The Semidefinite Relaxation of the k-Partition Polytope Is Strong*

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Abstract Radio frequency bandwidth has become a very scarce resource. This holds true in particular for the popular mobile communication system GSM. Carefully planning the use of the available frequencies is thus of great importance to GSM network operators. Heuristic optimization methods for this task are known, which produce frequency plans causing only moderate amounts of disturbing interference in many typical situations. In order to thoroughly assess the quality of the plans, however, lower bounds on the unavoidable interference are in demand. The results obtained so far using linear programming and graph theoretic arguments do not suffice. By far the best lower bounds are currently obtained from semidefinite programming. The link between semidefinite programming and the bound on unavoidable interference in frequency planning is the semidefinite relaxation of the graph minimum k-partition problem.

Here, we take first steps to explain the surprising strength of the semidefinite relaxation. This bases on a study of the solution set of the semidefinite relaxation in relation to the circumscribed k-partition polytope. Our focus is on the huge class of hypermetric inequalities, which are valid and in many cases facet-defining for the k-partition polytope. We show that a "slightly shifted version" of the hypermetric inequalities is implicit to the semidefinite relaxation. In particular, no feasible point for the semidefinite relaxation violates any of the facet-defining triangle inequalities for the k-partition polytope by more than $\sqrt{2}-1$ or any of the (exponentially many) facet-defining clique constraints by $\frac{1}{2}$ or more.

1 Introduction

Frequency planning is the key for GSM network operators to fully exploit the radio spectrum available to them. This spectrum is slotted into channels of 200 kHz bandwidth, characterized by their central frequency. Frequency planning has a significant impact on the quantity as well as on the quality of the radio communication services. Roughly speaking, radio communication requires a radio signal of sufficient strength which is not suffering too severely from interference by other signals. In a cellular system like GSM, these two properties, strong signals and little interference, are in conflict.

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1.1 Frequency Planning

The task of finding a "good" frequency plan can be described as follows:

> Given are the transmitters, the set of generally available frequencies together with the local unavailabilities, and three upper-diagonal square matrices. The first matrix specifies for every pair of transmitters how far apart their frequencies have to be at least. These are called separation constraints. The second matrix specifies for every transmitter pair how much co-channel interference is incurred in case they use the same frequency (where permissible), and the third matrix specifies how much adjacent-channel interference is incurred in case two transmitters use neighboring frequencies (where permissible). One frequency has to be assigned to every transmitter such that the following holds. All separation requirements are met, and all assigned frequencies are locally available. The optimization goal is to find a frequency assignment resulting in the least possible interference.

A precise mathematical model of our GSM frequency planning problem is given in [2, 12], for example. This model is widely accepted in practice, compare with [7], but there are alternative models as well.

Finding a feasible assignment is $\mathcal{NP}$-hard in general. This is easily derived by exploiting the connection to list coloring [13] or to T-coloring of graphs [20]. Neither can an approximation algorithms with reasonable performance exist unless $\mathcal{P} = \mathcal{NP}$, see [12].

Numerous heuristic and a few exact methods for solving the frequency assignment problem are known, see [2, 25] for surveys. Although we are nowadays in the situation that a number of heuristic methods performs well in practice, there is an uncomfortable lack of quality guarantees on the per-instance-level—not just in theory. Substantial efforts have been made to remedy this shortcoming over the last decade, see [1, 22, 23, 25, 31] but the results are not satisfactory. In particular, no exact method is presently known that is capable of solving large realistic planning problems in practice.

1.2 Quality Guarantees

A novel approach was recently proposed to derive a lower bound on the unavoidable co-channel interference: The frequency planning problem is simplified to a minimum graph k-partition problem, where the edge-weights represent co-channel interference or separation constraints and k is the number of overall available frequencies. A lower bound for the optimal k-partition partition (and, thus, for the unavoidable co-channel interference) is computed from a semidefinite relaxation of the resulting minimum graph k-partition problem.

Table 1 lists computational results for five realistic planning problems, which are publicly available over the Internet [14]. The first column gives the instance's name. The next five columns describe several characteristics of the instance, mostly in terms of underlying graph $G = (V, E)$. This graph contains one vertex for each transmitter, and two vertices are joined by an edge whenever at least one of the corresponding three matrix entries is non-zero. We list the number $|V(G)|$ of vertices in G, the edge density $\rho(G)$ of the graph, the maximum degree $\Delta(G)$ of a vertex, the clique number $\omega(G)$, and the number of globally available frequencies. (In the case of instance SIE2, the available spectrum consists of two contiguous bands of size 4 and 72. In all other cases there is just one band.) Columns 7 and 8 show the amount of co- and adjacent-channel interference incurred by the best presently known frequency assignment for each instance, see [12] for details. The last but one column contains the lower bound on the unavoidable co-channel interference. Finally, the last column gives the relative quality of the best known solution with respect to the bound on the unavoidable (co-channel) interference.

Table 1. Realistic frequency planning problems with quality guarantees

instance	characteristics					best assignment		lower	gap
	$\|V(G)\|$	$\rho(G)$	$\Delta(G)$	$\omega(G)$	freqs. (k)	co-ch.	ad.-ch.	bound	[%]
K	267	0.56	238	69	50	0.37	0.00	**0.1836**	102
B[4]	2775	0.13	1133	120	75	17.29	0.44	**4.0342**	339
B[10]	4145	0.13	1704	174	75	142.09	4.11	**54.0989**	170
SIE2	977	0.49	877	182	4 + 72	12.57	2.18	**6.9463**	112
SIE4	2785	0.11	752	100	39	71.09	9.87	**27.6320**	193

The lower bound figures are computed using the semidefinite programming solvers of Burer, Monteiro, and Zhang [4] and Helmberg [21]. Both are dual solvers, that is, their results are dual approximations of the optimal value without quality guarantee. The corresponding semidefinite programs involve a $|V| \times |V|$ square matrix variable and $\binom{|V|+1}{2}$ many linear constraints. Looking again at Tab. 1 reveals that these semidefinite programs are (very) large.

From the applications points of view, gaps of a few hundred percent are by far not satisfying. Nevertheless, for large realistic planning problems these are the first significant lower bounds to the best of our knowledge. We want to indicate why the (approximate) solution of the semidefinite relaxation yields stronger bounds than previous approaches.

1.3 Sketch of Results

In the remainder of this paper, we give first reasons for the surprisingly strong bounds on the unavoidable interference obtained from semidefinite programming. To this end, we study the relation between a class of polytopes associated to the minimum graph k-partition problem (via an integer linear programming formulation) and their circumscribing semidefinite relaxations. This is done in terms of the hypermetric inequalities, which form a huge class of valid and in many cases facet-defining inequalities for the polytopes. We show in Prop. 3

that a "slightly shifted version" of the hypermetric inequalities is implicit in the semidefinite relaxation. Propositions 5, 6, 7, and 8 state to which extent the "shift" may by reduced under specific conditions. We also show, see Prop. 9, that neither the semidefinite relaxation nor the linear relaxation, which is obtained from the standard integer linear programming formulation of minimum k-partition by dropping the integrality conditions, is generally stronger than the other. This is particularly noteworthy since the semidefinite relaxation contains only polynomially many linear constraints (in the size of the graph and k), whereas the linear relaxation is of exponential size.

1.4 Preliminaries

The definitions and notations used here are mostly taken from [32] for graph theory, from [30] for (integer) linear programming, and from [21] for semidefinite programming. A thorough exposition of the notions used is contained in [12].

2 The Minimum k-Partition Problem

The minimum graph k-partition problem or MINIMUM K-PARTITION problem we want to solve is formally defined as follows.

Definition 1. *An instance of the* MINIMUM K-PARTITION problem *consists of an undirected graph $G = (V, E)$, a weighting $w\colon E \to \mathbb{Q}$ of the edges, and a positive integer k. The objective is to find a partition of V into at most k disjoint sets $V_1, \ldots, V_p$ such that the value $\sum_{l=1}^{p} \sum_{vw \in E(G[V_l])} w(vw)$ is minimized.*

The MINIMUM K-PARTITION problem is studied in [5, 6]. The complementing MAXIMUM K-CUT problem is investigated in [8, 9, 15, 17, 24]. Recall that both problems have quite different characteristics in terms of their approximability. Whereas the MAXIMUM K-CUT can be solved $(1 - k^{-1})$–approximately in polynomial time, the MINIMUM K-PARTITION problem is not $\mathcal{O}(|E|)$–approximable in polynomial time unless $\mathcal{P} = \mathcal{NP}$. Results on the approximation of the MAXIMUM K-CUT problem are obtained in [15, 24]. These results are, however, of no use for the bounding of unavoidable interference in GSM frequency planning. The optimal cut value is underestimated so that the value of the MINIMUM K-PARTITION is overestimated. No lower bound is supplied that way. Due to the opposite sign restriction, the results from [17] do not apply either.

The class of polytopes associated with the MINIMUM K-PARTITION is defined through the convex hull of the feasible solutions to the integer linear programming formulation (1)–(4) of the problem. This formulation is only valid for complete graphs K_n with $n \geq k \geq 2$. (The graph $G = (V, E)$ with edge weights w is completed to $K_{|V|}$, and the edge weighting is extended to all new edges by assigning a weight of zero.) We assume for notational convenience that the vertex set of K_n is $\{1, \ldots, n\}$ and that the edge set is $\{ij \mid 1 \leq i < j \leq n\}$. One binary variable is used for every edge of the complete graph. The intended meaning is that $z_{ij} = 1$ if and only if the vertices i and j are in the same partite set of the partition.

$$\min \sum_{i,j \in V} w_{ij}\, z_{ij} \tag{1}$$

$$z_{ih} + z_{hj} - z_{ij} \leq 1 \qquad \forall\, h, i, j \in V \tag{2}$$

$$\sum_{i,j \in Q} z_{ij} \geq 1 \qquad \forall\, Q \subseteq V \text{ with } |Q| = k+1 \tag{3}$$

$$z_{ij} \in \{0, 1\} \tag{4}$$

The triangle inequalities (2) require the setting of the variables to be consistent, that is, transitive. The clique constraints (3) impose that at least two from a set of $k+1$ vertices have to be in the same partite set. Together with the constraints (2) this implies that there are at most k partite sets. In total, there are $3\binom{|V|}{3}$ many triangle and $\binom{|V|}{k+1}$ many clique inequalities. The number of clique inequalities grows roughly as fast as $|V|^k$ as long as $2k \leq |V|$. We come back to this point at the end of this section.

The well-known integer linear programming formulation of Maximum k-Cut is obtained by complementing the variables.

For the sake of simplicity, we assume $k \geq 3$ in the following. Clearly, if $k = 1$, then there is only one "partition;" in the case of $k = 2$, the Minimum 2-Partition problem is equivalent to the well-known Maximum Cut problem, see [11, 28] for surveys.

For all $3 \leq k \leq n$, we define the polytope

$$\mathcal{P}_{\leq k}(K_n) = \operatorname{conv}\Big\{z \in \{0,1\}^{E(K_n)} \mid z_{hi} + z_{ij} - z_{hj} \leq 1 \quad \forall\, h, i, j \in V; \sum_{i,j \in Q} z_{ij} \geq 1 \quad \forall\, Q \subseteq V, |Q| = k+1\Big\}\ .$$

$\mathcal{P}_{\leq k}(K_n)$ is the set of convex combinations of all feasible solutions to the integer linear program (1)–(4). The resulting class of polytopes is studied in [6, 8, 9, 12] and the special case $k = n$ is also investigated in [29]. Here, we merely highlight a few points.

The polytopes are full-dimensional in the space spanned by the edge variables, and none contains the origin. All valid inequalities for $\mathcal{P}_{\leq k}(K_n)$ that are violated by the origin have large support. To be more precise, given some inequality $a^T z \geq a_0$, the *support graph* of $a^T z \geq a_0$ is the subgraph of K_n induced by all edges ij with $a_{ij} \neq 0$. The following holds.

Proposition 1 ([12]). *Let $a^T z \leq a_0$ be a valid inequality for $\mathcal{P}_{\leq k}(K_n)$ and H its support graph. If H is k-partite, then $a_0 \leq 0$, and if $a_0 > 0$, then $|\{ij \in E \mid a_{ij} \neq 0\}| \geq \binom{k+1}{2}$.*

The hypermetric inequalities form a large and important class of valid and in many cases facet-defining inequalities for $\mathcal{P}_{\leq k}(K_n)$ [6]. They sometimes do separate the origin from $\mathcal{P}_{\leq k}(K_n)$, and they generalize a number of previously known inequalities, such as the triangle inequalities [19], the 2-partition inequalities [6, 19], the (general) clique inequalities [5], and the claw inequalities [29]. The right-hand sides of the hypermetric inequalities involve the function $f_{hm}(\eta, k) = \max \{\sum_{1\leq i<j\leq k} x_i x_j \mid \sum_{i=1}^{k} x_i = \eta, x_i \in \mathbb{Z}_+\}$, which depends on two integral parameters η and k, $\eta \geq 0$, $k \geq 1$. For given η and k, the value $f_{hm}(\eta, k)$ is the maximum number of edges in a k-partite graph with η many vertices. For our purposes, the following equivalent definition is more convenient:

$$\begin{aligned} f_{hm}(\eta, k) \;&=\; \binom{\eta \bmod k}{2} \left\lceil \frac{\eta}{k} \right\rceil^2 + \binom{k - \eta \bmod k}{2} \left\lfloor \frac{\eta}{k} \right\rfloor^2 \\ &\quad + \; (\eta \bmod k)(k - \eta \bmod k) \left\lceil \frac{\eta}{k} \right\rceil \left\lfloor \frac{\eta}{k} \right\rfloor \ . \end{aligned}$$

The hypermetric inequalities for $\mathcal{P}_{\leq k}(K_n)$ are defined as follows.

Definition 2. *Given $k \geq 3$ and a complete graph K_n and vertex weights $b_v \in \mathbb{Z}$ with $\eta = \sum_{v\in V(K_n)} b_v \geq 0$. The associated* hypermetric inequality *is*

$$\sum_{vw\in E(K_n)} b_v b_w \, z_{vw} \geq \sum_{vw\in E(K_n)} b_v b_w - f_{hm}(\eta, k) \ . \tag{5}$$

The hypermetric inequalities are shown to be valid for $\mathcal{P}_{\leq k}(K_n)$ in [6, 10]. Let us consider the simple case in which all vertex weights are equal to one. Then (5) bounds the number of edges that run within partite sets from below. The corresponding inequality for the Maximum k-Cut polytopes bounds the number of edges in the k-cut from above.

Before concluding this section, let us return to the integer linear program (1)–(4). A linear programming relaxation is obtained by replacing the feasible variable values $\{0, 1\}$ by their convex hull $[0, 1]$. We call the resulting polytope $\mathcal{P}^{LP}_{\leq k}(K_n)$. Recall that all facets separating the origin from the polytope have a support of at least $\binom{k+1}{2}$. The smallest such examples are the exponentially many clique inequalities (3). Among others, these inequalities with large support make linear programming relaxation of (1)–(4) hard to deal with. Not surprisingly, no major successes in solving Minimum k-Partition instances with a few hundred vertices and $k \gg 2$ using the classical branch-and-cut approach are reported in literature.

3 A Semidefinite Relaxation of Min k-Partition

Semidefinite programming is the task of minimizing (or maximizing) a linear objective function over the convex cone of positive semidefinite matrices subject to linear constraints. Here, a square matrix X with real-valued entries is positive semidefinite ($X \succeq 0$) if it is symmetric and its eigenvalues are non-negative. We assume that the reader is familiar with semidefinite programming, see, for example, the introductions/surveys given in [3, 21, 33].

An alternative formulation of the MINIMUM K-PARTITION problem can be seen as a semidefinite program with "integrality" constraints, compare with [15, 24]. The next lemma is the basis for this formulation.

Lemma 1 ([12, 15, 24]). *For all integers n and k satisfying $2 \leq k \leq n+1$ the following holds:*

1. *k unit vectors $\bar{u}_1, \ldots, \bar{u}_k \in \mathbb{R}^n$ exist such that $\langle \bar{u}_i, \bar{u}_j \rangle = \frac{-1}{k-1}$ for all $i \neq j$;*
2. *any given k unit vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ satisfy $\sum_{i<j} \langle u_i, u_j \rangle \geq -\frac{k}{2}$ and if $\langle u_i, u_j \rangle \leq \delta$ for all $i \neq j$, then $\delta \geq \frac{-1}{k-1}$.*

According to the lemma, we may fix a set $U = \{u_1, \ldots, u_k\} \subseteq \mathbb{R}^n$ of unit vectors with $\langle u_i, u_j \rangle = \frac{-1}{k-1}$ for $i \neq j$ (and $\langle u_i, u_i \rangle = 1$ for $i = 1, \ldots, k$). These k vectors are used as labels (or representations) for the k partite sets. The MINIMUM K-PARTITION problem is then the task of finding an assignment $\phi \colon V \mapsto U$ that minimizes the expression $\sum_{i,j \in V(K_n)} c_{ij} \frac{(k-1)\langle \phi(i), \phi(j) \rangle + 1}{k}$. The quotient in the summands evaluates to either 1 or 0, depending on whether the same vector or distinct vectors are assigned to the respective two vertices.

We assemble the scalar products $\langle \phi(i), \phi(j) \rangle$ into a square matrix X, being indexed row- and column-wise by V. The matrix X has the following properties: all entries on the principal diagonal are ones, all off-diagonal elements are either $\frac{-1}{k-1}$ or 1, and X is positive semidefinite. Conversely, every matrix X satisfying the above properties defines a k-partition of V in the same way as ϕ does [12]. Hence,

$$\min \textstyle\sum_{ij \in E(K_n)} c_{ij} \, \frac{(k-1)\, X_{ij} + 1}{k} \tag{6}$$

s. t.

$$X_{ii} = 1 \qquad \forall\, i \in V \tag{7}$$

$$X_{ij} \in \{\frac{-1}{k-1}, 1\} \qquad \forall\, i, j \in V \tag{8}$$

$$X \succeq 0 \tag{9}$$

is an alternative formulation of the MINIMUM K-PARTITION problem.

Replacing the constraints (8) by $\frac{-1}{k-1} \leq X_{ij} \leq 1$ yields a semidefinite program. Notice that $X_{ij} \leq 1$ can be dropped as constraint since it is enforced implicitly by X being positive semidefinite and $X_{ii} = 1$. The semidefinite programming relaxation obtained from (6)–(9) is ϵ-approximately solvable in polynomial time [18].

This type of relaxation of a combinatorial optimization problem was first introduced to compute the Shannon capacity of a graph [27] and later formed the basis for the famous 0.878-approximation algorithm for the MAXIMUM CUT problem [16]. The approximation algorithm was generalized to the MAXIMUM K-CUT problem with positive edge-weights in [15] and [24] independently. Notice that the semidefinite relaxation of MAXIMUM K-CUT differs from that of (6)–(9) only in the objective function. Due the previously cited inapproximability of MINIMUM K-PARTITION, however, a comparable approximation result as that for MAXIMUM K-CUT is not to be expected for MINIMUM K-PARTITION.

As stated before, we want to investigate the strength of the semidefinite relaxation. In order to do so, we relate the solution set of the semidefinite relaxation to the polytope $\mathcal{P}_{\leq k}(K_n)$. This is done by considering a projection of an affine image of the solution set into $\mathbb{R}^{\binom{n}{2}}$ in such a way that the objective function values are preserved. The image of the projection is called $\Theta_{k,n}$ and contains $\mathcal{P}_{\leq k}(K_n)$.

Let $\Psi_{k,n}$ denote the set of feasible solutions of the semidefinite relaxation of (6)–(9), that is, $\Psi_{k,n} = \big\{X \in \mathbb{R}^{n\times n} \mid X \succeq 0, X_{ii} = 1, X_{ij} \geq \frac{-1}{k-1}, i,j \in \{1,\ldots,n\}\big\}$. In the case of $k = 2$, $\Psi_{k,n}$ is the *elliptope* $\mathcal{E}_n = \big\{X \in \mathbb{R}^{n\times n} \mid X \succeq 0, X_{ii} = 1, i \in \{1,\ldots,n\}\big\}$. For $k > 2$, $\Psi_{k,n}$ is obtained by intersecting the elliptope $\mathcal{E}_n$ with the half-spaces defined by $X_{ij} \geq \frac{-1}{k-1}$ for all $i,j \in \{1,\ldots,n\}$. The elliptope is studied extensively in the literature, see [11] for a survey. It will later be exploited that the elliptope can also be characterized as

$$\begin{aligned}\mathcal{E}_n = \Big\{X \in S_n \mid X_{ii} = 1 \text{ for } i = 1,\ldots,n;\\ 2\sum_{1\leq i<j\leq n} b_i b_j X_{ij} \geq -\sum_{i=1}^{n} b_i^2 \quad \text{for all } b \in \mathbb{Z}^n\Big\} .\end{aligned} \tag{10}$$

Next, we define an affine mapping that projects $\Psi_{k,n}$ into $\mathbb{R}^{\binom{n}{2}}$. Let $T_k\colon \mathbb{R} \to \mathbb{R}$ be the affine transformation $x \mapsto \frac{k-1}{k}\,x + \frac{1}{k}$, mapping 1 onto 1 and $\frac{-1}{k-1}$ onto 0. The affine transformation T_k is extended to the set S_n of $n \times n$ symmetric real-valued matrices (which is isomorphic to $\mathbb{R}^{\binom{n+1}{2}}$) by letting $T_k\colon S_n \to S_n$, $S \mapsto \frac{k-1}{k}\,S + \frac{1}{k}\,E(n,n)$. Here, $E(n,n)$ is the $n \times n$ matrix with all entries being equal to one. Let $\zeta_{k,n}\colon S_n \to \mathbb{R}^{\binom{n}{2}}$, $X \mapsto \zeta_{k,n}(X) = z$ with $z_{ij} = \big(T_k(X)\big)_{ij}$ for $i < j$, and consider

$$\Theta_{k,n} = \zeta_{k,n}(\Psi_{k,n}) = \big\{\zeta_{k,n}(X) \mid X \in \Psi_{k,n}\big\} .$$

The restriction of $\zeta_{k,n}$ onto $\Psi_{k,n}$ is one-to-one, and $\zeta_{k,n}|_{\Psi_{k,n}}\colon \Psi_{k,n} \to \Theta_{k,n}$ is an affine bijection. Moreover, for any given $X \in \Psi_{k,n}$ and any given $w \in \mathbb{R}^{\binom{n}{2}}$ the identity $\frac{1}{2}\langle W, T_k(X)\rangle = \langle w, \zeta_{k,n}(X)\rangle$ holds. Here, W is the symmetric matrix obtained from w by letting $W_{ii} = 0$ and $W_{ij} = w_{ij}$ for all $1 \leq i < j \leq n$. $\langle\cdot,\cdot\rangle$ denotes the ordinary scalar products on the vector spaces S_n and $\mathbb{R}^{\binom{n+1}{2}}$, respectively.

A direct consequence of our definitions is that the optimization problems

$$\min \frac{1}{2}\langle W, T_k(X)\rangle \text{ s. t. } X \in \Psi_{k,n} \quad \text{and} \quad \min\langle w, z\rangle \text{ s. t. } z \in \Theta_{k,n} \tag{11}$$

are equivalent.

The affine image $\Theta_{k,n}$ of the truncated elliptope $\Psi_{k,n}$ contains the polytope $\mathcal{P}_{\leq k}(K_n)$ and is itself contained in the hypercube $[0,1]^{\binom{n}{2}}$, see [12]. A related connection between the MAXIMUM CUT polytope and the elliptope is studied in [26]. We call $\Theta_{k,n}$ a *semidefinite relaxation* of $\mathcal{P}_{\leq k}(K_n)$. $\Theta_{k,n}$ and $\mathcal{P}_{\leq k}(K_n)$ contain the same integral points.

Proposition 2. *Given integers k, n with $2 \leq k < n$, then $\Theta_{k,n}$ and $\mathcal{P}_{\leq k}(K_n)$ contain the same integral points.*

Proof. Let $\bar{z}$ be an integral (binary) vector in $\Theta_{k,n}$. Let $\bar{X}$ denote the pre-image of $\bar{z}$ under the mapping $\zeta_{k,n}$. All entries of the positive semidefinite matrix $\bar{X}$ are either $\frac{-1}{k-1}$ or $+1$.

No triangle constraint (2) is violated, because such a violation would imply that $\bar{X}$ has one of the matrices

$$\begin{bmatrix} 1 & \frac{-1}{k-1} & 1 \\ \frac{-1}{k-1} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & \frac{-1}{k-1} \\ 1 & 1 & 1 \\ \frac{-1}{k-1} & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{-1}{k-1} \\ 1 & \frac{-1}{k-1} & 1 \end{bmatrix}$$

as a principal sub-matrix. In all cases, the determinant is $-\left(\frac{k}{k-1}\right)^2 < 0$. Hence, none of these matrices may appear as a principal sub-matrix of $\bar{X}$.

According to Lemma 1, no subset Q of size larger than k can induce a sub-matrix $\bar{X}_{QQ}$ with all its off-diagonal elements equal to $\frac{-1}{k-1}$. Thus, at least one off-diagonal element in $\bar{X}_{QQ}$ equals 1 for each set Q of size $k+1$, and, consequently, no clique constraint (3) is violated by $\bar{z}$. □

4 The Semidefinite Relaxation $\Theta_{k,n}$ and $\mathcal{P}_{\leq k}(K_n)$

In this section, we investigate the relation between the semidefinite relaxation $\Theta_{k,n}$ and the polytope $\mathcal{P}_{\leq k}(K_n)$ in more detail. The hypermetric inequalities (5) play the central role. Our first result shows that a "slightly shifted version" of the hypermetric inequality is valid for $\Theta_{k,n}$. We also show, that neither $\mathcal{P}^{LP}_{\leq k}(K_n)$ nor $\Theta_{k,n}$ is generally contained in the other.

Proposition 3. *Given an integer $k \geq 3$ and an integral weight b_i for every vertex $i \in V(K_n)$. The following inequality is valid for $\Theta_{k,n}$:*

$$\sum_{ij \in E(K_n)} b_i \, b_j \, z_{ij} \;\geq\; \frac{1}{2k}\Big(\big(\sum_{i \in V(K_n)} b_i\big)^2 - k \sum_{i \in V(K_n)} b_i^2\Big) \; . \tag{12}$$

Proof. The function $\zeta_{k,n}$ maps any matrix $X \in \Psi_{k,n}$ to a vector $z \in \Theta_{k,n}$ such that $X_{ij} = \frac{1}{k-1}(k\, z_{ij} - 1)$ holds for every $1 \leq i < j \leq n$. Plugging this into (10) yields

$$2 \sum_{1 \leq i < j \leq n} b_i b_j \frac{k\, z_{ij} - 1}{k-1} = \frac{2k}{k-1} \sum_{1 \leq i < j \leq n} b_i b_j z_{ij} - \frac{2}{k-1} \sum_{1 \leq i < j \leq n} b_i b_j \geq -\sum_{i=1}^{n} b_i^2 \; .$$

This is equivalent to (12). □

The difference between the right-hand side of hypermetric inequalities (5) and the right-hand side of (12) is bounded by a term depending on the relation between k and the sum of the vertex weights b_i.

Proposition 4. *Given an integer $k \geq 3$ and an integral weight b_i for every vertex $i \in V(K_n)$. Then the difference between the right-hand side of the hypermetric inequalities* (5) *for the polytope $\mathcal{P}_{\leq k}(K_n)$ and the right-hand side of the hypermetric inequalities* (12) *for $\Theta_{k,n}$ is*

$$((\sum_{i \in V(K_n)} b_i) \bmod k) \frac{k - (\sum_{i \in V(K_n)} b_i) \bmod k}{2k} \quad .$$

The upper bounded $\frac{k}{8}$ for this expression is attained if $(\sum_{i \in V(K_n)} b_i) \bmod k = \frac{k}{2}$.

Proof. Let $p = \left\lfloor (\sum_{i \in V(K_n)} b_i)/k \right\rfloor$ and $r = \left(\sum_{i \in V(K_n)} b_i\right) \bmod k$, then simply plugging these parameters into the right-hand side of (5) yields

$$\begin{aligned} & f_{hm}(\sum_{i \in V(K_n)} b_i, k) + ((\sum_{i \in V(K_n)} b_i) \bmod k) \frac{k - (\sum_{i \in V(K_n)} b_i) \bmod k}{2k} \\ = \; & f_{hm}(pk + r, k) + r \frac{k - r}{2k} \quad . \end{aligned}$$

We expand this expression:

$$\begin{aligned} & f_{hm}(pk + r, k) + r\frac{k - r}{2k} \\ = \; & \left(-\frac{k\,p^2}{2} + \frac{k^2\,p^2}{2} - \frac{r}{2} - p\,r + k\,p\,r + \frac{r^2}{2}\right) + \left(\frac{r}{2} - \frac{r^2}{2\,k}\right) \\ = \; & \frac{1}{2k}\left(-k^2\,p^2 + k^3\,p^2 - k\,r - 2\,k\,p\,r + 2k^2\,p\,r + k\,r^2 + k\,r - r^2\right) \\ = \; & \frac{1}{2k}\left(k^3\,p^2 + 2k^2\,p\,r + k\,r^2 - k^2\,p^2 - 2\,k\,p\,r - r^2\right) \\ = \; & \frac{(k-1)\left(p\,k + r\right)^2}{2k} \quad . \end{aligned} \tag{13}$$

The right-hand side of (12) is

$$\begin{aligned} & \frac{1}{2k}\Big(\big(\sum_{i \in V(K_n)} b_i\big)^2 - k \sum_{i \in V(K_n)} b_i^2\Big) \\ = \; & \frac{1}{2k}\Big(2k \sum_{ij \in E(K_n)} b_i b_j - k\big(\sum_{i \in V(K_n)} b_i\big)^2 + \big(\sum_{i \in V(K_n)} b_i\big)^2\Big) \\ = \; & \sum_{ij \in E(K_n)} b_i b_j - \frac{(k-1)\left(\sum_{i \in V(K_n)} b_i\right)^2}{2k} \\ \overset{(13)}{=} \; & \sum_{ij \in E(K_n)} b_i b_j - f_{hm}(\sum_{i \in V(K_n)} b_i, k) \\ & \quad - ((\sum_{i \in V(K_n)} b_i) \bmod k) \frac{k - (\sum_{i \in V(K_n)} b_i) \bmod k}{2k} \quad . \end{aligned}$$

The first part of the claim follows from this. As far as the second part is concerned, we observe that $r\frac{k-r}{2k}$ is a quadratic polynomial in r. Its maximum of $\frac{k}{8}$ is attained for $r = \frac{k}{2}$. This completes the proof. □

In addition to the binary restrictions on the variables, the integer linear programming formulation (1)–(4) linked to $\mathcal{P}_{\leq k}(K_n)$ contains only constraints on triangles and on cliques of size $k+1$. Both classes of constraints are facet-defining hypermetric inequalities for $\mathcal{P}_{\leq k}(K_n)$ [5]. For these two types of inequalities, the following holds with respect to $\Theta_{k,n}$. Notice that (14) improves on the general hypermetric inequality (12) for $\Theta_{k,n}$.

Proposition 5. *Given the complete graph K_n and an integer k with $4 \leq k \leq n$, then for every $z \in \Theta_{k,n}$*

$$z_{ij} + z_{jl} - z_{il} \;\leq\; 1 + \frac{\sqrt{2(k-2)(k-1)} - (k-2)}{k} \quad \left[< \sqrt{2}\right] \tag{14}$$

holds for every triangle and

$$\sum_{ij \in E(Q)} z_{ij} \;\geq\; 1 - \frac{k-1}{2k} \quad \left[> \frac{1}{2}\right] \tag{15}$$

holds for every clique Q of size $k+1$ in K_n. Both bounds are tight.

All triangle inequalities (2) and all clique constraints (3) are "more than half satisfied" by every point in $\Theta_{k,n}$ in the sense that the violation is bounded by $\frac{1}{2}$ rather than by the worst possible 1.

In both cases, tightness follows from more general statements in Props. 6 and 8. These results, together with that from Prop. 7, are proven by means of semidefinite programming duality in Sec. 5.

Proposition 6. *Given the complete graph K_n and an integer k with $3 \leq k < n$. Let Q be a clique in K_n of size larger than k. Then*

$$\sum_{ij \in E(Q)} z_{ij} \geq \frac{|Q|}{2k}\big(|Q| - k\big) \tag{16}$$

is valid for $\Theta_{k,n}$, and there is a point $\bar{z} \in \Theta_{k,n}$ satisfying (16) at equality.

Under certain conditions on the relation among k, $|S|$, and $|T|$, a "shifted 2-partition inequality" is tight for $\Theta_{k,n}$.

Proposition 7. *Given the complete graph K_n, $n \geq 4$, and an integer k with $4 \leq k \leq n$. Let S and T be non-empty, disjoint subsets of $V(K_n)$ with $|S| \leq |T|$. Then*

$$z(E(S)) + z(E(T)) - z([S,T]) \geq \frac{1}{2k}\Big(\big(|T| - |S|\big)^2 - k\big(|T| + |S|\big)\Big) \tag{17}$$

is valid for $\Theta_{k,n}$. Furthermore, there is a point $\bar{z} \in \Theta_{k,n}$ satisfying (17) at equality if one of the following conditions holds:

1. $|S| = 1$ *and* $|T| \geq k-1$*;*
2. $|S| \geq 2$*,* $|S| + |T| \leq k$ *and either* $|T| \leq |S|^2$*, or* $|T| > |S|^2$ *together with* $k \leq \frac{|T|^2 - |S|^2}{|T| - |S|^2}$*.*

The treatment of the case $|S| = 1$ in Prop. 7 is not entirely satisfying, because the most prominent representative of the 2-partition inequalities, namely, the triangle inequalities, is not covered. The case of $|S| = 1$ and $2 \leq |T| \leq k-2$ is therefore considered separately.

Proposition 8. *Given the complete graph* K_n *and an integer* k *with* $4 \leq k \leq n$*. Let* S *and* T *be disjoint subsets of* $V(K_n)$ *with* $1 = |S| < |T| \leq k-2$*. Then*

$$\begin{aligned} z(E(S)) + z(E(T)) - z([S,T]) \\ \geq -1 - \frac{\sqrt{t(k-t)(k-1)} - (k-t)}{k} \quad \left[> -\sqrt{t} \right] \end{aligned} \tag{18}$$

is valid for $\Theta_{k,n}$*, and a point* $\bar{z} \in \Theta_{k,n}$ *fulfills* (18) *at equality.*

Finally, we show that $\Theta_{k,n}$ and the polytope $\mathcal{P}^{LP}_{\leq k}(K_n)$ are incomparable in general. Recall that $\mathcal{P}^{LP}_{\leq k}(K_n)$ is associated to the linear program, which is obtained from (1)–(4) by replacing $[0,1]$ for $\{0,1\}$ in (4). It follows from Prop. 5 that $\Theta_{k,n}$ contains points which are not contained in $\mathcal{P}^{LP}_{\leq k}(K_n)$, that is, $\Theta_{k,n} \not\subset \mathcal{P}^{LP}_{\leq k}(K_n)$. In general, the reverse inclusion does not hold either.

In order to see this, we fix integers k and n such that $4 \leq k < \sqrt{n}$. Let $\tilde{z} \in \mathbb{R}^{\binom{n}{2}}$ be the vector with all coordinates equal to $\frac{1}{k+1}$. Then $\tilde{z} \in \mathcal{P}^{LP}_{\leq k}(K_n)$ because $0 < \tilde{z}_{ij} < 1$ for all ij and $\tilde{z}$ satisfies all triangle inequalities (2) as well as all clique inequalities (3). The vector $\tilde{z}$ is, however, not contained in $\Theta_{k,n}$, because the valid inequality (12) with $b_i = 1$ for every vertex i is violated by $\tilde{z}$:

$$\sum_{ij \in E(K_n)} \tilde{z}_{ij} = \binom{n}{2} \frac{1}{k+1} = \frac{n\,(n-1)}{2\,(k+1)} \not\geq \frac{n\,(n-k)}{2k} = \frac{1}{2k}\left(n^2 - kn\right) .$$

This follows from $k\,(n-1) < (k+1)(n-k) \iff 0 < n - k^2$ and our assumption $\sqrt{n} > k$. In summary, the following holds.

Proposition 9. *Given two integers* k *and* n *with* $4 \leq k < \sqrt{n}$*, then neither* $\Theta_{k,n}$ *is contained in* $\mathcal{P}^{LP}_{\leq k}(K_n)$ *nor is the converse true.*

5 Proving Tightness Results Using Duality Theory

Two types of matrices are of importance in our proofs of Prop. 6, 7, and 8. Let $E(m,n) \in \mathbb{R}^{m \times n}$ be the matrix with all entries equal to 1. Let $D^{\alpha,\beta}(n)$ denote the symmetric square matrix of order $n \geq 1$ with all entries on the principal diagonal equal to α and all other entries equal to β. Some basic properties of $D^{\alpha,\beta}(n)$ are easily observed.

Proposition 10. *For $n \geq 1$, the determinant of $D^{\alpha,\beta}(n)$ is given by*

$$\det(D^{\alpha,\beta}(n)) = \big(\alpha - \beta\big)^{n-1} \big(\alpha + (n-1)\beta\big) \ .$$

For $\beta \notin \{\frac{-\alpha}{n-1}, \alpha\}$ the matrix $D^{\alpha,\beta}(n)$ is regular and its inverse is

$$D^{\alpha,\beta}(n)^{-1} = \frac{1}{\big(\alpha - \beta\big)\big(\alpha + (n-1)\beta\big)} \, D^{\alpha+(n-2)\beta,-\beta}(n) \ .$$

$D^{\alpha,\beta}(n)$ is positive semidefinite if and only if $\alpha \geq \beta \geq \frac{-\alpha}{n-1}$; it is positive definite if and only if strict inequality holds in both cases. (In case $n = 1$, $D^{\alpha,\beta}(n) = \big[\alpha\big]$ and $\beta = 0$ is assumed. The condition "$\beta \geq \frac{-\alpha}{n-1}$" becomes void.)

We need to know the conditions on α, β, γ, δ, ε, and s, t under which the matrix

$$A = \begin{bmatrix} D^{\alpha,\beta}(s) & \gamma E(s,t) \\ \gamma E(t,s) & D^{\delta,\varepsilon}(t) \end{bmatrix}$$

is positive semidefinite. These conditions can be derived by means of the Schur Complement Theorem. (This theorem states that the composite matrix with matrices X and Z as diagonal blocks and Y and Y^T as off-diagonal blocks is positive semidefinite if and only if $Z \succeq Y^T X^{-1} Y$ when X is a positive definite $(n \times n)$–matrix, Z is a symmetric $(m \times m)$–matrix, and Y is a $(m \times n)$–matrix.)

Proposition 11. *Given integers $s, t \geq 1$, the matrix*

$$A = \begin{bmatrix} D^{\alpha,\beta}(s) & \gamma E(s,t) \\ \gamma E(t,s) & D^{\delta,\varepsilon}(t) \end{bmatrix}$$

is positive semidefinite if and only if $D^{\alpha,\beta}(s)$, $D^{\delta,\varepsilon}(t)$ are both positive semidefinite and $\big(\alpha + (s-1)\beta\big)\big(\delta + (t-1)\varepsilon\big) \geq st\gamma^2$ holds.

Our tightness proofs are based on a pair of dual semidefinite programs. We denote with $E^{ij}(n) \in S_n$ the symmetric $(n \times n)$–matrix with entries 1 at positions (i,j) and (j,i), and zeros elsewhere. We simply write E^{ij} if the dimension is clear from the context.

Remark 1. For every matrix $C \in S_n$, the semidefinite programs

$$\begin{aligned} &\min \sum_{1 \leq i,j \leq n} C_{ij} X_{ij} \qquad \text{s.t.} \\ &\langle E^{ii}, X_{ii} \rangle = 1, \quad \langle E^{ij}, X_{ij} \rangle \geq \frac{-1}{k-1}, \quad \forall\, i,j \in \{1,\dots,n\}, i < j \\ &X \in S_n^+ \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\max \sum_{i=1}^{n} y_{ii} \ - \sum_{1 \leq i < j \leq n} \frac{y_{ij}}{k-1} \qquad \text{s.t.} \\ &C - \sum_{1 \leq i \leq j \leq n} y_{ij}\, E^{ij} \ \in \ S_n^+, \quad y_{ii} \in \mathbb{R}, \quad y_{ij} \in \mathbb{R}_+ \end{aligned} \tag{20}$$

are dual to each other. They are both strictly feasible.

The dual variable associated to the primal constraint $\langle E^{ii}, X\rangle = 1$ is y_{ii} and that associated to the primal constraint $\langle E^{ij}, X\rangle \geq \frac{-1}{k-1}$ is y_{ij}.

Proof. Duality is easily checked. We merely prove the strict feasibility here.

The identity matrix I_n is positive definite and strictly meets all inequality constraints of (19). Hence, I_n is in the relative interior of the solution space, and the first program is strictly feasible.

The vector $y \in \mathbb{R}^{\binom{n}{2}}$ with $y_{ii} = -\sum_{j=1}^{n} |C_{ij}| - n$ for all i and $y_{ij} = 1$ for all $i < j$ is a feasible dual solution. All sign restrictions on y are strictly met, and the matrix $C - \sum_{1\leq i \leq j \leq n} y_{ij}\, E^{ij}$ is positive definite, because it is strictly diagonally dominant. Therefore, the program (20) is strictly feasible, too. □

The Prop. 6, 7, and 8 can all be proven by exhibiting primal and dual feasible solutions of matching objective function values for an appropriate semidefinite program of the form given in Remark 1.

We give a complete proof only for Prop. 8, because in this case more than just a tightness result for (12) for a special case is needed. The right-hand side of (18) is strictly larger that of (12), and a new validity proof is required. In the other two cases, we merely give the matrix C for the objective function as well as suitable primal and dual feasible solutions. In fact, since the objective function values of the primal solutions attain the bound from (12), (new) dual solutions are strictly speaking not necessary.

Proof (of Prop. 8). We have to prove that (18) is valid and tight for $\Theta_{k,n}$. This is done by considering the optimization problem (19) with the left-hand side of (18) as objective function and by showing that the right-hand side of (18) is the optimal solution value.

Let $\tilde{b}$ be the edge weights obtained by setting $\tilde{b}_{ij} = b_i b_j$ with $b_i = 1$ if $i \in T$, $b_i = -1$ if $i \in S$, and $b_i = 0$ otherwise. Moreover, let $\tilde{B}$ denote the symmetric matrix with $\tilde{b}$ on its off-diagonal positions and zeros on the principal diagonal. Recall from (11) that solving $\min \sum_{ij\in E(K_n)} b_i b_j\, z_{ij}$ s. t. $z \in \Theta_{k,n}$ is equivalent to solving $\min \frac{1}{2}\langle \tilde{B}, T_k(X)\rangle$ s. t. $X \in \Psi_{k,n}$. It suffices to consider the subproblem on $S \cup T$, because every pair of dual solutions for the reduced problem can be extended to the full problem without changing the objective function value.

We give solutions X and y to the dual programs (19) and (20), respectively, with matching objective function values for the primal cost matrix $\tilde{B}_{S\cup T, S\cup T} = C^{1,t}$. We then compute $\frac{k-1}{2k}\langle C^{1,t}, X\rangle + \frac{1}{2k}\langle C^{1,t}, E(1+t, 1+t)\rangle$ and show that this is the desired value.

Let us first consider the maximization problem (20). We fix $a = \sqrt{\frac{k-1}{t(k-t)}}$. A short computation reveals that $0 < a \leq 1$ provided $1 \leq t \leq k-2$. Let $y_{11} = -\frac{1}{a}$, $y_{ii} = -a$ for $i = 2, \ldots, 1+t$, $y_{1j} = y_{j1} = 0$ for all $j = 2, \ldots 1+t$, and $y_{ij} = 1-a$ for all $i, j \in \{2, \ldots, 1+t\}, i < j$. Then, the vector y is a feasible solution because $y_{ij} \geq 0$ for all $i < j$ and

$$C^{1,t} - \sum_{1\leq i<j\leq n} y_{ij} E^{ij} = \begin{bmatrix} 1/a & -E(1,t) \\ -E(t,1) & D^{a,a}(t) \end{bmatrix} \succeq 0 \ .$$

The latter is a direct consequence of Prop. 11.

The objective function evaluates to

$$\begin{aligned}
&\sum_{i=1}^{n} y_{ii} - \frac{1}{k-1}\sum_{i\neq j} y_{ij} \\
&= 1\left(-\sqrt{\frac{k-1}{t(k-t)}}\right)^{-1} + t\left(-\sqrt{\frac{k-1}{t(k-t)}}\right) - \frac{t(t-1)}{k-1}\left(1-\sqrt{\frac{k-1}{t(k-t)}}\right) \\
&= -\sqrt{\frac{t(k-t)}{k-1}} - \underbrace{\frac{t(k-1)-t(t-1)}{k-1}\sqrt{\frac{k-1}{t(k-t)}}}_{=\sqrt{\frac{t(k-t)}{k-1}}} - \frac{t(t-1)}{k-1} \\
&= -2\sqrt{\frac{t(k-t)}{k-1}} - \binom{t}{2}\frac{2}{k-1}\ .
\end{aligned}$$

Next, we argue that the matrix

$$X = \begin{bmatrix} 1 & \sqrt{\frac{k-t}{t(k-1)}}\,E(1,t) \\ \sqrt{\frac{k-t}{t(k-1)}}\,E(t,1) & D^{1,-1/(k-1)}(t) \end{bmatrix}$$

is a primal feasible solution. Given that $1 < t \leq k-2$, all off-diagonal entries are at least as large as $\frac{-1}{k-1}$. By Prop. 11, X is positive semidefinite. We check the only condition that is not trivially fulfilled, namely,

$$1\left(1+(t-1)\frac{-1}{k-1})\right) = \frac{k-t}{k-1} = 1\,t\left(\sqrt{\frac{k-t}{t(k-1)}}\right)^2 .$$

The corresponding objective function value is

$$\langle C^{1,t}, X\rangle = -2t\sqrt{\frac{k-t}{t(k-1)}} - \binom{t}{2}\frac{2}{k-1} = -2\sqrt{\frac{t(k-t)}{k-1}} - \binom{t}{2}\frac{2}{k-1}\ .$$

For the dual transformed objective function value we obtain

$$\begin{aligned}
&\frac{k-1}{2k}\left(-2\sqrt{\frac{t(k-t)}{k-1}} - \binom{t}{2}\frac{2}{k-1}\right) + \frac{t(t-1)-2t}{2k} \\
&= -\frac{k-1}{k}\sqrt{\frac{t(k-t)}{k-1}} - \binom{t}{2}\frac{1}{k} - \frac{t}{k} + \binom{t}{2}\frac{1}{k} \\
&= -1 - \frac{\sqrt{t(k-t)(k-1)} - (k-t)}{k}\ .
\end{aligned}$$

This proves the claims concerning the validity and tightness of (18).

Finally, we show that $-\sqrt{t}$ bounds the above term from below. An application of l'Hôspital's rule yields that the expression $-1-\frac{\sqrt{t(k-t)(k-1)}-(k-t)}{k}$ converges to $-\sqrt{t}$ as k goes to infinity. It remains to check that the value of the expression is bounded from below by $-\sqrt{t}$:

$$
\begin{aligned}
& -\sqrt{t} < -1 - \frac{\sqrt{t(k-t)(k-1)} - (k-t)}{k} \\
\iff\ & k\sqrt{t} - t > \sqrt{t(k-t)(k-1)} \\
\overset{k \geq t \geq 1}{\iff}\ & tk^2 - 2tk\sqrt{t} + t^2 > t(k-t)(k-1) \quad [= tk^2 - t(t+1)k + t^2] \\
\overset{t \geq 0}{\iff}\ & -2k\sqrt{t} > -t(t+1)k \\
\overset{k \geq 0}{\iff}\ & \sqrt{t} < \frac{t+1}{2} \ .
\end{aligned}
$$

The last inequality holds for all $t \geq 2$. This completes the proof. □

Finally, we give hints for proving Prop. 6 and Prop. 7 analogously to the previous proof.

Proof (of Prop. 6, Sketch). Use $C = D^{0,1}(q)$ as objective and $X = D^{1,\frac{-1}{q-1}}(q)$ as primal solution. The resulting optimal value is $\frac{q}{2k}(q-k)$. This value is also attained for the dual solution y with $y_{ii} = 1$ for all i and $y_{ij} = 0$ for all $i \neq j$. □

Proof (of Prop. 7, Sketch). Use the objective $C^{s,t} = \begin{bmatrix} D^{0,1}(s) & -E(s,t) \\ -E(t,s) & D^{0,1}(t) \end{bmatrix}$. In case of $s = 1$, use $X = \begin{bmatrix} 1 & \frac{1}{t}E(1,t) \\ \frac{1}{t}E(t,1) & D^{1,-1/t}(t) \end{bmatrix}$ as primal solution, and $X = \begin{bmatrix} D^{1,\alpha}(s) & \gamma E(s,t) \\ \gamma E(t,s) & D^{1,\beta}(t) \end{bmatrix}$ in case of $s \geq 2, k \geq s+t$ and either $t \leq s^2$, or $t > s$ together with $k \leq \frac{t^2-s^2}{t-s^2}$. The resulting optimal value is $\frac{(t-s)^2 - k(t+s)}{2k}$. This value is also attained for the dual solution y with $y_{ii} = 1$ for all i and $y_{ij} = 0$ for all $i \neq j$. □

6 Conclusion

The semidefinite relaxation from Section 3 of the combinatorial Minimum k-Partition problem has been known for several years. The fact that such a semidefinite program is (ϵ-approximately) solvable in polynomial time is known even longer. But just within the last one or two years SDP solvers have matured to the point, where the semidefinite programs associated to graphs of sizes in the order of a few hundred vertices are computationally tractable in practice. Recall, however, that the semidefinite relaxation of Minimum k-Partition cannot serve as the basis for a constant-factor approximation for Minimum k-Partition unless $\mathcal{P} = \mathcal{NP}$.

Our motivation has been to derive lower bounds on the unavoidable interference in GSM frequency planning. In that sense, Minimum k-Partition is already a relaxations of our original problem. Nevertheless, the numerical bounds shown in Tab. 1 are surprisingly strong. Clearly, similar investigations should be performed for Minimum k-Partition directly.

We attribute the strength of the numerical bounds to a large extent to the "shifted hypermetric inequalities" (12), which are implicit in the semidefinite relaxation. Hence, with respect to the integer linear programming formulation (1)–(4) of the MINIMUM K-PARTITION problem, all triangle constraints (2) are violated by at most $\sqrt{2} - 1$ and all clique constraints (3) by less than $\frac{1}{2}$. This may serve as a preliminary explanation, but the relation between the MINIMUM K-PARTITION and its semidefinite relaxation is certainly not yet settled.

Moreover, the substantial progress in the development of SDP solvers and our promising computational results call for fathoming the following two questions: Which semidefinite-programming-based k-partitioning heuristics perform well in practice? (Simply applying randomized rounding may not be sufficient.) How successful is a branch-and-cut approach to the MINIMUM K-PARTITION problem based on semidefinite programming?

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