\}^{m \times n}$) and denote by $J$ the set of the column indices. Prove that if $A$ possesses the property:

\[
\forall j, k, \ell \in J : j \leq k \leq \ell, \text{ if } a_{ij} = 1 \text{ and } a_{i\ell} = 1 \text{ then } a_{ik} = 1
\]

then $A$ is totally unimodular.

Exercise 5

Proposition 2. Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix such that each of its column has at most two non-zero entries.

The matrix $A$ is totally unimodular if and only if there exists a partition $(I_1, I_2)$ of the set of rows $I$ of $A$ such that:

\[
a_{i_1,j} \cdot a_{i_2,j} \leq 0 \iff \{i_1, i_2\} \subseteq I_1 \text{ or } \{i_1, i_2\} \subseteq I_2
\]

for all $i_1, i_2 \in I$ with $i_1 \neq i_2$ and $a_{i_1j}, a_{i_2j} \neq 0$.

Derive a polynomial time algorithm that can test whether a generic matrix $A$ satisfies the above condition.

Hint: checking that a graph is bipartite can be checked in polynomial time.
Exercise 6

Let $A$ be an $m \times n$ binary matrix (i.e., $A \in \{0, 1\}^{m \times n}$). The matrix $A$ possesses the \textit{(column-wise) consecutive-ones property} when, after a possible reordering of the rows of $A$, the non-zero entries appear consecutively in each column.

Prove that $A$ possessing the (column-wise) consecutive-ones property is totally unimodular by exploiting the following proposition.

\textbf{Proposition 3.} Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix. $A$ is totally unimodular if for any subset $I'$ of the rows, there exists a partition $(I_1, I_2)$ such that each column $j \in J$ satisfies:

$$\left| \sum_{i \in I_1} a_{ij} - \sum_{i \in I_2} a_{ij} \right| \leq 1$$