Technische Universität Berlin - Institut für Mathematik

Describing Polyhedra by Polynomial Inequalities

Diplomarbeit

Betreut von
Prof. Martin Grötschel
Prof. Martin Henk

Angefertigt von Hartwig Bosse
Stargarderstr. 37a
10437 Berlin
Matrikelnummer: 152583

Berlin, im Januar 2003

Hinweis des Autors an Diplomanden

Diese Arbeit fußt auf einem Ergebnis von Bröcker und Scheiderer im Bereich der algebraischen Geometrie. Die Hauptaussage ist, dass jedes Polyeder in $\mathbb{R}^d$ durch $2d-1$ viele polynomielle Ungleichungen beschrieben werden kann, die Arbeit soll eine Konstruktion für diese Polynome liefern.

Dass dies hier erwähnt werden muß, ist der größte Kritikpunkt an dieser Diplomarbeit:
Sie ist unübersichtlich, es fehlt der rote Faden. Obwohl sich das Ergebnis inhaltlich durchaus sehen lassen kann, ist die präsentiete Form mangelhaft.

Meine Ratschläge an Sie sind in aller Kürze: Reduziern Sie die Dinge auf das Wesentliche, fangen Sie früh an zu schreiben und besprechen Sie das Geschriebene früh mit einem Korrektor.

Die nachfolgenden Ratschläge sind nicht mit dem Anspruch auf Volständigkeit oder genereller Richtigkeit formuliert:

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- Legen Sie lieber die Latte niedrig und springen hoch darüber. Ein kleines Ergebnis wohlformuliert ist sicher besser als ein großes, unlesbares.

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- Teilen Sie das Schreiben in mindestens drei zeitlich getrennte Vorgänge: Gliederung, Textskizze, Feinschliff.
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Das Folgende gehört zu den selbstverständlichen Grundvoraussetzungen normalen Arbeitens:

- Sorgen sie für eine geeignete Arbeitsumgebung. Dazu gehört
  - Ein Editor aus dem heraus Sie LaTeX kompilieren können (z.B. Win: Winedt, Linux: xemacs).
  - Das Lernen und Verwenden von Tastaturkürzelen.
  - Ein geeignetes Versionsmanagement. Unter Linux ist CVS wertvoll, xemacs beherrscht den VC-modus.
  - Eine geeignete Backupstrategie.
- Falls Sie jetzt noch nicht mit LaTeX vertraut sind, müssen Sie es jetzt werden! Das Buch: "LaTeX Band 1: Einführung" von Helmut Kopka ist empfehlenswert.
- Falls Sie Bilder benötigen werden, beschäftigen Sie sich mit Grafikprogrammen, die eps-Dateien produzieren (Win: Coreldraw, Photoshop(begrenzt), Linux: xfig)

Viel Erfolg!

Contents

1. Introduction

This thesis presents a new result in both the field of real algebraic geometry and in the theory on polyhedra.

The present work is motivated by a striking result of Bröcker and Scheiderer ([Sch89a], [Brö91] and [BCR98]) in the field of Real Algebraic Geometry on basic closed semi-algebraic sets. Basic closed semi-algebraic sets are subsets of a real closed field, built by the intersection of the result sets of finitely-many non-strict polynomial inequalities.

Bröcker and Sheiderer showed that any basic closed semi-algebraic set $\mathcal{V}$ can be represented by at most $d(d+1)/2$ polynomial inequalities where d denotes the algebraical dimension of the set $\mathcal{V}$.

In the case that $\mathcal{V}$ is an open semi-algebraic set, where the non-strict inequalities are replaced by strict inequalities, this number reduces even to d.

Therefore surprisingly any d-dimensional, convex polyhedron can be described by at most $d(d+1)/2$ polynomial inequalities, *independently of geometrical and combinatorial structure* of the given polyhedron. The reason is that any convex polyhedron P in $\mathbb{R}^d$ given as the intersection of finitely-many linear half spaces, is a basic closed semi-algebraic set described using polynomials of degree 1. Further the definition of the algebraical dimension of P meets the definition of $\dim(P)$ as the lowest dimension of all affine subspaces containing P.

Unfortunately up to now, no explicit general construction for such a set of polynomials is known. However, for some subclasses of polyhedra there are results.

In [Brö91], Example 2.10, or in [ABR96], Example 4.7, a description for a regular convex n-gon in $\mathbb{R}^2$ using two polynomials is given. This result was generalized to arbitrary convex polygons by vom Hofe in [Hof91] using three polynomial inequalities, which was improved by Bernig showing in [Ber98] that any convex polygon may be described using two polynomial inequalities. Grötschel and Henk give a constructive description in [GH] of simple convex polytopes of any dimension d using $O(d^d)$-many polynomial inequalities.

In this paper we will show the following:

Theorem 1.0.1 (Main Theorem). *Any d-dimensional convex polyhedron can be described using at most $(2d)$-many polynomials.*

Further we provide an explicit construction for the necessary polynomials, for the case that the polyhedron is a pointed cone.

From the viewpoint of Real Algebraic Geometry the given thesis presents a constructive proof for a stricter version of the results of Bröcker and Scheiderer on a subclass of basic closed semi-algebraic sets in $\mathbb{R}^d$, specifically those constructed on polynomials of degree 1 in $\mathbb{R}[X_1, \ldots, X_d]$, the field of all real polynomials in d-unknowns.

Further replacing the polynomial inequalities by strict inequalities on the same polynomials yields the interior of the given polyhedron. Thus we provide a description using $2d$-many polynomial inequalities for any open semi-algebraic set in $\mathbb{R}^d$ given by polynomials of degree 1.

1.1 Structure

This thesis is devided into three main parts, in the first part we introduce special properties on selected semi-algebraic sets and motivated by this we construct approximations to basic polyhedra in the second part, which in the last part are used to prove the Main Theorem.

We prove the Main Theorem by presenting a construction of the necessary polynomials in the case that the given polyhedon is a pointed cone. From this we deduce the general case making use of the concept of homogenization.

Preparation

In Chapter 3 we provide a brief overview of the methods and ideas that that are used in this thesis.
In Chapter 4 we present a method reducing the amount of polynomial inequalities needed to describe special semi-algebraic sets by multiplying the polynomials in the given description. Using this we provide a description of the crosspolytope in $\mathbb{R}^3$ based on 3 polynomial inequalities.
On behalf of this concept we will show in Chapter 5 that it is possible to deduct the Main Theorem for any pointed, polyhedral cone C, if for each face F of C a semi-algebraic set $\mathcal{S}^F$ with special properties is constructed.

Construction

In Chapter 6 to 11 we construct approximations $\mathcal{S}^F$ to cones $C^{[F]}$ which are representants to the faces F of a pointed polyhedral cone C. We discuss the relations of such sets $\mathcal{S}^{F_1}$, $\mathcal{S}^{F_2}$ based on different faces F_1, F_2 of C.

Conclusion

From this we conclude in Chapter 12 that the constructed $\mathcal{S}^F$ meet the restrictions outlined in Chapter 5, proving the Main Theorem.
In Chapter 13 we give the full construction for pointed cones C from which we deduce a construction for general polyhedra making use of the concept of homogenization.
Finally we review the given construction and present ideas for possible other methods suitable to solve the problem for special classes of polyhedra.

1.2 Conventions of Notation

Polyhedra

Whenever polyhedra are given as intersection of halfspaces

$$P = \left\{ x \in \mathbb{R}^d \ : \ a_i^T x \leq b_i \ , i=1,\ldots,m \right\} ,$$

unless mentioned otherwise we assume $\|a_i\| = 1$ and we will assume always that the given description is irreducible. So $P \cap \{x \in \mathbb{R}^d \ : \ a_i^T x = b_i\}$ is always a facet of P for $i=1,\ldots,m$. In particular $b_i = h_P(a_i)$, where h_P denotes the support function of P:

$$h_P(a) := \sup\{a^T x : \ x \in P\} \quad .$$

When unambiguous which polyhedron is meant, h_P is abbreviated to h. Notice that we will use h_P only such that its value is finite. For any $u \in \mathbb{R}^d$ by the statement "*u induces a facet of P*" we express that $P \cap \{x \in \mathbb{R}^d \ : \ u^T x = h_P(u)\}$ is a facet of P.

Faces

We denote the set of all proper faces of any polyhedron P by $\mathcal{F}(P)$, where a face F of P is called proper if $F \neq \emptyset$ and $F \neq P$. By $\mathcal{F}_k(P)$ we denote the set of all $F \in \mathcal{F}(P)$ with $\dim(F) = k$, $k = 0, \ldots, d-1$.

Further $[F] \subset \{1, \ldots, m\}$, the equality set, is the set of all indices $i \in \{1, \ldots, m\}$ such that $a_i^T x = b_i$ holds for all $x \in F$.

Cones

Throughout this paper we will work mostly with convex, full-dimensional, pointed polyhedral cones $C \subset \mathbb{R}^d$. Thus wherever nothing else is stated, we assume that $C := \{x \in \mathbb{R}^d \ : a_i^T x \leq 0 \ , i=1,\ldots,m\}$ is a d-dimensional pointed cone.

We distinguish between two different kinds of descriptions of cones as convex closures. There is either $\mathrm{cone}(W) + \mathrm{lin}(U)$ or $\mathrm{cone}(V) \oplus \mathrm{lin}(U)$. In both cases we will always assume that $U, V, W \subset \mathbb{R}^d$ are built on finitely many normalized elements, e.g. $W = \{ w_1, \ldots, w_n \}$, $w_i \in \mathbb{R}^d$ and $\|w_i\| = 1$ for all $i \in \{1, \ldots, n\}$.

Here both $U_1 + U_2$ and $U_1 \oplus U_2$ denote the Minkowski sum of two sets $U_1, U_2 \subset \mathbb{R}^d$ which in case of using $\oplus$ have to be perpendicular to each other, denoted $U_1 \perp U_2$.

Auxiliary Objects

To condense notation we define a hyper plane for $c \in \mathbb{R}^d$ by

$$H(c, \mu) \quad := \quad \{x \in \mathbb{R}^d \ : \ c^T x = \mu\}$$

and a closed ball wrapping $y \in \mathbb{R}^d$ with a radius $\delta \geq 0$

$$B(y, \delta) \;:=\; \{x \in \mathbb{R}^d \,:\, \|y - x\| \leq \delta\},$$

as well as an open ball with radius $\varepsilon > 0$ by

$$\overset{\circ}{B}(y, \varepsilon) \;:=\; \{x \in \mathbb{R}^d \,:\, \|x - y\| < \varepsilon\}.$$

2. Main Theorem

Our intention is to provide a constructive description of a convex polyhedron P by a basic closed semi-algebraic set $P = \{ x \in \mathbb{R}^d : \mathfrak{P}^1(x) \geq 0, \ldots, \mathfrak{P}^{2d}(x) \}$ using $2d$ polynomials $\mathfrak{P}^1, \ldots, \mathfrak{P}^{2d} \in \mathbb{R}[X_1, \ldots, X_d]$.

The core of this thesis is a construction for the special case that P is a pointed full-dimensional polyhedral cone, as from this construction the general case can be subducted by the using the homogenization of P. In this chapter we first provide a theorem for pointed polyhedral cones, which to prove will be the work of the rest of this paper. Then we prove that the Main Theorem follows directly from this construction using homogenization.

2.1 Main Theorem for Pointed Cones

The objects we will construct depend on the choice of a support vector for each face of the given cone C. To express this property we introduce the following object:

Definition 2.1.1. Let P be any convex polyhedron.
And let $c : \mathcal{F}(P) \longrightarrow \mathbb{R}^d$ be such that $c(F)$ is a *normalized* support vector of F for every $F \in \mathcal{F}(P)$.
Then the set of all support vectors $c(\mathcal{F}(C)) := \{c(F) : F \in \mathcal{F}(P)\}$ is called a choice of support vectors of P. Whenever the chosen polyhedron is unambiguous $c(\mathcal{F}(C))$ is abbreviated to $c(\mathcal{F})$.

The next theorem is the center of our construction:

Theorem 2.1.2. *Let C be a pointed polyhedral cone and $c(\mathcal{F})$ any choice of support vectors of C. Then for each face $F \in \mathcal{F}(C)$ there is a polynomial $\mathfrak{p}^F \in \mathbb{R}[X_1, \ldots, X_d]$ such that:*

$$C \quad = \bigcap_{k=0,..,d-1} \left\{ x \in \mathbb{R}^d : \mathfrak{P}_1^k(x) \geq 0 \, , \mathfrak{P}_2^k(x) \geq 0 \right\} \quad , \tag{2.1}$$

where

$$\mathfrak{p}_1^F(x) := 0 \quad \forall F \in \mathcal{F}_{d-1}(C) \cup \mathcal{F}_{d-2}(C) \tag{2.2}$$

and

$$\mathfrak{P}_1^k(x) \; := \prod_{F \in \mathcal{F}_k(C)} \mathfrak{p}^F(x) \qquad k = 0, \ldots, d-1 \quad , \tag{2.3}$$

$$\mathfrak{P}_2^k(x) \; := \prod_{F \in \mathcal{F}_k(C)} -c(F)^T x \qquad k = 0, \ldots, d-1 \quad . \tag{2.4}$$

Such that the two polynomials $\mathfrak{P}_1^{d-1}(x) = \mathfrak{P}_1^{d-2}(x) = 0$ *are redundant and listed in (2.1) on behalf of the compactness of description.*

2.2 Deducing the Main Theorem

We will deduct Main Theorem 1.0.1 from Theorem 2.1 in two steps: First full-dimensional polyhedra of the form $P = \mathrm{conv}(V) + \mathrm{cone}(W)$ where $\mathrm{cone}(W)$ is pointed are discussed, then we examine unbound polyhedra of the form $P = (\mathrm{conv}(V) + \mathrm{cone}(W)) \oplus \mathrm{lin}(U)$.

Lemma 2.2.1. *Let* $P := \mathrm{conv}(V) + \mathrm{cone}(W) \subset \mathbb{R}^d$ *be a* d*-dimensional polyhedron where* $V = \{ v_1, \ldots, v_k \}$, $W = \{ w_1, \ldots, w_n \}$ *and* $\mathrm{cone}(W)$ *is pointed.*
Then there is a description of P *as a basic closed semi-algebraic set using* $(2d)$*-many polynomial inequalities.*

Proof: We will use the homogenization $\mathrm{homog}(P) \subset \mathbb{R}^d \times \mathbb{R}$ of P (cf. [Zie98][1]) to make use of theorem 2.1.2:

$$\mathrm{homog}(P) := \left\{ t \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in P, t > 0 \right\} \cup \left\{ \begin{pmatrix} y \\ 0 \end{pmatrix} : y \in \mathrm{cone}(W) \right\} \quad .$$

This set is a pointed, polyhedral cone of dimension $d+1$, thus according to Theorem 2.1.2 we find a description of $\mathrm{homog}(P)$ in maximally $(2(d+1)-2) = 2d$ non-redundant polynomial inequalities:

$$\mathrm{homog}(P) \quad = \bigcap_{k=0,..,d} \left\{ x \in \mathbb{R}^{d+1} : \mathfrak{P}_i^k(x) \geq 0 \,, i = 1,2 \right\} \quad . \tag{2.5}$$

To retrieve P from $\mathrm{homog}(P)$ we use

$$x \in P \Leftrightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathrm{homog}(P) \quad ,$$

and thus we conceive a description for P by fixing $x_{d+1} := 1$ in (2.5):

$$P \quad = \bigcap_{k=0,..,d} \left\{ x \in \mathbb{R}^{d} : \mathfrak{P}_i^k(x,1) \geq 0 \,, i = 1,2 \right\} \quad , \tag{2.6}$$

where the polynomials $\mathfrak{P}_1^{d-1}(x) = \mathfrak{P}_1^{d-2}(x) = 0$ are redundant in the description of P given by (2.6) and thus P can be described using at most $2d$ polynomial inequalities. □

[1] The present formula for $\mathrm{homog}(P)$ is given differently from [Zie98] in consultation with the author.

Polyhedra Containing Affine Subspaces

Lemma 2.2.2. *Let* $P := (\mathrm{conv}(V)+\mathrm{cone}(W))\oplus\mathrm{lin}(U) \subset \mathbb{R}^d$ *be a* d*-dimensional polyhedron where* $V = \{v_1,\ldots,v_k\}$, $W = \{w_1,\ldots,w_n\}$, $U = \{u_1,\ldots,u_l\}$ *and* $\mathrm{cone}(W)$ *is pointed. Then there is a description for* P *as basic closed semi-algebraic set involving* $2(d-\dim(U)) \leq 2d$ *polynomial inequalities.*

Proof: Notice that the set $\mathrm{conv}(V)+\mathrm{cone}(W)$ is contained in some subspace of $\mathbb{R}^d$, namely

$$U^\perp := \left\{ x\in\mathbb{R}^d \,:\, u^T x = 0 \;\forall u \in U \right\}.$$

Let $k := \dim(U^\perp)$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}^k$, $x \mapsto Bx$, $B \in \mathbb{R}^{k\times d}$ be a linear orthogonal mapping such that $f(U^\perp) = \mathbb{R}^k$ and $f(U) = \{\mathbf{0}\}$. Then f maps the polyhedron $P \cap U^\perp$ bijectively onto $P' := f(P\cap U^\perp)$, which is a full-dimensional polyhedron in $\mathbb{R}^k$ of the form $P' = \mathrm{conv}(V') + \mathrm{cone}(W')$. So there is a description

$$P' = \{x \in \mathbb{R}^k \,:\, \mathfrak{p}_1(x) \geq 0, \ldots, \mathfrak{p}_{2k}(x) \geq 0\}$$

of P' as solution set of $(2k)$-many polynomial inequalities. The set

$$S := \left\{ x\in\mathbb{R}^d \,:\, \mathfrak{p}_1(Bx) \geq 0, \ldots, \mathfrak{p}_{2k}(Bx) \geq 0 \right\}$$

is a description of P in polynomial inequalities: Clearly $\mathfrak{p}_i(Bx)$, $i \in \{1,\ldots,2k\}$ are polynomials, and as $Bx = B(x+u)$ for any $x \in \mathbb{R}^d$ and $u \in \mathrm{lin}(U)$, we find $x \in S \Leftrightarrow x+u \in S$ such that $S = (S\cap U^\perp)\oplus\mathrm{lin}(U)$. Additionally $P\cap U^\perp = S\cap U^\perp$, because for every $x \in U^\perp$ we have

$$(x \in P) \;\Leftrightarrow\; Bx \in P' \;\Leftrightarrow \mathfrak{p}_i(Bx) \geq 0 \;\forall\, i \in \{1,\ldots,2k-1\} \;\Leftrightarrow x \in S \quad .$$

Thus there is a description for P using $2k$ polynomial inequalities, where $k = \dim(U^\perp) = d-\dim(U)$. $\square$

3. Motivation

In the next sections we will be mainly constructing special semi-algebraic sets which are approximations of certain classes of polyhedra. For these approximations we prove some rather peculiar properties. Now we outline how these properties are used.

3.1 Multiplying Polynomial Inequalities

Any cone $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i{=}1,\ldots,m\}$ already is a semi-algebraic set, so our main objective here is to discuss how to reduce the amount of polynomial inequalities used to describe a given semi-algebraic set.

Definition 3.1.1. Let $\mathcal{S} := \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_1(x) \geq 0,\ldots,\mathfrak{p}_m(x) \right\}$, $m \in \mathbb{N}$ be some semi-algebraic set.
Then we define the Zariski closure of $\mathcal{S}$, a special algebraical bound, by

$$\operatorname{clos}_Z(\mathcal{S}) \;:=\; \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_1(x) = 0,\ldots,\mathfrak{p}_m(x) = 0 \right\} \tag{3.1}$$

and the algebraical interior of $\mathcal{S}$ by

$$\operatorname{int}_Z(\mathcal{S}) \;:=\; \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_1(x) > 0,\ldots,\mathfrak{p}_m(x) > 0 \right\} \quad . \tag{3.2}$$

Corollary 3.1.2. *Directly from the definition we conclude – using that the polynomials $\mathfrak{p}_i$ are continuous – that $\operatorname{int}_Z(\mathcal{S}) \subset \mathring{\mathcal{S}}$ where $\mathring{\mathcal{S}}$ denotes the topological interior of $\mathcal{S}$ defined by the topology of open sets.*

Please derive from $\mathcal{S} = \{x \in \mathbb{R} : x^2 \geq 0\}$ that the opposite does not necessarily hold. For detailed properties of the Zariski closure we refer to [ABR96] and [BCR98].

Let $\mathcal{S} = \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_i(x) \geq 0,\; i = 1,2,3 \right\}$ be a semi-algebraic set. Let further $\mathcal{S}_i := \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_i(x) \geq 0 \right\}$ and let

$$\mathcal{S}_3 \subset \operatorname{int}_Z(\mathcal{S}_1) \cup \operatorname{int}_Z(\mathcal{S}_2) \;\cup (\operatorname{clos}_Z(\mathcal{S}_1) \cap \operatorname{clos}_Z(\mathcal{S}_2)\,)\,.$$

Then it is possible to multiply $\mathfrak{p}_1$ and $\mathfrak{p}_2$ without changing $\mathcal{S}$ as follows:

$$\mathcal{S} = \left\{ x{\in}\mathbb{R}^d \,:\, \mathfrak{p}_1(x)\mathfrak{p}_2(x) \geq 0 \,, \mathfrak{p}_3(x) \geq 0 \right\} \quad .$$

This is motivated by the the two sketches in Figure 3.1: Points in the set marked on the right-hand-side fulfill $\mathfrak{p}_1(x)\mathfrak{p}_2(x) \geq 0$, denoting the set where both polynomials are

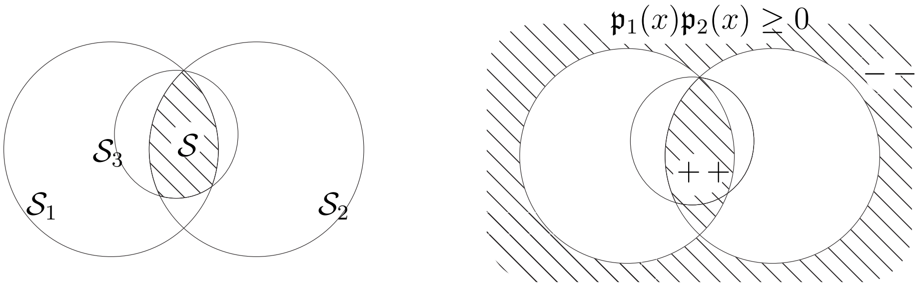

Fig. 3.1: The containment restriction on $\mathcal{S}_3$ allows multiplying $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

positive (negative) by $++$ ($--$). The intersection of all boundaries (circles) is the point $\operatorname{clos}_Z(\mathcal{S}_1) \cap \operatorname{clos}_Z(\mathcal{S}_2)$.

The most interesting about this is, that this restriction is *not* a restriction to the sets $\mathcal{S}_1$, $\mathcal{S}_2$, whose inequalities we are about to multiply.

Thus given a fixed set $\mathcal{S} = \bigcap_{i \in I} \mathcal{S}_i$, $I = \{1, \dots, m\}$, we might find some new sets 'fitting' into the given $\mathcal{S}_i$, according to the given restriction. In fact in the next section we prove that if all $\mathcal{S}_i$ are built on the same number n of polynomials and for every pair $\mathcal{S}_i$, $\mathcal{S}_j$ we have a 'corrective set' Q_{ij} such that

$$Q_{ij} \subset \operatorname{int}_Z(\mathcal{S}_i) \cup \operatorname{int}_Z(\mathcal{S}_j) \; \cup (\operatorname{clos}_Z(\mathcal{S}_i) \cap \operatorname{clos}_Z(\mathcal{S}_j)) \quad ,$$

then for the set $\mathcal{S}' = \mathcal{S} \cap \bigcap_{i \in I \times I} Q_{ij}$ the inequalities used to describe the $\mathcal{S}_i$ can be multiplied into n polynomials without changing $\mathcal{S}'$.

3.2 The Use of Combinatorics

The core task of this thesis is to find such 'correction' sets for a given cone

$$C = \{x \in \mathbb{R}^d \, : \, a_i^T x \leq 0 \, , i{=}1,\dots,m\}.$$

First we define for every facet $G \in \mathcal{F}_{d-1}(C)$ a semi-algebraic set $\mathcal{S}^G := \{x \in \mathbb{R}^d \, : \, -a_i^T x \geq 0, \, i \in [G]\}$ such that $C = \bigcap_{G \in \mathcal{F}_{d-1}(C)} \mathcal{S}^G$.

Then we deal with finding additional semi-algebraic sets $\mathcal{S}$ containing C and fulfilling the special restriction. Hereby on one hand we reduce the amount of polynomials by multiplying some polynomials, on the other hand adding semi-algebraic sets to the description of C clearly introduces even more polynomials.

To solve this dilemma we provide a geometrical invariable based on combinatorics, specifically for each proper face of C we construct a single set.

In the following sections for every $F \in \mathcal{F}(C)$, $\dim(F) < d-1$, we construct a semi-algebraic set $\mathcal{S}^F = \left\{ x \in \mathbb{R}^d \, : \, \mathfrak{p}_i^F(x) \geq 0 \, , \, i = 1, 2 \right\}$ containing C, such that for each pair of faces $F_1, F_2 \in \mathcal{F}_k(C)$, of identical dimension $0 < k \leq d-1$, intersecting in some face $F_3 = F_1 \cap F_2 \in \mathcal{F}(C)$ the following containment restriction holds

$$\mathcal{S}^{F_3} \subset \operatorname{int}_Z(\mathcal{S}^{F_1}) \cup \operatorname{int}_Z(\mathcal{S}^{F_2}) \ \cup \big(\operatorname{clos}_Z(F_1) \cap \operatorname{clos}_Z(\mathcal{S}^{F_2})\big) \quad .$$

Using the previous argumentation on the set $\mathcal{S} := \bigcap_{F \in \mathcal{F}(C)} \mathcal{S}^F$ – which here equals C – provides an inductive argument:

Each two facets G_1, G_2 of C intersect in some lower-dimensional face $F \in \mathcal{F}_k(C)$. Using the according $\mathcal{S}^F$ as correction sets as discussed above, the polynomials used to describe all $\mathcal{S}^G$, $G \in \mathcal{F}_{d-1}(C)$, can be multiplied into 2 polynomials, without changing $\mathcal{S}$. The same then holds for faces $F_1, F_2 \in \mathcal{F}_k(C)$, $k > 0$ and thus by reverse induction on k we end when reaching $k = 0$, where there is only the single vertex of C in $\mathcal{F}_0(C)$ and thus only one 'original' $\mathcal{S}^F$ is left. Therefore we end up with a constant number of inequalities per dimension.

So the fact that the basic sets used in this thesis are based on two polynomials is the major reason why the proposed construction uses $(2d)$-many polynomials. The explicit proof that the Main Theorem follows from the mentioned properties of the $\mathcal{S}^F$ is given in Section 5.2, based on the results of the next chapter. That the construction of such sets $\mathcal{S}^F$ is possible will be the core objective of this thesis.

4. Reducing the Number of Polynomials in Semi-Algebraic Sets

In this chapter we provide a characterization of those polynomials in a given description of a semi-algebraic set $\mathcal{S}$, which can be multiplied into a single polynomial, without changing $\mathcal{S}$ (see Figure 3.1, page 18). Directly from this characterization the construction for a description involving three polynomials of the crosspolytope C_3 in $\mathbb{R}^3$ is derived. In this example a technique is used which is different form the main approach followed in this thesis. A discussion of such procedure using colorings of facets is found at the end in Section 14.1.

Using the following theorem, in Chapter 5 a proof for the Main Theorem is given, based on the properties of objects postulated there and constructed in later chapters.

4.1 Multiplication of Polynomial Inequalities

In this thesis we denote the set of all integers excluding 0 by $\mathbb{N}$.

Theorem 4.1.1. *Let $M \in \mathbb{N}$ be a given fixed integer.*
Let $\mathcal{S}^i$, $i \in \{1,\dots,m\}$ and $\mathcal{Q}^k$, $k \in \{1,\dots,n\}$ be two sets of basic closed semi-algebraic sets where the $\mathcal{S}^i$ are built on M polynomial inequalities

$$\mathcal{S}^i := \left\{\, x \in \mathbb{R}^d \,:\, \mathfrak{p}_1^i(x) \geq 0, \dots, \mathfrak{p}_M^i(x) \geq 0 \,\right\} \quad .$$

Let the $\mathcal{Q}^k$ be such that for any pair $\mathcal{S}^i$, $\mathcal{S}^j$ there is a $\mathcal{Q}^l$ fulfilling the containment restriction

$$\mathcal{Q}^l \subset \operatorname{int}_Z(\mathcal{S}^i) \cup \operatorname{int}_Z(\mathcal{S}^j) \;\cup \left(\operatorname{clos}_Z(\mathcal{S}^i) \cap \operatorname{clos}_Z(\mathcal{S}^j)\right) \quad .$$

Then when we multiply the polynomials $\mathfrak{p}_i^j$ into M many polynomials $\mathfrak{P}_1, \dots, \mathfrak{P}_M$ as follows

$$\mathfrak{P}_j := \prod_{i=1}^{m} \mathfrak{p}_j^i$$

we obtain

$$\bigcap_{i=1,\dots,m} \mathcal{S}^i \;\cap \bigcap_{k=1,\dots,n} \mathcal{Q}^k \;=\; \left\{\, x \in \mathbb{R}^d \,:\, \mathfrak{P}_1(x) \geq 0, \dots, \mathfrak{P}_M(x) \geq 0 \,\right\} \;\cap \bigcap_{k=1,\dots,n} \mathcal{Q}^k \quad .$$

Proof: Let $\mathcal{S} = \bigcap_{i=1,\dots,m} \mathcal{S}^i \;\cap\; \bigcap_{k=1,\dots,n} \mathcal{Q}^k$ and let

$$\mathcal{S}' = \;\left\{\, x \in \mathbb{R}^d \,:\, \mathfrak{P}_1(x) \geq 0, \dots, \mathfrak{P}_M(x) \geq 0 \,\right\} \;\cap \bigcap_{k=1,\dots,n} \mathcal{Q}^k \quad .$$

then $\mathcal{S} \subset \mathcal{S}'$ holds as for $x \in \mathcal{S}$ we find $\mathfrak{p}^i_j(x) \geq 0$ for any $(i,j) \in \{1,\dots,m\} \times \{1,\dots,M\}$, and thus $\mathfrak{P}_j(x) \geq 0$ for any $j \in \{1,\dots,M\}$. Assume $\mathcal{S} \neq \mathcal{S}'$, then there must be some $y \in \mathcal{S}'$ with $y \notin \mathcal{S}$ and thus $y \notin \mathcal{S}^i$ for some $i \in \{1,\dots,m\}$. So we might assume w.l.o.g. $\mathfrak{p}^1_1(y) < 0$. As $y \in \mathcal{S}'$ we find $\mathfrak{P}_1(x) \geq 0$ thus there is at least one $j \in \{1,\dots,m\}$ such that $\mathfrak{p}^j_1(y) \leq 0$. Thus we have

$$\left(\mathfrak{p}^1_1(y) < 0 \text{ and } \mathfrak{p}^j_1(y) \leq 0\right) \quad . \tag{4.1}$$

For the pair $\mathcal{S}^1$, $\mathcal{S}^j$ there is some $\mathcal{Q}^l$ such that

$$\mathcal{Q}^l \subset \mathrm{int}_Z(\mathcal{S}^1) \cup \mathrm{int}_Z(\mathcal{S}^j) \ \cup \left(\mathrm{clos}_Z(\mathcal{S}^1) \cap \mathrm{clos}_Z(\mathcal{S}^j)\right) \quad . \tag{4.2}$$

As $y \in \mathcal{S}'$ we therefore find $y \in \mathcal{Q}^l$ and such using (4.2) we find

$$\left(\mathfrak{p}^1_1(y) > 0 \text{ or } \mathfrak{p}^j_1(y) > 0\right) \text{ or } \left(\mathfrak{p}^1_1(y) = \mathfrak{p}^j_1(y) = 0\right) \quad ,$$

which contradicts (4.1), and thus we have $\mathcal{S} = \mathcal{S}'$. □

Just for the generality it is worth mentioning that any semi-algebraic set $\mathcal{S} := \{x \in \mathbb{R}^d : \mathfrak{p}_i(x) \geq 0 \ , i \in \{1,\dots,k\}\}$ with $k < M$ can be transferred into a semi-algebraic set described by M inequalities without changing $\mathrm{int}_Z(\mathcal{S})$ by defining $\mathfrak{p}_i(x) := 1$ for $i = M-k+1,\dots,M$.

4.2 Examples

4.2.1 The Crosspolytope in $\mathbb{R}^3$

In this example we use a technique different to the approach in the rest of the paper. Here we first separate the face-defining inequalities into two groups before we multiplied them. Implicitly we make use of a two-coloring, where facets sharing an edge are separated. This approach is discussed at the end in Section 14.1.

Example 4.2.1. Let $C_3 := \{x \in \mathbb{R}^3 : a^T x \leq 1 \ , \forall a \in \{1,-1\}^3\}$ be the crosspolytope in $\mathbb{R}^3$. Then using the sign $\sigma(a) := \prod_{i=1}^3 (a)_i$ of each of the vectors $a \in \{-1,1\}^3$ they are split up into two groups such that $A_1 := \{a \in \{-1,1\} : \ \sigma(a) = 1\}$ and A_{-1} is define accordingly. Using the correction set $\mathcal{Q} := \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ it is possible to multiply the facet-defining inequalities of C_3 split by $\sigma(a)$ separately into two polynomials:

$$C_3 = \{x \in \mathbb{R}^3 : \|x\| \geq 0 \ , \prod_{a \in A_k} (1 - a^T x) \geq 0 \text{ for each } k = -1, 1\} \quad .$$

The proof for this relies on the theorem 4.1.1. We clearly find $C_3 \in \mathcal{Q}$, it is thus left to prove that for each pair $a_1, a_2 \in \{-1,1\}^3$, $a_1 \neq a_2$ the two sets $\mathcal{S}^i := \{x \in \mathbb{R}^d : 1 - a_i^T x \geq 0\}$, $i = 1,2$ fulfill the containment restriction whenever a_1, a_2 have the same sign, in detail:

$$\mathcal{Q} \subset \mathrm{int}_Z(\mathcal{S}^1) \cup \mathrm{int}_Z(\mathcal{S}^2) \ \cup \left(\mathrm{clos}_Z(\mathcal{S}^1) \cap \mathrm{clos}_Z(\mathcal{S}^2)\right) \quad .$$

Let $y \in \mathbb{R}^3$ be such that it is not contained in the set on the right-hand-side of this restriction. Then we find that y violates both strict inequalities $1 - a_i^T x > 0$ and at least one of the equalities $1 - a_i^T x = 0$. We find $1 - a_i^T y \leq 0$ where $<$ holds for at least one $i = 1, 2$. Thus we have to prove the following for $k = -1, 1$:

$$\forall a_1, a_2 \in A_k,\, a_1 \neq a_2 : \quad [(1 - a_1^T y < 0) \wedge (1 - a_2^T y \leq 0)] \Rightarrow y \notin \mathcal{Q} \quad .$$

Let $a_1, a_2 \in A_k$ and $a_1 \neq a_2$ then it is clear that both vectors differ in exactly two inputs and have one input in common, e.g. $a_1 = (1,1,1)^T, a_2 = (-1,-1,1)^T$. Thus we might assume w.l.o.g. that the a_i share the third input $(a_1)_3 = (a_2)_3$, and therefore we find $(a_1)_1 = -(a_2)_1$ and $(a_1)_2 = -(a_2)_2$. Therefore simply adding the corresponding inequalities we find that

$$[(1 - a_1^T y < 0) \wedge (1 - a_2^T y \leq 0)] \Rightarrow 2 - 2(a_1)_3(y)_3 < 0$$

Thus we have $|a_3(y)_3| > 1$ and therefore $\|y\| > 1$, which is proof to our assumption.

4.2.2 Parallel Halfspaces

Here we take a new look at some very well known example. Although trivial, it offers some new perspective on the given theorem.

Example 4.2.2. We know that $\{x \in \mathbb{R} : -1 \leq x \leq 1\} = \{x \in \mathbb{R} : x^2 \leq 1\}$. In the notion of the lemma we express this using
$\mathcal{S}_1 := \{x \in \mathbb{R} : 1+x \geq 0\}$ and $\mathcal{S}_2 := \{x \in \mathbb{R} : 1-x \geq 0\}$ and stating $\mathcal{S}_1 \cap \mathcal{S}_2 = \{x \in \mathbb{R} : (1+x)(1-x) \geq 0\}$.
Why could these inequalities be multiplied without a correction set $\mathcal{Q}$?
The answer is simple if we check that the appropriate correction set is $\mathbb{R}$ by

$$\mathbb{R} \subset \operatorname{int}_Z(\mathcal{S}_1) \cup \operatorname{int}_Z(\mathcal{S}_2) = \{x \in \mathbb{R} : x < 1\} \cup \{x \in \mathbb{R} : x > -1\} \quad .$$

As $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathbb{R}$ the theorem may be applied.

4.2.3 Removing Redundant Polynomials

Remind that in general it is not possible to reduce the total amount of polynomials by 'inventing' new $\mathfrak{p}_0 = 0$ and multiplying these to other polynomials. Whenever such is possible some of the inequalities used had been redundant already:

Example 4.2.3. Let $\mathcal{S}^0 := \{x \in \mathbb{R}^d : 0 \geq 0\}$ and $\mathcal{S}, \mathcal{Q}$ be semi-algebraic sets, then clearly $\mathcal{S} \cap \mathcal{Q} = \mathcal{S}^0 \cap \mathcal{S} \cap \mathcal{Q}$. Intending to multiply the polynomials $\mathfrak{p}_i$ defining $\mathcal{S}$ by 0 we have to check

$$\mathcal{Q} \subset \operatorname{int}_Z(\mathcal{S}) \cup \operatorname{int}_Z(\mathcal{S}^0) \cup \left(\operatorname{clos}_Z(\mathcal{S}) \cap \operatorname{clos}_Z(\mathcal{S}^0)\right) \quad .$$

By $\operatorname{clos}_Z(\mathcal{S}^0) = \mathbb{R}^d$, $\operatorname{int}_Z(\mathcal{S}^0) = \emptyset$ this restriction boils down to $\mathcal{Q} \subset \operatorname{int}_Z(\mathcal{S}) \cup \operatorname{clos}_Z(\mathcal{S}) \subset \mathcal{S}$. If this holds, then *clearly* $\mathcal{S}$ is obsolete in $\mathcal{Q} \cap \mathcal{S}$ and multiplication of the polynomials $\mathfrak{p}_i$ by 0 is possible without changing $\mathcal{S} \cap \mathcal{Q}$.

5. Target of Further Construction

In the first section, we provide a 'correction-set' $\mathcal{Q}$ which allows us to multiply the polynomials $b_i - a_i^T x$ which define a polyhedron given by $P = \{x \in \mathbb{R}^d \,:\, b_i - a_i^T x \geq 0\}$. This semi-algebraic set $\mathcal{Q}$ is the intersection of linear halfspaces induced by a support vector of a ridge of P, where we call any $(d-2)$-dimensional face of P a ridge.

Based on this description of P, in the second section we derive restrictions for further correction sets, which we construct in later chapters. Finally an inductive proof shows that the Main Theorem derives directly from the properties we postulate.

5.1 Start of Induction

Theorem 5.1.1. *Let $P = \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i = 1, \ldots, m \right\}$ be a d-dimensional polyhedron, where $m > 1$ and $d \geq 2$. Further let $c(\mathcal{F}(P))$ be any choice of support vectors for C. Then we find*

$$P = \{x \in \mathbb{R}^d \,:\, \prod_{i=1}^{m} \left(b_i - a_i^T x\right) \geq 0\} \;\cap \bigcap_{F \in \mathcal{F}_{d-2}(P)} \{x \in \mathbb{R}^d \,:\, h(c(F)) - c(F)^T x \geq 0\}. \quad (5.1)$$

Remark 5.1.2. If $m = 1$ there is nothing to multiply, and if $d < 2$, P has no ridges.

This formulation looks quite familiar to the one in the last section (see Theorem 4.1.1), but the situation is a bit different here. So this theorem is proved 'by hand', using the following lemma, more specialized found in [GH], Lemma 3.1 .

For every $y \in \mathbb{R}^d$ which which is part of the positive product of all inequalities of P but not in P, there is a ridge G of P, whose defining inequalities both are violated by y, at least one of them strictly:

Lemma 5.1.3. *Let $P = \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i = 1, \ldots, m \right\}$ be a d-dimensional polyhedron, where $d \geq 2$.*
Let further $y \in \mathbb{R}^d$ such that $y \notin P$ and

$$y \in \{x \in \mathbb{R}^d \,:\, \prod_{i=1}^{m} \left(b_i - a_i^T x\right) \;\geq\; 0\} \quad (5.2)$$

Define the facettes F_i of P by $F_i := P \cap \left\{ x \in \mathbb{R}^d \,:\, a_i^T x = b_i \right\}$, $i = 1, \ldots, m$.
Then there is a ridge $F_k \cap F_l \in \mathcal{F}_{d-2}(P)$ with

$$(b_k - a_k^T y < 0) \;\; \text{and} \;\; (b_l - a_l^T y \leq 0) \quad . \quad (5.3)$$

Remark 5.1.4. The restriction to a full-dimensional polyhedron P is necessary as otherwise any ridge of P is incident to more than two inequalities defining P. With some small amount of work this lemma can be generalized.

The proof uses the set $\mathrm{int}_Z(P) := \{x \in \mathbb{R}^d \,:\, b_i - a_i^T x > 0\}$ which – as P is full-dimensional – is non-empty and coincides with the topological interior $\mathring{P}$ of P. Further using an index-set $I \subset \{1, \ldots, m\}$ we will define a set $P^I := \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i \in I \right\}$, which we will discuss in detail in Section 9.1.

Proof: We will prove the lemma for y by showing: $\neg(5.3) \Rightarrow \neg(5.2)$. If P has no ridges, then as $\dim(P) \geq 2$ it is either a hyperplane or the intersection of two parallel halfspaces. In both cases for every $y \in \mathbb{R}^d$ from (5.2) follows directly $y \in P$, and so lemma 5.1.3 holds as it's preliminaries never are fulfilled.

Thus we might assume that P has at least one ridge, therefore $m \geq 2$. Let $y \in \mathbb{R}^d$ be any point on the outside of P, then w.l.o.g. we have $y \in \{x \in \mathbb{R}^d \,:\, b_1 - a_1^T x < 0\}$. Let I be the set

$$I := \{i \in \{2, \ldots, m\} : F_i \cap F_1 \in \mathcal{F}_{d-2}(P)\},$$

then if (5.3) does not hold for y then $b_i - a_i^T y > 0 \,\forall i \in I$. Hence to prove the lemma we need to show:

$$y \in \left\{ b_1 - a_1^T x < 0 \right\} \cap \left\{ b_i - a_i^T x > 0 \,, i \in I \right\} \quad \Rightarrow \left(\prod_{i=1}^{m} \left(b_i - a_i^T y\right) < 0 \right), \tag{5.4}$$

and therefore we claim:

$$\begin{aligned} & \left\{ b_1 - a_1^T x < 0 \right\} \cap \left\{ b_i - a_i^T x > 0 \,, i \in I \right\} \\ = \; & \left\{ b_1 - a_1^T x < 0 \right\} \cap \left\{ b_i - a_i^T x > 0 \,, i \in \{2, \ldots, m\} \right\} \quad . \end{aligned} \tag{5.5}$$

The proof for (5.4) follows directly from (5.5), as any y in the set of the right-hand-side of (5.5) must fulfill $\prod_{i=1}^m \left(b_i - a_i^T y\right) < 0$. We will prove (5.5) by contradiction using the set $P^I := \{x \in \mathbb{R}^d \,:\, b_i - a_i^T x \geq 0 \,, i \in I\}$, where $\mathrm{int}_Z(P^I) := \{x \in \mathbb{R}^d \,:\, b_i - a_i^T x > 0 \,, i \in I\}$. Assume (5.5) does not hold, then there is $k \in \{2, \ldots, m\} \setminus I$ such that:

$$\left(\mathrm{int}_Z(P^I) \cap \left\{ b_1 - a_1^T x < 0 \right\}\right) \cap \left\{ b_k - a_k^T x \leq 1 \right\} \neq \emptyset \quad . \tag{5.6}$$

From this we derive that there is y' in the set in (5.6) with $b_k - a_k^T y' < 0$ by the following: We know $\mathrm{int}_Z(P^I) \cap \{b_1 - a_1^T x < 0\}$ is an open set, containing y and thus non-empty. So there is some ε-ball $B(y, \varepsilon) \subset \mathrm{int}_Z(P^I) \cap \{b_1 - a_1^T x < 0\}$. Clearly $B(y, \varepsilon) \not\subset \{b_k - a_k^T x = 0\}$ as otherwise $\mathbb{R}^d = \mathrm{lin}(B(y, \varepsilon)) \subset \{b_k - a_k^T x = 0\}$, therefore there is $y' \in B(y, \varepsilon)$ with $a_k^T y' > b_k$.

This leads to a contradiction as follows:

$$\begin{aligned} (\quad & \exists y' \in \left(\mathrm{int}_Z(P^I) \cap \left\{ b_1 - a_1^T x < 0 \right\}\right) : \quad b_k - a_k^T y' < 0 && (5.7) \\ \text{and} \quad & \exists y'' \in \left(\mathrm{int}_Z(P^I) \cap \left\{ b_1 - a_1^T x > 0 \right\}\right) : \quad b_k - a_k^T y'' < 0 \;) && (5.8) \\ \Rightarrow \quad & \exists y''' \in \left(\mathrm{int}_Z(P^I) \cap \left\{ b_1 - a_1^T x = 0 \right\}\right) : \quad b_k - a_k^T y''' < 0 && (5.9) \\ \Rightarrow \quad & \exists y''' \in F_1 : \quad y''' \notin P && (5.10) \end{aligned}$$

Here (5.8) must hold as follows: F_k is a face of $P \subsetneq (P^I \cap \{x \in \mathbb{R}^d \,:\, b_1 - a_1^T x \geq 0\})$ and therefore there is $y'' \in \mathrm{int}_Z(P^I) \cap \{x \in \mathbb{R}^d \,:\, b_1 - a_1^T x > 0\}$ with $y'' \notin P$ which is 'cut off' by the inequality $a_k^T x \leq b_k$. Thus we have $a_k^T y'' > b_i$.

Finally (5.9) must hold for some $y''' \in [y', y'']$ due to the convexity of $P^I \cap \big\{\, b_k - a_k^T x < 0 \,\big\}$ and as the function $(b_1 - a_1^T x)$ is continuous. Therefore we conclude (5.10) from noticing that by the special definition of I we have

$$\begin{aligned} F_1 &:= \{x \in \mathbb{R}^d \,:\, a_1^T x = b_1\} \cap P \\ &= \{x \in \mathbb{R}^d \,:\, a_1^T x = b_1\} \cap \{x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \; i \in I\} \quad , \end{aligned}$$

as the polyhedron F is fully defined by its facets, which are all ridges of P contained in F.

So we have proved the lemma by receiving a contradiction to the assumption that (5.5) does not hold. $\square$

Proof of Theorem 5.1.1

Proof: Please recall that we have to prove $P = \mathcal{S}$ with

$$\mathcal{S} := \{x \in \mathbb{R}^d \,:\, \prod_{i=1}^{m} \big(b_i - a_i^T x\big) \geq 0\} \cap \bigcap_{F \in \mathcal{F}_{d-2}(P)} \{x \in \mathbb{R}^d \,:\, h(c(F)\,) - c(F)^T x \geq 0\} \quad (5.11)$$

Clearly the inclusion $P \subset \mathcal{S}$ must hold.
So assume (5.11) does not hold, then there is $y \in \mathcal{S}$ and $y \notin P$.

We have $\prod_{i=1}^{m} \big(b_i - a_i^T y\big) \;\geq\; 0$ and $y \notin P$, thus from lemma 5.1.3 we know that there are $k, l \in \{1, \ldots, m\}$ such that $G := P \cap \{x \in \mathbb{R}^d \,:\, a_k^T x = b_k,\; a_l^T x = b_l\}$ is a ridge of P and

$$(a_k^T y > b_k) \;\text{ and }\; (a_l^T y \geq b_l) \quad . \qquad (5.12)$$

Let $c(G)$ be any support vector of G then as P is full-dimensional we know that $c(G) = \lambda\, a_k + \mu\, a_l$ where $\lambda, \mu > 0$, and so we find $h(c(G)\,) = \lambda\, b_k + \mu\, b_l$.
Thus using (5.12) we find

$$c(G)^T y = \lambda\, a_k^T y + \mu\, a_l^T y > \lambda\, b_k + \mu\, b_l = h(c(G)\,)$$

and thus $h(c(G)\,) - c(G)^T y < 0$ which contradicts $y \in \mathcal{S}$. Therefore $P = \mathcal{S}$ must hold. $\square$

5.2 The Properties of Target Objects

In this section we provide the proof for the Main Theorem based on the postulated properties of objects we will construct in further sections. Please recall that our aim is to construct one single set $\mathcal{S}^F$ per face $F \in \mathcal{F}(C)$. In the previous chapter sets $\mathcal{S}^F = \{x \in \mathbb{R}^d : -c(F)^T x \geq 0\}$ where introduced for $F \in \mathcal{F}_k(C)$, $k \geq d-2$. For faces with other dimensions these sets will be different, so from now on we will always distinguish between $k \geq d-2$ and $k \leq d-3$. We claim the following:

Proposition 5.2.1 (Target of Construction). *Let $C \subset \mathbb{R}^d$, $d > 1$ be a convex pointed polyhedral cone, and let $c(\mathcal{F}(C))$ be any choice of support vectors for C.*
Further for every $F \in \mathcal{F}_{d-2}(P)$ define $\mathcal{S}^F := \left\{ x \in \mathbb{R}^d : -c(F)^T x \geq 0 \right\}$.

Then for each proper face $F \in \mathcal{F}_k(C)$ of dimension $k < d-2$ there is a semi-algebraic set $\mathcal{S}^F = \left\{ x \in \mathbb{R}^d : \mathfrak{p}_i^F(x) \geq 0 \, , \, i = 1, 2 \right\}$ with $C \subset \mathcal{S}^F$ and such that the following holds:
For every pair of proper faces $F, G \in \mathcal{F}_n(P)$ of equal dimension $0 < n \leq d-2$ we find

$$\mathcal{S}^{F \cap G} \quad \subset \quad \mathrm{int}_Z(\mathcal{S}^F) \cup \mathrm{int}_Z(\mathcal{S}^G) \cup \left(\mathrm{clos}_Z(\mathcal{S}^F) \cap \mathrm{clos}_Z(\mathcal{S}^G) \right) \quad . \tag{5.13}$$

Remark 5.2.2. Please note that clearly each two faces $F, G \in \mathcal{F}(C)$ of dimension $n > 0$ intersect in some face $F \cap G \in \mathcal{F}(C)$ of lower dimension.

5.3 Proving the Main Theorem using the Target Objects

From these postulated properties of $\mathcal{S}^F$ and with the previous theorem 5.1.1 we prove the Main Theorem for pointed cones using the following theorem. The proof for this comes through Theorem 4.1.1 which deals with the 'relation of intersection and multiplication' in semi-algebraic sets.

Remark 5.3.1. In the following theorem some of the polynomials $\mathfrak{p}_1^F$ (defined there) are set to zero, to improve compactness of notation. Notice that these $\mathfrak{p}_1^F$ are only multiplied to 'their kind'. For further information on trivial polynomials see Example 4.2.3 (p. 23).

Theorem 5.3.2. *Let $C \subset \mathbb{R}^d$, $d > 1$ be a convex, pointed, polyhedral cone. And for every proper face $G \in \mathcal{F}(C)$ let $\sup(G)$ be any support vector for G.*
Assuming that Proposition 5.2.1 holds, for any $F \in \mathcal{F}(C)$, $\dim(F) < d-2$ let $\mathfrak{p}_1^F, \mathfrak{p}_2^F$ be the polynomials postulated there.
Then the Main Theorem for Cones is proven,
as defining $\mathfrak{p}_2^F(x) := -c(F)^T x$ and $\mathfrak{p}_1^F := 0$ for every $F \in \mathcal{F}_k(C)$, $k \geq d-2$ leads to

$$C \quad = \quad \bigcap_{k=0}^{d-1} \bigcap_{F \in \mathcal{F}_l(C)} \{x \in \mathbb{R}^d : \prod_{F \in \mathcal{F}_k(C)} \mathfrak{p}_i^F(x) \geq 0, \; i = 1, 2\} \quad . \tag{5.14}$$

Proof for 5.3.2: Let C be given by $C = \{x \in \mathbb{R}^d : a_i^T x \leq 0 \;, i = 1,\ldots,m\}$ and define $\mathcal{S}^F := \{x \in \mathbb{R}^d : \mathfrak{p}_i^F \geq 0,\, i = 1,2\}$, for every $F \in \mathcal{F}(C)$.
Start of induction:
For every proper face $F \in \mathcal{F}_{d\text{-}2}(C)$ we derive from the definition of $\mathfrak{p}_1^F, \mathfrak{p}_2^F$ that $\mathcal{S}^F = \{x \in \mathbb{R}^d : -c(F)^T x \geq 0\}$.
Further we recall that for any of facet $G \in \mathcal{F}_{d\text{-}1}(C)$ the only possible normalized support vector $c(G)$ is the a_i defining G and therefore

$$\prod_{G \in \mathcal{F}_{d-1}(C)} \mathfrak{p}_2^G(x) = \prod_{i=1}^m -a_i^T x \;.$$

Thus the result of the previous Theorem 5.1.1 in the notation used here combined it with the restriction $C \subset \mathcal{S}^F$ leads to the following:

$$\begin{aligned} C &= \{x \in \mathbb{R}^d : \prod_{i=1}^m -a_i^T x \geq 0\} && \cap \bigcap_{F \in \mathcal{F}_{d\text{-}2}(C)} \mathcal{S}^F \\ &= \{x \in \mathbb{R}^d : \prod_{F \in \mathcal{F}_{d\text{-}1}(C)} \mathfrak{p}_i^F(x) \geq 0 \,;\, i = 1,2\} && \cap \bigcap_{k=0}^{d\text{-}2} \bigcap_{F \in \mathcal{F}_k(C)} \mathcal{S}^F \quad . \end{aligned} \tag{5.15}$$

Inductive step:
In order to prove the theorem we prove that for any $n \in \{1,\ldots,d-2\}$ using $S(n) := \bigcap_{k=0}^n \bigcap_{F \in \mathcal{F}_k(C)} \mathcal{S}^F$ we find

$$S(n) \;=\; \{x \in \mathbb{R}^d : \prod_{F \in \mathcal{F}_k(C)} \mathfrak{p}_i^F(x) \geq 0 \,;\, i = 1,2\} \cap S(n-1) \quad . \tag{5.16}$$

Let $n \in \{1,\ldots,d-2\}$, then for every pair $F, G \in \mathcal{F}_n(C)$ there is a face $F \cap G \in \mathcal{F}(C)$ with $\dim(F \cap G) < n$ as C is pointed. By the restrictions in (5.13) we conclude that

$$\mathcal{S}^{F \cap G} \;\subset\; \mathrm{int}_Z(\mathcal{S}^F) \cup \mathrm{int}_Z(\mathcal{S}^G) \cup \big(\mathrm{clos}_Z(\mathcal{S}^F) \cap \mathrm{clos}_Z(\mathcal{S}^G)\,\big) \tag{5.17}$$

and thus Theorem 4.1.1 directly proves (5.16). Applying the inductive argument (5.16) $(d-2)$ times to (5.15) we find

$$C \;=\; \bigcap_{k=1}^d \{x \in \mathbb{R}^d : \prod_{F \in \mathcal{F}_{d\text{-}1}(C)} \mathfrak{p}_i^F(x) \geq 0 \,;\, i = 1,2\} \;\cap\; S(0) \quad .$$

Knowing that the cone C only has one vertex v_0, namely the origin, we find that there is nothing we have to 'multiply':

$$\begin{aligned} S(0) &= \{x \in \mathbb{R}^d : \mathfrak{p}_i^{v_0}(x) \geq 0 \; i = 1,2\} \\ &= \{x \in \mathbb{R}^d : \prod_{F \in \mathcal{F}_0(C)} \mathfrak{p}_i^F(x) \geq 0 \,;\, i = 1,2\} \quad . \end{aligned}$$

This proves the theorem.

6. Approximation of Non-Conic Polyhedra by a Single Inequality

In this chapter we provide an approximation for a polyhedron of the form $P := \mathrm{conv}(V) \oplus \mathrm{lin}(U) \subset \mathbb{R}^d$ where $V = \{ v_1, \ldots, v_k \}$ and $U = \{ u_1, \ldots, u_l \}$.

With this construction given, in later chapters we construct an approximation for special cones $C^{[F]}$ representing the faces F of a pointed cone C. From these objects we will finally prove that they meet the restrictions given in Proposition 5.13, which then proves the Main Theorem.

Here we will at first introduce the lineality space of a set and discuss the measurement of distances before we finally construct a convex semi-algebraic set based on a single polynomial inequality approximating a given polytope. From this the unbound case follows directly through a simple projection.

6.1 Lineal Spaces

To deal with linear subspaces contained in polyhedra will be a common topic in this particular paper. Thus whenever we introduce a new geometrical object or an algebraic construction of such, you will find a short discussion on their lineality space.

So first of all we would like to recall the definition and some properties of the lineality space of a set:

Definition 6.1.1. Let $U \subset \mathbb{R}^d$ be any subset of $\mathbb{R}^d$. Then the *lineality space* of U is defined as

$$\mathrm{lineal}(U) := \left\{ x \in \mathbb{R}^d \,:\, y + tx \in U \,, \forall y \in U, t \in \mathbb{R} \right\} \quad . \tag{6.1}$$

For example $\mathrm{lineal}(P) = \{a_1, \ldots, a_m\}^\perp$ holds for any polyhedron $P = \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i = 1, \ldots, m \right\}$.

Directly from the definition we may derive that $\mathrm{lineal}(U)$ is a linear subspace of $\mathbb{R}^d$, and as $U = U + \mathrm{lineal}(U)$, U can be decomposed to the Minkowski sum

$$U \;=\; (U \cap \mathrm{lineal}(U)^\perp) \;\oplus\; \mathrm{lineal}(U) \quad , \tag{6.2}$$

where $U^\perp := \{x \in \mathbb{R}^d \,:\, u^T x = 0 \,\forall\, u \in U\}$. For any set U set with a non-trivial lineality space we will be working with the cross-section $U \cap \mathrm{lineal}(U)^\perp$ as a representative of U, measuring all important values in $U^\perp$.

Definition 6.1.2. For polyhedra of the form $P = \mathrm{conv}(V) + \mathrm{lin}(U)$ we will call

$$\mathrm{diam}(P \cap \mathrm{lineal}(P)^{\perp})$$

the *realive diameter* of P where $\mathrm{diam}(P')$ denotes the diameter of any polytope P'.

Further we would like to remind that the lineality space is conservative with respect to the topological interior of a set, we find if y lies in the topological interior $\overset{\circ}{S}$ of S, then we have $y + \mathrm{lineal}(S) \subset \overset{\circ}{S}$.

Finally we would like to prove two short Lemmata:

Lemma 6.1.3. *For every set $S \subset \mathbb{R}^d$, $S \neq \emptyset$ with $S = S \cap \left\{ x \in \mathbb{R}^d \,:\, c^T x \geq 0 \right\}$ we find the lineality space of S perpendicular to c, thus $c^T y = 0$ for any $y \in \mathrm{lineal}(S)$. Further we have*

$$\mathrm{lineal}(S) \quad = \quad \bigcap_{\mu \geq 0} \mathrm{lineal}\left(S \cap \{x \in \mathbb{R}^d \,:\, c^T x = \mu\}\right)$$

Proof: Assume $u \in \mathrm{lineal}(S)$. Then as $S \subset \{x \in \mathbb{R}^d \,:\, c^T x \geq 0\}$ we must have $c^T u = 0$ otherwise let $y \in S$ then for the the point $y' := y - (c^T y - 1)u/(c^T u)$ we find $c^T y' = 1 > 0$ and thus $y' \notin S$ although $y' \in S + \mathrm{lineal}(S)$.

Thus we find

$$\begin{aligned} & u \in \mathrm{lineal}\,(S) \\ \Leftrightarrow\ & \forall y \in S: \ \ y + \mathrm{lin}(u) \subset S \ \wedge\ c^T u = 0 \\ \Leftrightarrow\ & \forall y \in S: \ \ y + \mathrm{lin}(u) \subset S \cap \{x \in \mathbb{R}^d \,:\, c^T x = c^T y\} \\ \Leftrightarrow\ & \forall \mu \geq 0: \ u \in \mathrm{lineal}(S \cap \{x \in \mathbb{R}^d \,:\, c^T x = \mu\}) \quad , \end{aligned}$$

where the last equality derives from the insight that for all $y \in S$ we have $c^T y \geq 0$ and therefore $y \in \{x \in \mathbb{R}^d \,:\, c^T x = \mu\}$ for some $\mu \geq 0$. □

The next Lemma states that all cross-sections of a set perpendicular to it's (non trivial) lineality space are equal.

Lemma 6.1.4. *Let $S \subset \mathbb{R}^d$, $S \neq \emptyset$ with* $\mathrm{lineal}(S) = \mathrm{lin}(U) \oplus \mathrm{lin}(c)$, $c \in \mathbb{R}^d$, $c \neq 0$. *Then for every $\mu \in \mathbb{R}^d$ defining $H^\mu := \left\{ x \in \mathbb{R}^d \,:\, c^T x = \mu \right\}$ we obtain* $\mathrm{lineal}(S \cap H^\mu) = \mathrm{lin}(U)$.

Proof: Clearly we have $\mathrm{lineal}(S \cap H^\mu) \subset \{c\}^\perp$, since for any $u \in \mathrm{lineal} S \cap H^\mu$ from $x \in H^\mu$ and $x + u \in H^\mu$ follows $\mu = c^T x = c^T(x + u)$ and thus $c^T u = 0$. Further $\mathrm{lin}(U) \subset \mathrm{lineal}(S \cap H^\mu)$ must hold, since for $x \in S \cap H^\mu$ and $u \in \mathrm{lin}(U)$ we have due to the previous argumentation $x + u \in H^\mu$ and $x + u \in S$.

Such assume $\mathrm{lineal}(S \cap H^\mu) \neq \mathrm{lin}(U)$. Then as the lineality space is a subspace of $\mathbb{R}^d$ there is $u \in \mathrm{lin}(U)^\perp$ such that $u \in \mathrm{lineal}(S \cap H^\mu)$.

Let y be any point in S then defining $\lambda := \mu - c^T y$ we know that the point $x = y + \lambda c$ is in H^μ and in S as $c \in \mathrm{lineal}(S)$. Thus we have $u + y = u + x + \lambda c \in S$ as $u + x \in S$ by $x \in S \cap H^\mu$ and $u \in \mathrm{lineal}(S \cap H^\mu)$, and as $\lambda c \in \mathrm{lineal}(S)$. As y was chosen randomly we have $u \in \mathrm{lineal}(S) = \mathrm{lin}(U) \oplus \mathrm{lin}(c)$ which is a contradiction to the assumption $u \perp \mathrm{lin}(U)$. □

6.2 Distances

In the next section will s construct an approximation $\mathcal{K}(P, \varepsilon)$ containing P, which has a maximum distance $\varepsilon > 0$ to P. And P has a guaranteed non-zero minimum distance $\Delta > 0$ to the boundary of $\mathcal{K}(P, \varepsilon)$.

Thereby the maximum distance between two compact bodies K, K' is measured by the Hausdorff distance

$$\mathrm{dist}(K, K') \ := \ max\left\{ \max_{x\in K} \min_{y\in K'} \|x-y\| \ , \ \max_{y\in K'} \min_{x\in K} \|y-x\| \right\} \tag{6.3}$$

and the minimum distance is measured by $\mathrm{d}(K, K') := \min_{x\in K} \min_{x'\in K'} \|x - x'\|$.

Please recall that if $\mathrm{lineal}(K) = \mathrm{lineal}(K') = \mathrm{lin}(U)$ then by definition of $\mathrm{dist}(\cdot, \cdot)$ and $\mathrm{d}(\cdot, \cdot)$ we find:

$$\begin{aligned} \mathrm{dist}(K, K') &= \mathrm{dist}(K \cap U^{\perp}, K' \cap U^{\perp}) \\ \mathrm{d}(K, K') &= \mathrm{d}(K \cap U^{\perp}, K' \cap U^{\perp}) \quad . \end{aligned}$$

This holds as follows: For any $x \in K$ denote the point in K' closest to x by y. Then $x - y \in U^{\perp}$ holds as y is the point in the affine subspace $y + \mathrm{lin}(U)$ closest to x. Knowing that there is $x' \in K \cap U^{\perp}$ and $u \in \mathrm{lin}(U)$ such that $x–u = x'$, we conclude $\|x - y\| = \|x' - (y - u)\|$ where $y - u \in K'$ holds by the choice of $\mathrm{lin}(U)$ as well as $y–u \in U^{\perp}$ because $y–u = y–x + x'$.

Thus for any such pair x, y there is a pair $x' \in K \cap U^{\perp}$ and $y' \in K' \cap U^{\perp}$ with $\|x - y\| = \|x' - y'\|$, proving the assumption.

Here we will mostly use these distances indirectly:
In order to express that every point $y \in P$ has a minimum distance $\Delta < \mathrm{d}(y, x)$ to any point x in the complement of $\mathcal{K}(P, \varepsilon)$, we simply state $B(y, \Delta) \subset \mathcal{K}(P, \varepsilon)$ where $B(y, \Delta) := \left\{ x \in \mathbb{R}^d : \|y - x\| \leq \Delta \right\}$. Further we will express $\mathrm{dist}(P, \mathcal{K}(P, \varepsilon)) \leq \varepsilon$ by $\mathcal{K}(P, \varepsilon) \subset \bigcup_{y\in P} B(y, \varepsilon)$.

This is the form of notation we will need later, it is clear that the expressions are equivalent.

6.3 Approximation of Non-Conic Polyhedra

This section provides the basics for the coming chapters. Thus we will give a brief summary on it's contents in the next Theorem.

We will show that for any polyhedron $P = \mathrm{conv}(V) \oplus \mathrm{lin}(U)$, we may construct a convex body $\mathcal{K}(P, \varepsilon)$ containing P, which has a maximum distance $\varepsilon > 0$ to P. But as well any point in P has a granted non-zero minimum distance $\Delta > 0$ to the boundary of $\mathcal{K}(P, \varepsilon)$, so the approximation is not 'too perfect'. We will discuss the reason for this in chapter 7 when approximating cones.

The construction given here is a specialized form of the solution found in [GH], following an approach of [Ham63]. Further results can be found in [Fir74] and [Wei75].

The construction is generalized such that it suffices to know some bounds on the diameter of the cross-section $\mathrm{conv}(V) = P \cap \mathrm{lineal}(P)^{\perp}$.

Theorem 6.3.1. *Let* $P := \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i = 1, \ldots, m \right\} = \mathrm{conv}(V) \oplus \mathrm{lin}(U)$ *be a* d*-dimensional polyhedron with* $d > 0$ *where* $\dim(U) \leq d - 1$.
Let further $\varepsilon > 0$ *and* $0 < r \leq \mathrm{diam}(P') \leq R$.
Then there is a polynomial

$$\mathfrak{p}(x) \;=\; \sum_{i=1}^{m} \lambda_i (a_i^T x - d_i)^{2k} \qquad d_i \in \mathbb{R},\ \lambda_i > 0\,, k \in \mathbb{N} \quad , \tag{6.4}$$

such that for the convex set $\mathcal{K}(P, \varepsilon) := \left\{ x \in \mathbb{R}^d \,:\, \mathfrak{p}(x) \leq 1 \right\}$ *we find* $P \subset \mathcal{K}(P, \varepsilon)$ *and*

$$\bigcup_{y \in P} B(y, \Delta) \;\subset\; \mathcal{K}(P, \varepsilon) \;\subset\; \bigcup_{y \in P} B(y, \varepsilon) \quad , \tag{6.5}$$

where $\Delta := \frac{r}{R} \frac{\varepsilon}{(d+1)^3}$.

Remark 6.3.2. Please note that although $\mathrm{diam}(P)$ is used in the proof, knowing some bounds r, R to $\mathrm{diam}(P)$ makes it obsolete for the construction. Nevertheless if $\mathrm{diam}(P)$ is known it clearly can be used instead of r and R, the lower bound then boils down to $\Delta := \frac{\varepsilon}{(d+1)^3}$.

Further we would like to recall that the expression in (6.5) corresponds to the distances discussed in the previous section.

6.3.1 The Construction

Theorem 6.3.3 (Construction I). *Let* $P := \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i=1,\ldots,m \right\}$ $= \mathrm{conv}(V) \oplus \mathrm{lin}(U)$ *be a d-dimensional polyhedron with* $d > 0$ *where* $\dim(U) \leq d-1$. *For* $i=1,\ldots,m$ *let*

$$\mathfrak{v}_i(x) := \frac{2a_i^T x - h(a_i) + h(-a_i)}{h(a_i) + h(-a_i)} \quad . \tag{6.6}$$

Further let $\varepsilon > 0$ *and define* r, R *such that* $0 < r \leq \mathrm{diam}(\mathrm{conv}(V)\,) \leq R$.
Finally using $\delta := \frac{\varepsilon}{R(d+1)}$ *and* $k \in \mathbb{N} :\ k > m\left(2\ln(1+\frac{\delta}{1+\delta})\right)^{-1}$ *define*

$$\mathfrak{p}_\varepsilon(x) := \sum_{i=1}^{m} \frac{1}{m} \left(\frac{\mathfrak{v}_i(x)}{1+\delta} \right)^{2k} \quad . \tag{6.7}$$

Then for the convex set $\mathcal{K}(P,\varepsilon) := \left\{ x \in \mathbb{R}^d \,:\, \mathfrak{p}_\varepsilon(x) \leq 1 \right\}$ *we find* $P \subset \mathcal{K}(P,\varepsilon)$ *and*

$$\bigcup_{y \in P} B(y,\Delta) \;\subset\; \mathcal{K}(P,\varepsilon) \;\subset\; \bigcup_{y \in P} B(y,\varepsilon) \quad , \tag{6.8}$$

where $\Delta := \frac{r}{R} \frac{\varepsilon}{(d+1)^3}$.

Remark 6.3.4. Please notice that although $P = \mathrm{conv}(V) \oplus \mathrm{lin}(U)$ might be unbound, $h(a_i)$ and $h(-a_i)$ are finite as $a_i \perp \mathrm{lin}(U)$ for every $i = 1,\ldots,m$. Thus in any case $\mathfrak{v}_i$ is well defined (its denominator is non-zero by the restrictions on P).
Further the given a_i do not need to be normalized, as $\mathfrak{v}_i$ does not depend on the notation of a_i, nor do any of the other characteristics of P we use.

Proof for the *previous* Theorem 6.3.1: Clearly Theorem 6.3.3 directly proves Theorem 6.3.1, if we show that the polynomial $\mathfrak{p}_\varepsilon$ in (6.7) meets the restrictions on the polynomial $\mathfrak{p}$ in (6.4). Defining

$$\lambda_i := \frac{1}{[1+\delta]^{2k}} \frac{2^{2k}}{m\,[h(a_i)+h(-a_i)]^{2k}} \qquad d_i := \frac{h(a_i) - h(-a_i)}{2}$$

we find

$$\mathfrak{p}_\varepsilon(x) \;=\; \sum_{i=1}^{m} \lambda_i (a_i^T x - d_i)^{2k} \qquad d_i \in \mathbb{R},\ \lambda_i > 0\,, k \in \mathbb{N} \quad , \tag{6.9}$$

and so to prove the Theorem 6.3.1 it suffices to prove Theorem 6.3.3. □

The proof for Theorem 6.3.3 comes in two steps, at first we prove in a separate lemma that it holds in the special case that $P = \mathrm{conv}(V)$ is a bound polytope. Then we prove the general case $P = \mathrm{conv}(V) \oplus \mathrm{lin}(U)$.

Proof for bound Polytopes

Lemma 6.3.5. *The Theorem (6.3.3) holds when $P = \operatorname{conv}(V)$ is a bound polytope.*

Proof: Let $K_\varepsilon := \mathcal{K}(P, \varepsilon)$. Since $|\mathfrak{v}_i(x)| \leq 1$ for all $x \in P$ we certainly have $P \subset K_\varepsilon$, and as $\mathfrak{p}_\varepsilon$ is a convex polynomial clearly K_ε is a convex body.
Further $\operatorname{diam}(P) > 0$ must hold, since $\dim(P) = d > 0$ thus the restriction $0 < r \leq \operatorname{diam}(P)$ is feasible.

As the $\mathfrak{v}_i(x)$ are defined relative to P we might assume w.l.o.g. that the origin is the center of gravity of P.

Exterior restriction:
At first we prove $\operatorname{dist}(P, K_\varepsilon) \leq \varepsilon$. Let $\lambda := \varepsilon / \operatorname{diam}(P) = \delta(d+1)$ then we show $K_\varepsilon \subset P_\lambda$, where $P_\lambda := \{x \in \mathbb{R}^d \,:\, a_i^T x \leq (1+\lambda) b_i\}$. This proves the upper bound as $\operatorname{dist}(P, P_\lambda) \leq \lambda \operatorname{diam}(P) = \varepsilon$.

Let $y \notin P_\lambda$ then we may assume w.l.o.g. $a_1^T x > (1+\lambda) h(a_1)$ which implies

$$\mathfrak{v}_1(y) > 1 + 2\lambda \left(\frac{h(a_1)}{h(a_1) + h(-a_1)} \right) \;\geq\; 1 + 2\lambda \left(\frac{1}{d+1} \right) \;= 1 + 2\delta,$$

where the last inequality follows from the choice of the origin as the center of gravity of P (cf. [BF34]). Recalling $\delta > 0$ by definition and keeping the lower bound on k in mind this leads to

$$\left(\frac{\mathfrak{v}_1(y)}{1+\delta} \right)^{2k} \;>\; \left(1 + \frac{\delta}{1+\delta} \right)^{2k} \;>\; m \quad .$$

Thus $\mathfrak{p}(y) > 1$ holds, which shows $y \notin K_\varepsilon$, proving the right-hand expression in (6.8).

Interior restriction:
Let $y \in P$ and $x \notin K_\varepsilon$, then we prove now that $\mathrm{d}(y, x) \geq \Delta$.
Defining here

$$\lambda := \frac{\varepsilon}{\operatorname{diam}(P)(d+1)d} = \delta / d \quad ,$$

we show $P_\lambda \subset K_\varepsilon$. This proves the lower bound on $\mathrm{d}(y, x)$, since $x \notin P_\lambda$ must hold, which together with $P \subset P_\lambda$ implies

$$\mathrm{d}(y, x) \geq \operatorname{dist}(P, P_\lambda) \;\geq \lambda \operatorname{diam}(P)/(d+1) \;\geq \varepsilon/(d+1)^3 \quad ,$$

where the second inequality follows again as the origin was chosen as center of gravity of P (cf. [BF34]).

Let $y' \in P_\lambda$ then we have $-(1+\lambda) h(-a_i) \;\leq\; a_i^T y' \;\leq\; (1+\lambda) h(a_i)$ for all $i = 1, \ldots, m$ implying

$$-1 - 2\lambda \left(\frac{h(-a_i)}{h(a_i) + h(-a_i)} \right) \;\leq\; \mathfrak{v}_i(y') \;\leq\; 1 + 2\lambda \left(\frac{h(a_i)}{h(a_i) + h(-a_i)} \right) \quad .$$

And since the origin is contained in P by $h(-a_i), h(a_i) \geq 0$ we we conclude

$$|\mathfrak{v}_i(y')| \leq 1 + 2\lambda \left(\frac{\max\{h(-a_i), h(a_i)\}}{h(a_i)+h(-a_i)} \right) \; \leq \; 1 + 2\lambda \left(\frac{d}{2} \right) \; = \; 1 + \delta \quad .$$

Here again, the last inequality follows from the choice of the origin as the center of gravity (cf. [BF34]). Such we have

$$\mathfrak{p}(y) \leq \sum_{i=1}^{m} \frac{1}{m} \left(\frac{1+\delta}{1+\delta} \right)^{2k} = 1 \qquad \Rightarrow y \in K_\varepsilon \quad .$$

This proves $P_\lambda \subset \mathcal{K}(P, \varepsilon)$ and thus $\mathrm{d}(y, x) > \Delta$ which as x, y where chosen randomly proves the left-hand expression in (6.8).

Proof of Theorem 6.3.3

Here we will prove the general case, where $\mathrm{lineal}(P) \neq 0$ is possible.

Proof: Let P:=$\mathrm{conv}(V) \oplus \mathrm{lin}(U)$ fulfill the restrictions given in Theorem 6.3.3.

Define $n := \dim(U)$. By $\dim(U) \leq d-1$ we find $\dim(U^\perp) \geq 1$. Thus as P is full-dimensional $P' := P \cap U^\perp$ has dimension 1 or higher. Therefore P' meets the restrictions of the previous Lemma 6.3.5.

Using an appropriate linear, isometric transformation of coordinates we assume w. l. o. g. that $U^\perp = \mathrm{lin}(\{e_1, \dots, e_n\})$, where e_i denotes the i-th coordinate vector. By simply fixing the last $(d-n)$ coordinates we identify $U^\perp$ with the space $\mathbb{R}^n$. Constructing $\mathcal{K}(P', \varepsilon)$ for the polytope $P' := P \cap U^\perp$ we find that $\mathcal{K}'_\varepsilon := \mathcal{K}(P', \varepsilon) \cap U^\perp$ is a convex set, fulfilling the restrictions given in (6.5) relative to $U^\perp$, by using the previous Lemma 6.3.5.

Further constructing $\mathcal{K}(P, \varepsilon)$, all values used in the construction according to Theorem 6.3.1 are measured relative to $P \cap U^\perp$. Therefore by construction $\mathcal{K}(P, \varepsilon) = \mathcal{K}(P', \varepsilon)$ holds.

As $\{a_1, \dots, a_m\} \subset U^\perp$ we find that the polynomial $\mathfrak{p}_\varepsilon(x)$, defining $\mathcal{K}(P, \varepsilon)$, does not depend on the last $n-d$ inputs of x. Thus we have $\mathcal{K}(P, \varepsilon) = \mathcal{K}'_\varepsilon \oplus \mathrm{lin}(U)$, since for every $x \in \mathbb{R}^d$ and $u \in \mathrm{lin}(U)$ we find $x \in \mathcal{K}(P', \varepsilon) \Leftrightarrow x + u \in \mathcal{K}(P', \varepsilon)$.

Therefore $\mathrm{lineal}(\mathcal{K}(P, \varepsilon)) = \mathrm{lin}(U)$ must hold because $\mathcal{K}(P, \varepsilon) \cap ({}^\perp U) = \mathcal{K}'_\varepsilon$ is bound, since P' is bound and $\mathrm{dist}(P', \mathcal{K}'_\varepsilon) \leq \varepsilon$.

Therefore $\mathcal{K}(P, \varepsilon)$ is a convex set with lineality space $\mathrm{lin}(U)$.

Exterior restriction:
Recall from the general discussion on the lineality space, that the lineality space of $\mathcal{K}(P, \varepsilon)$ and its complement are equal.

Let $y \in P$ and denote by $x \in \mathbb{R}^d$ the point in the complement of $\mathcal{K}(P, \varepsilon)$ closest to y. Then $x + \mathrm{lin}(U) \not\subset \mathcal{K}(P, \varepsilon)$. Therefore

$$\mathrm{d}(y, x) \geq \mathrm{d}(y, x + \mathrm{lin}(U)) \geq \mathrm{d}(P, x + \mathrm{lin}(U)) > \Delta$$

where the last distance can be measured relative to $U^\perp$ since both sets share the same lineality space $\text{lin}(U)$.

Since $x + \text{lin}(U) \cap U^\perp$ is a subset of the complement of $\mathcal{K}(P, \varepsilon)$ the last strict inequality holds.

Interior restriction:
Clearly find

$$\text{dist}(\mathcal{K}(P, \varepsilon), P) = \text{dist}(\mathcal{K}(P, \varepsilon) \cap U^\perp, P \cap U^\perp)$$

as both sets P and $\mathcal{K}(P, \varepsilon)$ share the same lineality space. Using Lemma 6.3.5 we find that

$$\text{dist}(\mathcal{K}(P, \varepsilon) \cap U^\perp, P \cap U^\perp) = \text{dist}(\mathcal{K}(P', \varepsilon) \cap U^\perp, P') \leq \varepsilon. \quad \square$$

The Interior

Here we discuss the relation between the topological interior and the algebraical interior of the special semi-algebraic set $\mathcal{K}(P, \varepsilon)$ in a more generalized form, such that we can reuse the following Lemma later:

Lemma 6.3.6. *Let $\mu \geq 0$ and $\mathfrak{p} \in \mathbb{R}[X_1, \ldots, X_d]$ be of the following special form:*

$$\mathfrak{p}(x) := \textstyle\sum_{i=1}^m \lambda_i \left(a_i^T x - d_i\right)^{2k} \quad \textit{where} \quad \lambda_i > 0, \quad m, k \in \mathbb{N} \quad , \tag{6.10}$$

such that $\mathfrak{p}(x)$ is not constant and that the set $S := \left\{ x \in \mathbb{R}^d \,:\, \mathfrak{p}_(x) \leq \mu \right\}$ is non-empty. Then S has the following geometrical properties:

$$\text{lineal}(S) \;=\; \{a_1, \ldots, a_m\}^\perp \tag{6.11}$$

$$\overset{\circ}{S} \;=\; \text{int}_Z(S) \tag{6.12}$$

Proof: The proof for this is based on the fact that $\mathfrak{p}(x)$ is convex, we will have to show (6.11) in order to prove (6.12):

Proof for (6.11):
If $S \neq \emptyset$ then clearly we have $\{a_1, \ldots, a_k\}^\perp \subset \text{lineal}(S)$. If (6.11) does not hold then there is $u \in \text{lineal}(S) \setminus \{a_1, \ldots, a_k\}^\perp$. We might assume w.l.o.g. $a_1^T u \neq 0$ and thus especially $u \neq 0$.

Clearly we find if $x \in S$ then $\left|a_1^T x - d_1\right| \leq M$ where $M := (\mu/\lambda_1)^{1/(2k)} \geq 0$, otherwise we had $\lambda_1 (a_1^T x - d_1)^{2k} > \mu$ and thus $x \notin S$. Please note that $M = \mu = 0$ makes no difference here.
Let y be any point in $S \neq \emptyset$. Defining $u' := (2M+1)u/(a_1^T u)$ we find $a_1^T u' = 2M + 1$ and thus by the restrictions on $y \in S$ we find $\left|(a_1^T(y+u') - d_1)\right| \geq \left|a_1^T u'\right| - |M| = M + 1 > M$. Therefore $y + u'$ do not fulfill the given restriction, leading to the contradiction $y + u' \notin S$.

Proof for (6.12):

If $S = \mathbb{R}^d$ then certainly $\{a_1, \ldots, a_m\}^\perp = \text{lineal}(S) = \mathbb{R}^d$ and thus $a_i = \mathbf{0}$ for every $i \in \{1, \ldots, m\}$ and therefore the polynomial is constant. So assume $S \neq \mathbb{R}^d$, then $\text{lineal}S \neq \mathbb{R}^d$ and thus there is $a_i \neq 0$. So w.l.o.g. we might assume $a_1 \neq 0$.

Assume (6.12) does not hold, then as we know $\text{int}_Z(S) \subset \overset{\circ}{S}$ there is $y \in \overset{\circ}{S}$ such that $y \notin \text{int}_Z(S)$, thus $\mathfrak{p}(y) = \mu$.

As $y \in \overset{\circ}{S}$ we find $\varepsilon > 0$ such that the opposite points $y \pm \varepsilon a_1$ are in S and such $\mathfrak{p}(y \pm \varepsilon a_1) \leq \mu$. As $\mathfrak{p}$ is a convex polynomial, from $\mathfrak{p}(y) = \mu$ and $\mathfrak{p}(y \pm \varepsilon a_1) \leq \mu$ follows immediately that $\mathfrak{p}(y + ta_1) = \mu$ for every $t \in [-\varepsilon, \varepsilon]$. Thus this polynomial is constant in t and thus $\mathfrak{p}(y + ta_i) = \mu$ for every $t \in \mathbb{R}^d$.

But we know that the summand $(a_1^T(y + ta_1) - d_i)^{2k}$ in $\mathfrak{p}$ grows arbitrarily large with $t \longrightarrow \infty$, and all the other summands are positive. This leads to a contradiction. $\square$

7. Geometrical Approximation of Cones

7.1 Judging the Quality of Approximation

Before we define a basic closed semi-algebraic set which approximates some cone $C := \text{cone}(W) \oplus \text{lin}(U)$ we first introduce a measurement for the quality of such an approximation. Notice that for a given polyhedron $P = \text{conv}(V) \oplus \text{lin}(U)$ we constructed a semi-algebraic set $\mathcal{K}(P, \varepsilon)$ where we could control the distance $\text{dist}(P, \mathcal{K}(P, \varepsilon))$. We have

$$\text{dist}(P, \mathcal{K}(P, \varepsilon)) \leq \varepsilon \quad \text{which is equivalent to} \quad \mathcal{K}(P, \varepsilon) \subset \bigcup_{y \in P} B(y, \varepsilon) \quad ,$$

where the size of ε measures the quality of approximation. So there are two ways of expressing this quality for $\mathcal{K}(P, \varepsilon)$.
As $P \subset \mathcal{K}(P, \varepsilon)$ there is another way of putting this, namely expressing that the points in $\mathcal{K}(P, \varepsilon)$ are not 'too far' from P:

$$x \in \mathcal{K}(P, \varepsilon) \Longrightarrow \text{d}(P, x) \leq \varepsilon \quad .$$

To reduce complexity assume here, that the given objects are convex polyhedra of the form $P = \text{conv}(V)$ or $C = \text{cone}(W)$, where C is pointed.
Then the approximation $\mathcal{S}(C, \alpha)$ of such a cone C we construct is a *cone* with the *same apex* as C, such that the rays of $\mathcal{S}(C, \alpha)$ are not 'too far' from the rays of C. Here the 'distance' of two rays is measured by their angle. In brief we want:

$$v \in \mathcal{S}(C, \alpha) \text{ and } v \neq \mathbf{0} \quad \Longleftrightarrow \quad \exists w \in C,\, w \neq \mathbf{0} : \ \angle(v, w) \leq \alpha \quad .$$

Notice that the apex of C is an exceptional object here, and in further constructions will need to be viewed separately.
For the further work we use a description of the 'distance' between $\mathcal{S}(C, \alpha)$ and C which is alike $P \subset \bigcup_{x \in P} B(x, \varepsilon)$.

7.2 Basic Objects

To approximate polyhedral cones $C = \text{conv}(W) \oplus \text{lin}(U)$ we introduce new geometrical objects the approximation is based on. These will not be balls $B(y, \varepsilon)$ with a maximal distance to a point y, but cones $\text{K}(c, \alpha)$ with a maximal angle to a ray:

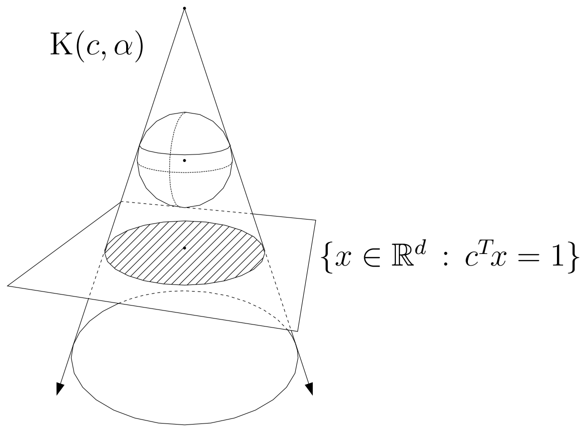

Fig. 7.1: The cone $\mathrm{K}(c,\alpha)$ has a spherical cross-section.

Definition 7.2.1. Let $c \in \mathbb{R}^d$, $c \neq 0$ and $\alpha \in [0, \pi/2)$. Then we define a cone with opening angle α along the ray $\mathrm{ray}(c) := \{\lambda\, c\,, \lambda \geq 0\}$ by

$$\mathrm{K}(c,\alpha) := \left\{\, x \in \mathbb{R}^d \,:\, \angle(x,c) \leq \alpha \,\right\} \quad . \tag{7.1}$$

Corollary 7.2.2. *By the previous definition we conclude using basic trigonometry, that if* $\|c\| = 1$*:*

$$\begin{aligned} \mathrm{K}(c,\alpha) &= \operatorname{cone}\left(B(c, \tan(\alpha)) \cap \{x \in \mathbb{R}^d \,:\, c^T x = 1\}\right) \\ &= \bigcup_{y \in \mathrm{ray}(c)} B(y, sin(\alpha)\, \|y\|) \end{aligned} \tag{7.2}$$

Thus $\mathrm{K}(c,\alpha)$ has a spherical cross-section with radius $\tan(\alpha)$ at $\{x \in \mathbb{R}^d \,:\, c^T x = 1\}$ and each point of the ray $\{\lambda c \,|\, \lambda \geq 0\}$ has a non-zero distance to $\mathrm{K}(c,\alpha)^{\mathrm{c}}$ except for the origin (see Figure 7.2)

7.2.1 Angles expressed by Distances

Assume again here $P = \operatorname{conv}(V)$ or $C = \operatorname{cone}(W)$, where C is pointed.
The semi-algebraic set $\mathcal{K}(P,\varepsilon)$ constructed in Lemma 6.3.5 (p. 36) also has a minimum distance Δ for every $y \in P$ to the boundary of $\mathcal{K}(P,\varepsilon)$ which we expressed $\bigcup_{x \in P} B(x,\delta) \subset \mathcal{K}(P,\varepsilon)$. Using $\mathcal{K}(P,\varepsilon)$ we will construct a similar object for cones C, a conic semi-algebraic set $\mathcal{S}(C,\alpha)$ with $C \subset \mathcal{S}(C,\alpha)$ and

$$\bigcup_{w \in C} \mathrm{K}(w,\beta) \subset \mathcal{S}(C,\alpha) \subset \bigcup_{w \in C} \mathrm{K}(w,\alpha) \quad , \tag{7.3}$$

for some fixed $0 < \beta \leq \alpha$. Later we will express this by

$$\bigcup_{w \in C} B(w,\Delta(w)) \subset \mathcal{S}(C,\alpha) \subset \bigcup_{w \in C} B(w,\Delta(w)) \quad , \tag{7.4}$$

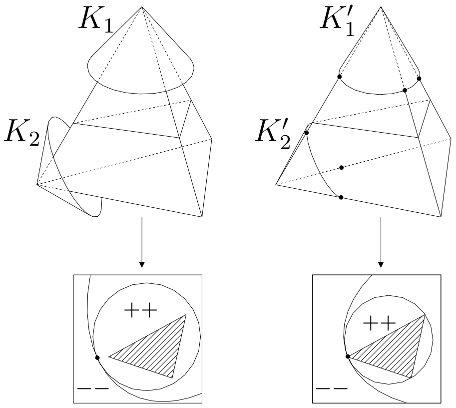

Fig. 7.2: The Cones must not be close to the Object

where $\Delta(y) := \sin(\beta)\,\|w\|$ and $\varepsilon(y) := \sin(\alpha)\,\|w\|$ using the basic properties of the sets $\mathrm{K}(w,\cdot)$.

Equation (7.4) implies the following: Judging C and $\mathcal{S}(C,\alpha)$ as a collection of rays implies that the rays of $\mathcal{S}(C,\alpha)$ are neither 'too far' from nor 'too close' to the rays of C, where 'distance' is measured by the angle.

The reason why we insist on the approximation being not too close is motivated by Figure 7.2.1. There we see on the left-hand side two cones K_1, K_2 with some 'safety distance' to the hull of a tetrahedron P. The cross-section below this shows, that the area where both the polynomials defining K_1 and K_2 are negative is 'far' from P.

On the right-hand side we see that the cones K_1', K_2' intersect with the hull of the polyhedron elsewhere as in the apex, and thus the set where both the polynomials defining K_1' and K_2' are negative is connected to P.

7.2.2 An Idea for the Approximation of Cones by Semi Algebraic Sets

The relations between C and $\mathcal{S}(C,\alpha)$ are quite similar to the relations of P and $\mathcal{K}(P,\varepsilon)$. Therefore, a way of constructing $\mathcal{S}(C,\alpha)$ based on the construction of some $\mathcal{K}(P,\varepsilon)$, is to use a non-conic representative of C which comes in by the construction of a vertex figure as found in [Zie98] (see Figure 7.2.2).

We define a reference hyperplane H such that $C\cap H$ as well as $\mathrm{K}(w,\alpha)\cap H$ where $w\in C\setminus\{\mathbf{0}\}$ are good representatives for C resp. $\mathrm{K}(w,\alpha)$. Given a normalized support vector $c\in\mathbb{R}^d$ of the apex of C, H is defined by $H := \left\{\, x\in\mathbb{R}^d \,:\, c^T x = -1 \,\right\}$ clearly we have $C = \mathrm{cone}(C\cap H)$.
With α chosen carefully $\mathrm{K}(w,\alpha) = \mathrm{cone}(\,\mathrm{K}(w,\alpha)\cap H\,)$ for every $w\in C$. As C is pointed, the polyhedron $P := C\cap H$ is of the form $P = \mathrm{conv}(V)$ and thus we build $\mathcal{K}(P\cap H^c,\varepsilon)$, which is an ε-approximation of the cross-section $C\cap H$. This $\mathcal{K}(P\cap H^c,\varepsilon)$ then is 'conified' in just the same way as the cylinder $\{x\in\mathbb{R}^3 \,:\, 1-x_1^2-x_2^2\geq 0\}$ becomes the double-cone $\{x\in\mathbb{R}^3 \,:\, x_3^2-x_1^2-x_2^2\geq 0\}$. Keeping the chosen support vector we find that the inequality

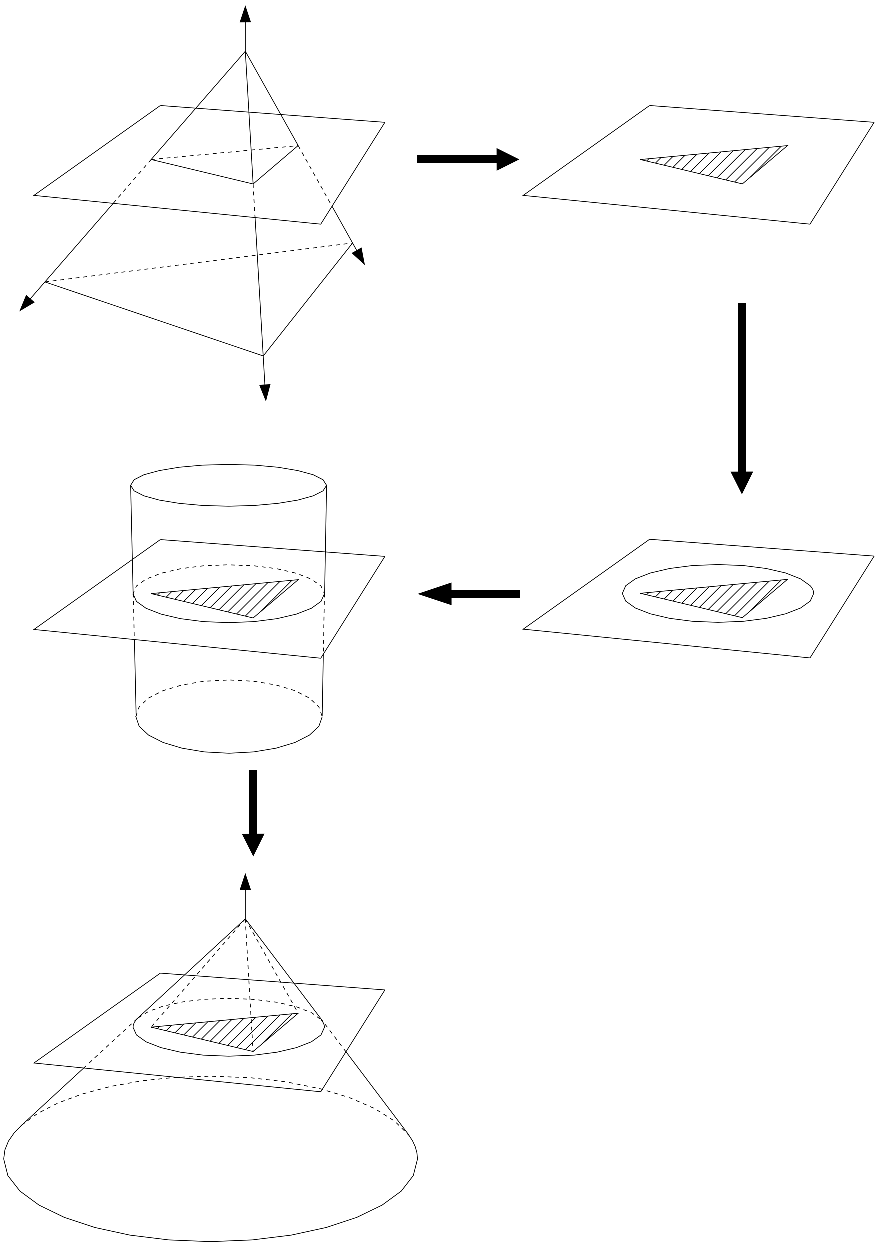

Fig. 7.3: The Approximation of a Polyhedral Cone by a Semi-Algebraic Set

$c^T x \leq 0$ cuts the double-cone to a conic approximation of C.

While for P there is no upper bound on ε when building $\bigcup_{y \in P} B(y, \varepsilon)$, the geometrical properties of C must be considered to ensure that the approximation still is a cone, specifically for all $w \in C \setminus \{\mathbf{0}\}$ we must have $\angle(c, w) + \alpha < \pi/2$.
Further to prove that the rays in the bound of $\mathcal{S}(C, \alpha)$ are 'far enough' from C, we will have to restrict α in a very special way, introduced in the next sections.

Finally the lineality space $\text{lineal}(C)$ of each cone C we will finally be working on needs to be considered in contrast to the construction of $\mathcal{K}(P, \varepsilon)$. Since C geometrically is represented by $C \cap \text{lineal}(C)^{\perp}$, which is pointed if C is not a hyper-plane, the final representative of C we will be working on is $C \cap H^c \cap \text{lineal}(C)^{\perp}$.

7.3 Basic Properties of Cones

7.3.1 Intersection of a Hyperplane with a Cone

Clearly we know that the intersection of a plane $H^c = \{x \in \mathbb{R}^d \,:\, c^T x = 1\}$ with a cone $K = \mathrm{K}(w, \alpha)$ with spherical cross-section results in a paraboloid if the angle α is chosen such that $\angle(w, c) + \alpha < \pi/2$. For further constructions we need an upper and a lower bound on the diameter of this intersection.

There is a huge amount of literature on intersections of such cones K with hyper-planes, but unfortunately the description is always such that the foot point c of the hyperplane is contained in C. Thus here a short lemma on intersection of cones.
The proof is purely analytical and thus we present it in the appendix, section 15.1, page 95. Here we will only check that all mentioned objects are well defined.

Lemma 7.3.1. *Let $H^c := \left\{\, x \in \mathbb{R}^d \,:\, c^T x = 1 \,\right\}$ be a hyperplane defined by $c \in \mathbb{R}^d$, $\|c\| = 1$. Let further $y \in H^c$ and define $\gamma := \angle(c, y)$.
Then if we restrict α by $0 \leq \alpha \leq (\pi/4 - \gamma/2)$ we find*

$$H^c \cap \mathrm{K}(y, \beta) \;\subset\; H^c \cap B(y, \sin(\alpha)) \;\subset\; H^c \cap \mathrm{K}(y, \alpha) \tag{7.5}$$

*where $\beta \in [0, \alpha]$ is defined by $sin(\beta) := \sin(\alpha)\ \left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right)$.
Further any ray of $\mathrm{K}(y, \alpha)$ intersects H^c, we have*

$$\mathrm{K}(y, \alpha) = \mathrm{cone}(H^c \cap \mathrm{K}(y, \alpha)\,) \quad , \tag{7.6}$$

where for β holds accordingly the same as $\beta \leq \alpha$.

Remark 7.3.2. The reason why we define β implicitly by $\sin(\beta)$, is that for our further constructions we will always need $\sin(\beta)$, never just the 'pure' β.
Further we would like to mention that later when approximating cones by semi-algebraic sets, the term $\left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right)$ will find entry into an important constant for this construction.

All objects are well defined

As $y \in H^c$ we know that $\angle(c, y) < \pi/2$, thus we have $\pi/4 - \gamma/2 > 0$ and thus the restrictions on α are feasible.
We find $0 \leq \alpha \leq \pi/4$ and thus $0 \leq \sin(\alpha)\ (1/2)\cos^2(\gamma)\cos(\alpha) \leq \sin(\alpha)$.
Therefore β is well defined and we have $0 \leq \beta \leq \alpha < \pi/4$ as the function sin is monotonous on $[0, \pi/4]$.
Further for any $v \in \mathrm{K}(y, \alpha)$, $v \neq \mathbf{0}$ we find $\angle(c, v) \leq \angle(c, y) + \angle(v, y) \leq \gamma + \alpha < \pi/2$ and thus $\mathrm{ray}(v)$ must intersect H^c. The same clearly holds for β as $0 \leq \beta \leq \alpha$, which proves (7.6).

7.3.2 Angle of a Support Vector to the Extremal Rays

Motivated by the previous lemma, for a pointed cone $C = \text{cone}(W)$ with a support vector $\bar{c}$ of its apex, the angle $\gamma = \max\{\angle(-\bar{c}, v) :\ v \neq 0,\ v \in C\}$ is important, as for any point $w \in \text{cone}(W)$, $w \neq \mathbf{0}$, the angle $\alpha = \pi/4 - \gamma/2$ fulfills the restrictions imposed there.

We define this special angle relative to $\bar{c} \in \mathbb{R}^d$ and a linear subset of $\mathbb{R}^d$ to improve usability on cross-sections of cones containing linear subspaces.

Definition 7.3.3. Let $C \subset \text{lin}(U) \subset \mathbb{R}^d$ be a pointed polyhedral cone contained in some linear subspace of $\mathbb{R}^d$, with $\dim(C) = \dim(\text{lin}(U)) > 1$. Further let $\bar{c} \in \text{lin}(U)$ be a support vector of the apex of C.
Then by

$$\angle_{max}(C, -\bar{c}) := \max\{\angle(-\bar{c}, v) :\ v \neq 0,\ v \in C\}$$

we define the *Maximal Relative Angle* of C, where 'relative' means relative to $-\bar{c}$.

Corollary 7.3.4. *Let C be a polyhedral cone and $\bar{c} \in \mathbb{R}^d$ meeting the restrictions of Definition 7.3.3. Then $\angle_{max}(C, -\bar{c})$ is well defined, and $0 < \angle_{max}(C, -\bar{c}) \leq \pi/2$.*

Proof: Clearly by choice of $\bar{c}$ as a support vector $\angle(w, \text{cone}(W)) < \pi/2$ holds for any $w \in \text{cone}(W)$. To prove the lower bound let $W := \{ w_1,\ldots, w_n \} \subset \mathbb{R}^d$ be such that the description $C = \text{cone}(W)$ is irreducible. Then from $\dim(C) > 1$ follows $n \geq 2$ and thus for $w_1 \neq w_2$ we have $w_i \neq \bar{c}$ for at least one $i = 1, 2$. Therefore $0 < \max(\angle(w_1, -\bar{c}), \angle(w_2, -\bar{c}))$. □

In the following lemma we prove the relative maximum angle is assumed in the extremal rays of C.

Lemma 7.3.5. *Let $C = \text{cone}(W) \subset \mathbb{R}^d$, $d > 0$ be a pointed polyhedral cone where $W = \{ w_1,\ldots, w_n \}$ and $\bar{c} \in \mathbb{R}^d$ is a normalized support vector the apex of C. Then*

$$\angle_{max}(C, -\bar{c}) \quad = \quad \max\{\angle(-\bar{c}, w) :\ w \neq 0,\ w \in W\} \quad . \tag{7.7}$$

Proof: Let $w_j \in W$ such that $\bar{c}^T w_j = \min\{\bar{c}^T w : w \in W \backslash W^F\}$ and let $w \in \text{cone}(W)$, $w \neq \mathbf{0}$ we show $\angle(\bar{c}, w) \leq \angle(\bar{c}, w_j)$. By definition of C as a convex cone, there is a representant $w' \in \text{conv}(W)$ (rem.: conv, not cone) with $w = \lambda\, w'$, $\lambda > 0$, implying $\angle(\bar{c}, w) = \angle(\bar{c}, w')$. This point w' is by definition given as $w' = \sum_{i=1}^k \lambda_i\, w_i$ with $\lambda_i \geq 0$ and $1 = \sum_{i=1}^k \lambda_i$. Therefore $\|w'\| \leq 1$ must hold and since $\text{cone}(W)$ is pointed the origin can not be in $\text{conv}(W)$, thus $\|w'\| \neq 0$. Using this we obtain

$$\cos(\angle(\bar{c}, w')) = \frac{\bar{c}^T w'}{\|w'\|} \geq \frac{\sum_{i=1}^k \lambda_i\, \bar{c}^T w_i}{1} \geq \bar{c}^T w_j = \cos(\angle(\bar{c}, w_j)) \quad .$$

By choice of $\bar{c}$ as a support vector of the vertex of C we must have
$\angle(\bar{c}, w), \angle(\bar{c}, w_j) \in [0, \pi/2)$ which leads to $\angle(\bar{c}, y) \leq \angle(\bar{c}, w_j)$ because the function cos is monotonous in this area. This directly proves (7.7). □

8. Approximation of Cones by a Semi-Algebraic Set

In this chapter we provide a construction for the approximation of cones by a semi-algebraic set which is defined by two polynomial inequalities. For this construction we define a representant for a cone similar to the known apex figure (cf. [Zie98]). According to the ideas brought forward in the previous chapter we will then build an approximation to the apex figure, which we 'lift' to an approximation $\mathcal{S}(C,\alpha)$ of the cone. Finally we discuss the topological interior of the constructed set.

8.1 Main Result

Theorem 8.1.1. *Let $C := \operatorname{cone}(W) \oplus \operatorname{lin}(U) \subset \mathbb{R}^d$ be a $d-$dimensional polyhedral cone where $\operatorname{cone}(W)$ is pointed and $\dim(U) < d-1$. Let further $c \in \mathbb{R}^d$ be a normalized support vector of the apex $\operatorname{lin}(U)$ of C.*
Define $P := \operatorname{cone}(W) \cap \{x \in \mathbb{R}^d \,:\, c^T x = -1\}$ and let $0 < r \leq \operatorname{diam}(P) \leq R$.
Restrict α to $\alpha \in \left(0, \frac{1}{2}(\frac{\pi}{2} - \gamma)\right]$ where $\gamma := \max\{\angle(-c, w) : w \in W \setminus \{\mathbf{0}\}\}$.
Then we are able to construct a semi-algebraic set

$$S(C,\alpha) := \left\{\, x \in \mathbb{R}^d \,:\, \mathfrak{p}_\alpha(x) \geq 0,\ c^T x \geq 0 \,\right\} \tag{8.1}$$

which is a convex cone containing C, with

$$\operatorname{int}_Z(\mathcal{S}(C,\alpha)) = \overset{\circ}{\mathcal{S}}(C,\alpha) \tag{8.2}$$

$$\operatorname{clos}_Z(\mathcal{S}(C,\alpha)) = \operatorname{lin}(U) \tag{8.3}$$

Further $\mathcal{S}(C,\alpha)$ fulfills the following containment restrictions:

$$\bigcup_{y \in C} B(y, \Delta(y)) \subset \mathcal{S}(C,\alpha) \subset \bigcup_{y \in C} B(y, \varepsilon(y)) \tag{8.4}$$

where $\varepsilon(y) := \sin(\alpha)\mathrm{d}(y, \operatorname{lin}(U))$ and

$$\Delta(y) := \sin(\alpha)\mathrm{d}(y, \operatorname{lin}(U)) \left(\frac{r}{R}\frac{1}{(d+1)^3}\right)\left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right)$$

The proof for this theorem is split into two parts, first the set $\mathcal{S}(C,\alpha)$ is constructed and the properties in (8.4) are proven. Then we show (8.2).

We will denote the properties in (8.4) 'containment restrictions' calling the inclusion $\bigcup_{y \in C} B(y, \Delta(y)) \subset \mathcal{S}(C,\alpha)$ the 'interior containment restriction' and accordingly the other inclusion the 'exterior containment restriction'.

8.2 The Apex Figure

Recalling the ideas stretched out in the chapter on geometric approximation of cones (see figure 7.2.2, page 44), we recall that we intend to build a non-conic representative for a given cone C by intersecting C with a standard hyperplane defined by a support vector of the apex of C.

The difference to well-known vertex figures is that we allow the cone to be non-pointed and in order to simplify later discussions we will use a *full-dimensional* polyhedron P as representative described by the following:

Lemma 8.2.1. *Let* $C := \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i=1,\ldots,m \right\} = \operatorname{cone}(W) \oplus \operatorname{lin}(U)$ *be a d-dimensional cone, where* $\operatorname{cone}(W)$ *is pointed and* $\dim(U) \leq d-2$.
Let further $\bar{c} \in \mathbb{R}^d$ *be a normalized support vector of the apex of* C*, define* $H^{\bar{c}} := H(\bar{c}, -1)$ *and* $V := \{-w/(c^T w) :\; w \in W\}$.

Then the following definitions for an apex figure P *of* C *are equal:*

$$\begin{aligned} P &:= (C \cap H^{\bar{c}}) \oplus \operatorname{lin}(\{\bar{c}\}) && (8.5)\\ &= \operatorname{conv}(V) \oplus \operatorname{lin}(U) \oplus \operatorname{lin}(\{\bar{c}\}) && (8.6)\\ &= \left\{ x \in \mathbb{R}^d \,:\, \bar{a}_i^T x \leq \bar{c}^T a_i \,, i=1,\ldots,m \right\}, \quad \text{where } \bar{a}_i := a_i - (\bar{c}^T a)\bar{c}, && (8.7) \end{aligned}$$

where the description in (8.7) is irreducible. Further the relative diameter of P*, given by*

$$\operatorname{diam}(P \cap \operatorname{lineal}(P)^{\perp}) = \operatorname{diam}(\operatorname{cone}(W) \cap H^{\bar{c}}) \quad , \qquad (8.8)$$

is finite as well as $h_P(\bar{a}_i)$ *and* $h_P(-\bar{a}_i)$ *are finite for every* $i \in \{1, \ldots, m\}$.

Proof: The full proof for the theorem is found in the appendix, Section 15.2 (pg. 101) and derived from a discussion on vertex figures in [Zie98].

Remark 8.2.2. In brief the last statements on P show that it fits the requirements for the construction of $\mathcal{K}(P, \varepsilon)$.

While (8.5) and (8.6) are clearly equal and (8.8) follows directly from (8.5), it is necessary to prove that (8.5) and (8.7) are equal, for which we provide a brief sketch of the proof here.

Let $P' := (C \cap H^{\bar{c}}) \oplus \operatorname{lin}(\{\bar{c}\})$ and $P'' := \left\{ x \in \mathbb{R}^d \,:\, \bar{a}_i^T x \leq \bar{c}^T a_i \,, i=1,\ldots,m \right\}$.

Then regarding the given description of C as intersection of halfspaces it is easy to check that $P'' \cap H^{\bar{c}} = C \cap H^{\bar{c}}$ using the definition of $\bar{a}_i$ and $H^{\bar{c}} = \left\{ x \in \mathbb{R}^d \,:\, c^T x = -1 \right\}$.

From this $P' = P''$ can be concluded, since by definition of $\bar{a}_i$ as projections to $\{\bar{c}\}^{\perp}$ we should find $\operatorname{lineal}(P'') = \operatorname{lineal}(C) \oplus \operatorname{lin}(\bar{c}) = \operatorname{lineal}(P)$. And thus $P' = P''$ must hold since they share the same cross-section $C \cap H^{\bar{c}}$, perpendicular to their equal lineality spaces.

8.3 Construction

In the following we construct a semi-algebraic set approximating a polyhedral cone, just as proposed in the previous chapter, where we discussed the basic ideas for this construction in detail. For a short glimpse of the geometrical properties of this cone see figure 7.2.2 (p.44).

Theorem 8.3.1 (Construction). *Let* $C := \left\{ x \in \mathbb{R}^d \ : \ a_i^T x \leq 0 \ , i = 1, \ldots, m \right\} = \operatorname{cone}(W) \oplus \operatorname{lin}(U)$ *be a* $d-$*dimensional polyhedral cone , where* $\operatorname{cone}(W)$ *is pointed and* $\dim(U) \leq d-2$. *Let further* $c \in \mathbb{R}^d$ *be a support vector of the apex* $\operatorname{lin}(U)$ *of* C.
Then construct a semi-algebraic set $\mathcal{S}(C, \alpha)$ *by the following steps:*

Define $\gamma := \max\{\angle(-c, w) : \ w \neq \mathbf{0}, \ w \in \operatorname{cone}(W)\}$.

Choose $\alpha \in \left(0, \frac{1}{2}(\frac{\pi}{2} - \gamma)\right]$ *and define* $\varepsilon := \sin(\alpha)$.

Define r, R *such that* $0 < r \leq \operatorname{diam}(\operatorname{cone}(W) \cap H(c, -1)) \leq R$.

Define $P := \left\{ x \in \mathbb{R}^d \ : \ \bar{a}_i^T x \leq c^T a_i \ , i = 1, \ldots, m \right\}$, *where* $\bar{a}_i := a_i - (c^T a) c$.

Then finally using r, R, ε *and* P *construct* $\mathcal{K}(P, \varepsilon) = \{x \in \mathbb{R}^d \ : \ \mathfrak{p}_\varepsilon(x) \geq 0\}$ *as done in Theorem 6.3.1, such that*

$$\mathfrak{p}_\varepsilon(x) \ := \ 1 - \sum_{i=1}^{m} \lambda_i \left(\bar{a}_i^T x - d_i\right)^{2k} \quad \text{where } \lambda_i > 0 \text{ and } k \in \mathbb{N} \quad .$$

Then the semi-algebraic set $S(C, \alpha) := \left\{ x \in \mathbb{R}^d \ : \ \mathfrak{p}_\alpha(x) \geq 0 \ \wedge \ c^T x \leq 0 \right\}$ *where*

$$\mathfrak{p}_\alpha(x) := (c^T x)^{2k} - \sum_{i=1}^{m} \lambda_i \left(\bar{a}_i^T x + (c^T x) d_i\right)^{2k}$$

is a convex cone holding the following containment restrictions:

$$\bigcup_{y \in C} B(y, \Delta(y)) \ \subset \ \mathcal{S}(C, \alpha) \ \subset \ \bigcup_{y \in C} B(y, \varepsilon(y)) \tag{8.9}$$

where $\varepsilon(y) := \sin(\alpha) \mathrm{d}(y, \operatorname{lin}(U))$ *and*

$$\Delta(y) := \sin(\alpha) \mathrm{d}(y, \operatorname{lin}(U)) \left(\frac{r}{R} \frac{1}{(d+1)^3} \right) \left(\frac{1}{2} \cos^2(\gamma) \cos(\alpha) \right).$$

Remark 8.3.2. Please note the sign change from "$-d_i$" in $\mathfrak{p}_\varepsilon$ to "$+(c^T x) d_i$" in $\mathfrak{p}_\alpha$, this guarantees that $\mathfrak{p}_\varepsilon(y) = \mathfrak{p}_\alpha(y)$ whenever $c^T y = -1$.

Remark 8.3.3. We recall that from Lemma 8.2.1 we know that P meets the restrictions on building $\mathcal{K}(P,\varepsilon)$. There we proved as well that $\operatorname{diam}(P \cap \operatorname{lineal}(P)^{\perp}) = \operatorname{diam}(\operatorname{cone}(W) \cap H^c)$ and thus the bounds r, R are correctly chosen in order to construct $\mathcal{S}(C,\alpha)$.
Further we remind that for the construction of $\mathcal{K}(P,\varepsilon)$ the $\bar{a}_i$ used to describe P do not need to be normalized. Finally we find that γ is well defined as clearly $\gamma = \max\{\angle(-c,w) : w \in \operatorname{cone}(W) \ \|w\| = 1\}$ and thus well defined as the maximum of a smooth function on a compact set.

We will prove this theorem in three steps, one lemma each:
At first we will prove the theorem in case that C is pointed, where in two separate lemma the containment restrictions (8.9) are proven. Then we discuss the situation $\operatorname{lin}(U) \neq \{\mathbf{0}\}$.

In the following we will shorten sets alike $\{x \in \mathbb{R}^d : c^T x = \mu\}$ to $H(c,\mu)$ in order to abbreviate the notation.

Definition 8.3.4. Let $C \subset \mathbb{R}^d$ be a polyhedral cone and $c \in \mathbb{R}^d$ a support vector of its apex. Let α, r, R fulfill the restrictions given in Theorem 8.3.1.
Then we call the set $\{C, c, \alpha, r, R\}$ feasible for construction.

8.3.1 Basic Properties

Lemma 8.3.5. *Let $C \subset \mathbb{R}^d$ be a pointed d-dimensional cone and c a support vector of its apex, further let $\alpha, r, R \in \mathbb{R}$ such that $\{C, c, \alpha, r, R\}$ is feasible for construction.*
Then the set $\mathcal{S}(C,\alpha)$ constructed in Theorem 8.3.1 using c, α, r and R is a convex cone with lineality space $\{\mathbf{0}\}$.

Proof: First of all clearly $S(C,\alpha)$ is a (possibly non-convex) cone: Let $y \in \mathcal{S}(C,\alpha)$ then for any $y' = \lambda\, y$, $\lambda \geq 0$ we find $c^T y' = \lambda\,(c^T y) \leq 0$ and $\mathfrak{p}_\alpha(y') = (\lambda)^{2k} \mathfrak{p}_\alpha(y) \geq 0$ which shows $y' \in \mathcal{S}(C,\alpha)$.

Lineality space: We find

$$\begin{array}{rcl} \mathcal{S}(C,\alpha) \cap H(c,0) & = & \{x \in \mathbb{R}^d : -\sum_{i=0}^m \lambda_i (\bar{a}_i^T x)^{2k} \geq 0,\ c^T x = 0\} \\ & = & \{x \in \mathbb{R}^d : \bar{a}_i^T x = 0,\ c^T x = 0\} = \{\mathbf{0}\} \quad , \end{array} \tag{8.10}$$

where the last equality holds as c is a support vector of the apex of C, which is $\mathbf{0}$.

Let $u \in \operatorname{lineal}(\mathcal{S}(C,\alpha))$, then if $c^T u \neq 0$, with $\mathbf{0} \in \mathcal{S}(C,\alpha)$ both $\mathbf{0} \pm u$ must be in $\mathcal{S}(C,\alpha)$, which is a subset of $\{x \in \mathbb{R}^d : c^T x \leq 0\}$. This leads to $\pm c^T u \leq 0$ which implies the contradiction $c^T u = 0$. Thus $c^T u = 0$, from which we conclude $u \in \operatorname{lineal}(\mathcal{S}(C,\alpha) \cap H(c,0)) = \{\mathbf{0}\}$. Thus $\operatorname{lineal}(\mathcal{S}(C,\alpha)) = \{\mathbf{0}\}$.

Convexity: Defining $P := C \cap H(c,-1)$ as done in Theorem 8.3.1 we derive

$$\mathcal{S}(C,\alpha) \cap H(c,-1) \;=\; \mathcal{K}(P,\varepsilon) \cap H(c,-1). \tag{8.11}$$

We show that $\mathcal{S}(C,\alpha)$ has the form

$$\mathcal{S}(C,\alpha) = \operatorname{cone}(\ \mathcal{S}(C,\alpha) \cap H(c,-1)\ ) \tag{8.12}$$

and thus is convex, as $\mathcal{S}(C,\alpha) \cap H(c,-1) = \mathcal{K}(P,\varepsilon) \cap H(c,-1)$ is convex due to the convexity of $\mathcal{K}(P,\varepsilon)$.

So all we need to show is that any ray of the cone $\mathcal{S}(C,\alpha)$ intersects $H(c,-1)$. Assume the contrary. Then we find $w \in \mathcal{S}(C,\alpha)$, $w \neq \mathbf{0}$ such that $\operatorname{ray}(w) \cap \{x \in \mathbb{R}^d : c^T x = -1\} = \emptyset$ which is equal to $c^T w \geq 0$. Therefore as

$$w \in \mathcal{S}(C,\alpha) \subset \{x \in \mathbb{R}^d : -c^T x \leq 0\}$$

we find $c^T w = 0$. With the use of (8.10) this leads to the contradiction $w = \mathbf{0}$ where $w \neq \mathbf{0}$ should hold. So we proved (8.12) and therefore $\mathcal{S}(C,\alpha)$ is convex. □

8.3.2 Interior Restriction

In case the cone C is pointed clearly $\operatorname{lineal}(C) = \{\mathbf{0}\}$ and thus the term $\mathrm{d}(y, \operatorname{lin}(U))$ in the containment restriction (8.9) of 8.3.1 changes to $\mathrm{d}(y,\mathbf{0}) = \|y\|$.

Lemma 8.3.6. *Let $C = \operatorname{cone}(W) \subset \mathbb{R}^d$ be a pointed d-dimensional cone and c a support vector of its apex, further let $\alpha, r, R \in \mathbb{R}$ such that $\{C, c, \alpha, r, R\}$ is feasible for construction. Then for the set $\mathcal{S}(C,\alpha)$ constructed in Theorem 8.3.1 using c, α, r and R the interior containment restriction $\bigcup_{x\in C} B(y,\Delta(y)) \subset \mathcal{S}(C,\alpha)$ holds for*

$$\Delta(y) := \sin(\alpha)\,\|y\| \left(\frac{r}{R}\frac{1}{(d+1)^3}\right)\left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right) \quad . \tag{8.13}$$

Please notice that in the following proof we will use $\Delta \in \mathbb{R}$ and $\Delta(y)$ which are not equal.

In the following proof our intention is to use the properties of some cone $\mathrm{K}(v,\beta)$, for which in Lemma 7.3.1 (p. 46), we proved that with β chosen well, $\mathrm{K}(v,\beta) \subset \operatorname{cone}(H^c \cap B(v, \sin(\alpha)))$ holds for the given α.

Proof: It clearly suffices to prove that $B(w,\Delta(w)) \subset \mathcal{S}(C,\alpha)$ for every $w \in \operatorname{cone}(W)$.

Therefore let $w \in \operatorname{cone}(W)$, then we find some representant $v \in \operatorname{cone}(W) \cap H^c$ such that $w \in \operatorname{ray}(v)$, where $w = \mathbf{0}$ is not an exception.

Let $P = C \cap H(c,-1) \oplus \operatorname{lin}(\{c\})$ be the apex figure of C relative to c. From the proof of the last lemma we recall

$$\mathcal{S}(C,\alpha) = \operatorname{cone}(\mathcal{K}(P,\varepsilon) \cap H^c) \quad , \tag{8.14}$$

and we know that by construction of $\mathcal{K}(P,\varepsilon)$, the set contains some ball $B(v,\Delta) \subset \mathcal{K}(P,\varepsilon)$ with

$$\Delta := \frac{r}{R}\frac{\varepsilon}{(d+1)^3} = \frac{r}{R}\frac{1}{(d+1)^3}\sin(\alpha)$$

where $0 < r \leq R$. Hereby $0 < \Delta < \sin(\alpha)$ must hold, as $0 < \sin(\alpha)$ by choice of $\alpha \in (0, \pi/4 - \gamma/2]$. Thus there is some $\beta \in (0, \pi/4 - \gamma/2]$ with $\Delta = \sin(\beta)$.

So considering (8.14) we find

$$\mathrm{cone}(B(v, \sin(\beta)) \cap H^c) \subset \mathcal{S}(C, \alpha) \quad . \tag{8.15}$$

Defining $\gamma' := \angle(c, v)$ we find we have $\gamma' \leq \gamma$ by definition of γ as the maximum angle between c and any $w' \in \mathrm{cone}(W) \setminus \{\mathbf{0}\}$, so we find $(\pi/4 - \gamma/2) \leq (\pi/4 - \gamma'/2)$ and thus we have $\beta \in (0, \pi/4 - \gamma'/2]$. Therefore we may apply Lemma 7.3.1 (p. 46) stating that there is $\beta' \in [0, \beta]$ with

$$\mathrm{K}(v, \beta') \subset \mathrm{cone}(B(v, \sin(\beta)) \cap H^c) \quad , \tag{8.16}$$

where

$$\sin(\beta') = \sin(\beta) \left(\frac{1}{2} \cos^2(\gamma) \cos(\alpha) \right) \quad .$$

Recalling the basic properties of $\mathrm{K}(v, \beta')$ and using $w \in \mathrm{K}(v, \beta')$ together with (8.16) and (8.15) we find

$$B(w, \sin(\beta') \, \|w\|) \subset \mathrm{K}(v, \beta) \subset \mathcal{S}(C, \alpha) \quad .$$

Remembering $\Delta = \sin(\beta)$ we calculate

$$\begin{array}{rcll} \sin(\beta') \, \|w\| & = & \sin(\beta) & \left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right) \;\; \|w\| \\ & = & \frac{r}{R} \frac{1}{(d+1)^3} \sin(\alpha) & \left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right) \;\; \|w\| \\ & = & \Delta(w) & \end{array}$$

So finally we receive $B(w, \Delta(w)) \subset \mathcal{S}(C, \alpha)$, which as w was chosen randomly proves the Lemma for every $w \in \mathrm{cone}(W)$. □

8.3.3 Exterior Restriction

Lemma 8.3.7. *Let $C = \mathrm{cone}(W) \subset \mathbb{R}^d$ be a pointed d-dimensional cone and c a support vector of its apex, further let $\alpha, r, R \in \mathbb{R}$ such that $\{C, c, \alpha, r, R\}$ is feasible for construction.*

Then for the set $\mathcal{S}(C, \alpha)$ constructed in Theorem 8.3.1 using c, α, r and R the exterior containment restriction $\mathcal{S}(C, \alpha) \subset \bigcup_{x \in C} B(y, \varepsilon(y))$ holds for

$$\varepsilon(y) := \sin(\alpha) \, \|y\| \quad .$$

Remark 8.3.8. Please notice that in this proof we will use $\varepsilon \in \mathbb{R}$ and $\varepsilon(y)$ which are not equal.

Proof: The main part of this proof will be to prove that

$$\mathcal{S}(C, \alpha) \subset \bigcup_{v \in C \cap H^c} \mathrm{K}(v, \alpha) \quad , \tag{8.17}$$

From which the lemma follows by basic properties of the cones $\mathrm{K}(v,\alpha)$.

Define $H^c := H(c,-1)$, $\varepsilon := \sin(\alpha)$ and

$$P := C \cap H^c + \mathrm{lin}(\{c\})$$

and let $\mathcal{K}(P,\varepsilon)$ be the object used in Theorem 8.3.1. Then we derive from construction of $\mathcal{K}(P,\varepsilon)$ that $\mathrm{dist}(P,\mathcal{K}(P,\varepsilon)) \leq \varepsilon$.

Let $y \in H^c \cap \mathcal{K}(P,\varepsilon)$ and denote by $y' \in P$ the point in P closest to y. Then recalling $\mathrm{lineal}(P) = \mathrm{lin}(\{c\})$, clearly y' is the point closest to y of all points in the affine subspace $y' + \mathrm{lin}(c) \subset P$, and thus $y' - y \in \{c\}^{\perp}$. Thus $y' \in H^c$ must hold, since $y \in H^c = \{x \in \mathbb{R}^d : c^T x = -1\}$ and therefore

$$c^T(y') = c^T(y'-y) + c^T(y) = c^T y = -1.$$

Thus we have $y \in B(y',\varepsilon) \cap H^c \subset B(y', \sin(\alpha)\,\|y\|)$, where we used that $\varepsilon = \sin(\alpha)$ and $\|x\| \geq 1$ for any $x \in H^c$.

As y was chosen randomly we find

$$\mathcal{K}(P,\varepsilon) \cap H^c \subset \bigcup_{v \in P \cap H^c} B(v, \sin(\alpha)\,\|v\|).$$

In proof of Lemma 8.3.5, we already showed

$$\mathcal{S}(C,\alpha) = \mathrm{cone}\,(\,\mathcal{K}(P,\varepsilon) \cap H^c\,) \quad , \tag{8.18}$$

and therefore using $\mathrm{K}(v,\alpha) = \mathrm{cone}\,(B(v,\sin(\alpha)\,\|v\|)\,)$ for any $v \neq \mathbf{0}$ we conclude:

$$\begin{aligned} \mathcal{S}(C,\alpha) \quad &\subset \quad \bigcup_{v \in P \cap H^c} \mathrm{cone}\,(B(v,\varepsilon\,\|v\|) \cap H^c\,) & (8.19)\\ &\subset \quad \bigcup_{v \in P \cap H^c} \mathrm{K}(v,\alpha) \quad . & (8.20) \end{aligned}$$

From this we derive the lemma by recalling basic properties of $\mathrm{K}(v,\alpha)$

$$\mathrm{K}(v,\alpha) = \bigcup_{w \in \mathrm{ray}(v)} B(w, \sin(\alpha)\,\|w\|) \quad ,$$

and by remembering that any ray of C intersects H^c as c is a support vector of the apex: $C = \bigcup_{v \in C \cap H^c} \mathrm{ray}(v)$. With these two properties and (8.17) we prove the lemma by

$$\begin{aligned} \mathcal{S}(C,\alpha) \quad &\subset \quad \bigcup_{v \in \mathrm{cone}(W) \cap H^c} \; \bigcup_{w \in \mathrm{ray}(v)} B(w, \sin(\alpha)\,\|w\|) \\ &= \quad \bigcup_{w \in C} B(w, \sin(\alpha)\,\|w\|) \quad . \end{aligned}$$

$\square$

8.3.4 Cones containing Lineality Spaces

Finally we prove Theorem 8.3.1 in the case that C is of the form $C = \operatorname{cone}(W) \oplus \operatorname{lin}(U)$.

Remark 8.3.9. In the following we assume $\dim(U) > 0$. For $\dim(U) = 0$ C is pointed and we refer to the previously shown lemma.

Lemma 8.3.10. *Let $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0\,, i{=}1,\ldots,m\} = \operatorname{cone}(W) \oplus \operatorname{lin}(U) \subset \mathbb{R}^d$ be a d-dimensional cone where $0 < \dim(U) \leq d-2$. Further let c be any support vector of its apex, and let $\alpha, r, R \in \mathbb{R}$ such that $\{C, c, \alpha, r, R\}$ is feasible for construction.*

Then the set $\mathcal{S}(C,\alpha)$ constructed in Theorem 8.3.1 using c, α, r and R, is a convex cone holding the following containment restrictions:

$$\bigcup_{y \in C} B(y, \Delta(y)) \quad \subset \quad \mathcal{S}(C,\alpha) \quad \subset \quad \bigcup_{y \in C} B(y, \varepsilon(y)) \tag{8.21}$$

where $\varepsilon(y) := \sin(\alpha)\mathrm{d}(y, \operatorname{lin}(U))$ and

$$\Delta(y) := \sin(\alpha)\mathrm{d}(y, \operatorname{lin}(U)) \, \left(\frac{r}{R}\frac{1}{(d+1)^3}\right)\left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right)$$

Proof: As $\dim(U) \leq d-2$ the cone C meets the restrictions given in Theorem 8.3.1 thus the construction of $\mathcal{S}(\operatorname{cone}(W), \alpha)$ is possible.

Let $n := \dim(U) > 0$. By $\dim(U) \leq d-2$ we find $\dim(U^\perp) \geq 2$. Thus as C is full-dimensional $\operatorname{cone}(W)$ has dimension 2 or higher. Therefore $\operatorname{cone}(W)$ meets the restrictions of the previous Lemmata 8.3.5, 8.3.6 and 8.3.7.

Using an appropriate linear, isometric transformation of coordinates we assume that $U^\perp = \operatorname{lin}(\{e_1, \ldots, e_n\})$ where e_i denotes the i-th coordinate vector. By fixing the the last $(d-n)$ coordinates to zero, we identify $U^\perp$ with the space $\mathbb{R}^n$. Constructing $\mathcal{S}(C,\alpha)$ relative to $C \cap U^\perp$ we find that $\mathcal{S}' := \mathcal{S}(C,\alpha) \cap U^\perp$ is a convex cone, fulfilling the restrictions given in (8.21), by using the previous Lemmata 8.3.5, 8.3.6 and 8.3.7.

By definition we find that all measurements used in Theorem 8.3.1 constructing $\mathcal{S}(C,\alpha)$ are measured relative to $U^\perp$. Further as $\{a_1, \ldots, a_m\} \subset U^\perp$ and $c \subset U^\perp$, we find that the polynomials defining $\mathcal{S}(C,\alpha)$, $\mathfrak{p}_\alpha(x)$ and $c^T x$ do not depend on the last $n-d$ inputs of x. Thus we have $\mathcal{S}(C,\alpha) = \mathcal{S}' \oplus \operatorname{lin}(U)$, and thus $\operatorname{lineal}(\mathcal{S}(C,\alpha)) = \operatorname{lin}(U)$.

Therefore $\mathcal{S}(C,\alpha)$ is a convex cone with lineality space $\operatorname{lin}(U)$.

Interior restriction:
Let $y \in C$ then there is y' in $\operatorname{cone}(W)$, $u \in \operatorname{lin}(U)$ such that $y = y' + u$. This implies that $\Delta(y) = \Delta(y')$ as $\mathrm{d}(y'+u, \operatorname{lin}(U)) = \mathrm{d}(y', \operatorname{lin}(U))$. Further as $\mathbf{0}, y' \in U^\perp$ and $\mathbf{0} \in \operatorname{lin}(U)$ we find $\|y'\| = \mathrm{d}(y', \operatorname{lin}(U))$. Thus using Lemma 8.3.6 we prove the following:

$$\begin{aligned} B(y', \Delta(y')) \quad &\subset \quad B(y', \Delta(y')) \cap U^\perp + \operatorname{lin}(U) \\ &\subset \quad \mathcal{S}' + \operatorname{lin}(U) \;=\; \mathcal{S}(C,\alpha) \end{aligned} \tag{8.22}$$

where the inclusion $B(y', \Delta(y')) \cap U^\perp \subset \mathcal{S}'$ in (8.22) was proved relative to $U^\perp$ previously in Lemma 8.3.6. Thus we find:

$$B(y, \Delta(y)) = B(y' + u, \Delta(y')) = B(y', \Delta(y')) + u \subset \mathcal{S}(C, \alpha) \quad ,$$

where the last inclusion follows from $u \in \text{lineal}(\mathcal{S}(C, \alpha))$ and $By'\Delta(y') \subset \mathcal{S}(C, \alpha)$. As y was chosen randomly, this proves the interior restriction in (8.21).

Exterior restriction:
Clearly $\mathcal{S}(C, \alpha) = \mathcal{S}' \oplus \text{lin}(U)$ holds and further for any pair $y \in \text{cone}(W)$ and $u \in \text{lin}(U)$ we have $\text{d}(y, \text{lin}(U)\,) = \text{d}(y + u, \text{lin}(U)\,)$ implying $\varepsilon(y) = \varepsilon(y + u)$. Further here again for any $y \in U^\perp$ we have $\text{d}(y, \text{lin}(U)) = \|y\|$. Thus using the exterior restriction calculated on $\mathcal{S}'$ relative to $(^\perp U)$ in Lemma 8.3.7 we find the following:

$$\begin{aligned}
\mathcal{S} \oplus \text{lin}(U) &\subset \bigcup_{w \in \text{cone}(W)} B(y, \varepsilon(y)) + \text{lin}(U) \\
&\subset \bigcup_{w \in \text{cone}(W)} \bigcup_{u \in \text{lin}(U)} B(y, \varepsilon(y)) + u \\
&= \bigcup_{w \in \text{cone}(W)} \bigcup_{u \in \text{lin}(U)} B(y + u, \varepsilon(y)) \\
&= \bigcup_{w \in C} B(w, \varepsilon(w)).
\end{aligned}$$

$\square$

8.4 Zariski Closure and Interior

Lemma 8.4.1. *Let $C := \operatorname{cone}(W) \oplus \operatorname{lin}(U) \subset \mathbb{R}^d$ be a d-dimensional polyhedral cone, where $\operatorname{cone}(W)$ is pointed and $\dim(U) < d-1$. Let further $\bar{c} \in \mathbb{R}^d$ be a support vector of the apex $\operatorname{lin}(U)$ of C and let $\mathcal{S} := \mathcal{S}(C, \alpha)$ be calculated properly, using $\bar{c}$, according to Theorem 8.3.1. Then the geometrical and the topological interior of $\mathcal{S}$ are equal:*

$$\begin{aligned} \operatorname{int}_Z(\mathcal{S}) &= \overset{\circ}{\mathcal{S}} & (8.23) \\ \operatorname{clos}_Z(\mathcal{S}) &= \operatorname{lineal}(C) & (8.24) \end{aligned}$$

Proof: By construction the semi-algebraic set $\mathcal{S}$ is of the following form, $\mathcal{S} = \{x \in \mathbb{R}^d : \mathfrak{p}_\alpha(x) \geq 0 \; \bar{c}^T x \geq 0\}$ where

$$\mathfrak{p}_\alpha(x) := (\bar{c}^T x)^{2k} - \sum_{i=1}^{m} \lambda_i \left(\bar{a}_i^T x - (\bar{c}^T x) d_i\right)^{2k} \quad .$$

For each $\mu \geq 0$ we find that $\mathcal{S} \cap \{x \in \mathbb{R}^d : \bar{c}^T x = -\mu\} = \mathcal{S}_\mu \cap \{x \in \mathbb{R}^d : \bar{c}^T x = -\mu\}$ where

$$\mathcal{S}_\mu := \{x \in \mathbb{R}^d : \mu^{2k} - \sum_{i=1}^{m} \lambda_i (\bar{a}_i - \mu\, d_i)^{2k} \geq 0\}$$

In Lemma 6.3.6 (p. 38) , we concluded from the special form $\mathcal{S}_\mu$ that $\operatorname{int}_Z(\mathcal{S}_\mu) = \overset{\circ}{\mathcal{S}}_\mu$ whenever $\mathcal{S}_\mu \neq \emptyset$. As any point $y \in \mathcal{S}(C, \alpha)$ fulfills $c^T y \leq 0$, any point y in the topological interior of $\mathcal{S}$ is contained in the relative topological interior of $\mathcal{S}_{\mu'} \cap \{x \in \mathbb{R}^d : \bar{c}^T x = -\mu'\}$ where $\mu' := -c^T y$ and thus $\mathfrak{p}_\alpha(y) > 0$, proving (8.23).

Further $\operatorname{clos}_Z(\mathcal{S})$ is clearly a subset of $\{x \in \mathbb{R}^d : \bar{c}^T x = 0\}$ and for $\mu = 0$ we obtain

$$\mathcal{S}_\mu = \{x \in \mathbb{R}^d : -\sum_{i=1}^{m} \lambda_i (\bar{a}_i)^{2k} \geq 0\} = \{\bar{a}_1, \ldots, \bar{a}_m\}^\perp$$

where we know that by construction in Theorem 8.3.1 we have $\{\bar{a}_1, \ldots, \bar{a}_m\}^\perp = \operatorname{lin}(U) \oplus \operatorname{lin}(\{\bar{c}\})$.

Thus $\operatorname{clos}_Z(\mathcal{S}) = \operatorname{lin}(U) \oplus \operatorname{lin}(\{\bar{c}\}) \cap \{x \in \mathbb{R}^d : \bar{c}^T x = 0\} = \operatorname{lin}(U)$. □

9. The Representant for a Face

We recall that our main ambition is, given a pointed cone C, to construct a semi-algebraic set $\mathcal{S}^F$ for each true face $F \in \mathcal{F}(C)$ such that all these $semi-algebraicset^F$ fulfill the containment restrictions postulated in Section 5.2, Proposition 5.2.1 (p. 28).

Each $\mathcal{S}^F$ will be an approximation $\mathcal{S}(C^{[F]}, \alpha^F)$ to a special cone $C^{[F]}$ (see Figure 9.1) which we will introduce and discuss in this chapter.

From the qualities of the sets $\mathcal{S}(C^{[F]}, \alpha^F)$ we prove in further chapters, that using a special set of α^F leads to the proof of the Main Theorem.

9.1 Relaxation-Cones of a Pointed Cone

Definition 9.1.1. Let $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i = 1, \ldots, m\}$ be a d-dimensional pointed cone and let $F \in \mathcal{F}(C)$ be a true face of C.
Then we define a relaxation cone $C^{[F]}$ of C by

$$C^{[F]} := \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0, \, i \in [F]\} \quad . \tag{9.1}$$

From the definition we find some general properties of $C^{[F]}$ in the following lemma and we see how to retrieve some of the faces of C in $C^{[F]}$.

Lemma 9.1.2. *Let $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i = 1, \ldots, m\} = \operatorname{cone}(W)$ be a d-dimensional pointed cone and let $F \in \mathcal{F}(C)$ be some proper face of C.*
Let further $W^F \subset W$ such that $F = \operatorname{cone}(W^F)$.

Then for $C^{[F]}$ we find

$$C^{[F]} = \operatorname{cone}(W \backslash W^F) + \operatorname{lin}(W^F) = C + \operatorname{lin}(F) \quad ,$$

and thus $\operatorname{lineal}(C^{[F]}) = \operatorname{lin}(F)$.
Further the faces $\mathcal{F}(C^{[F]})$ of $C^{[F]}$ are given by

$$\mathcal{F}(C^{[F]}) = \{G + \operatorname{lin}(F) : \; G \in \mathcal{F}(C) \,, F \subset G\} \quad .$$

The proof is basic theory on polyhedra and therefore we prove it – using Fourier-Motzkin-Elimination – in the appendix, Section 15.3, page 103.

The proof is motivated by the sketch in Figure 9.1.

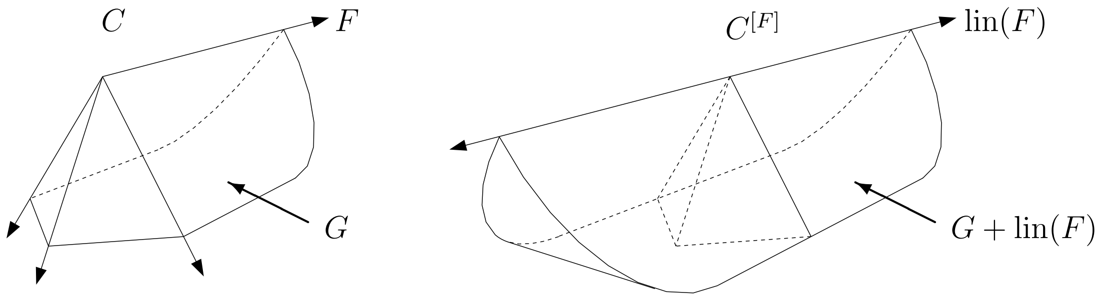

Fig. 9.1: The Relaxation Cone for an Edge

10. The Minimum interior Angle

As described in in Section 5.2, the containment-restrictions apply only for $\mathcal{S}^{F_1}, \mathcal{S}^{F_2}$ when the two faces $F_1, F_2 \in \mathcal{F}_k(C)$, $k > 0$ intersect. Therefore we examine the geometrical situation of the intersection $C^{[F_1]} \cap C^{[F_2]}$ in the second section, providing a connection between the interior of a set $\mathcal{S}^F$ and the outer boundary of some other $\mathcal{S}^G$.

In later sections we will examine the intersection of two sets $\mathcal{S}^{F_1} := \mathcal{S}(C^{[F_1]}, \alpha)$ and $\mathcal{S}^{F_2} := \mathcal{S}(C^{[F_2]}, \alpha)$ where both $F_i \in \mathcal{F}_k(C)$, $k > 0$ are faces intersecting in $G \in \mathcal{F}(C)$. Our intention is to fit $\mathcal{S}^G := \mathcal{S}(C^{[G]}, \alpha)$ 'into' the other two semi-algebraical sets according to the requirements discussed in section 5.2.

In the previous sections we have learned that $\mathcal{S}(C^{[G]}, \alpha)$ is contained in a set of ε-balls $B(y, \varepsilon(y))$ around the points $y \in C^{[G]}$.
On the other hand we learned that for every $y \in C^{[F]}$ we find some Δ-ball $B(y, \Delta(y, F))$ inside of $\mathcal{S}(C^{[F]}, \alpha)$.

To show that we have $\mathcal{S}^G \subset \mathrm{int}_Z(\mathcal{S}^{F_1}) \cup \mathrm{int}_Z(\mathcal{S}^{F_2}) \cup [\mathrm{clos}_Z(\mathcal{S}^{F_1}) \cup \mathrm{clos}_Z(\mathcal{S}^{F_2})]$ we will later simply prove that for any $y \in C^{[G]}$ we have

$$B(y, \varepsilon(y, G)) \subset B(y, \Delta(y, F_1)) \cup B(y, \Delta(y, F_2)) \quad .$$

where $\varepsilon(y, G)$, $\Delta(y, F_i)$ are the measurements provided by the containment restrictions on $\mathcal{S}^{F_i}$ and $\mathcal{S}^G$. These measurements depend on $\mathrm{d}(y, \mathrm{lin}(G))$ and $\mathrm{d}(y, \mathrm{lin}(F_i))$ respectively, therefore it is necessary to give a relation between these.

10.1 The Minimum Interior Angle of a Cone

In this section we examine the geometrical situation of the intersection $C^{[F_1]} \cap C^{[F_2]}$.

We will do so by using the face $G = F_1 \cap F_2$, remembering that each two faces of a pointed cone with equal dimension $k > 0$ intersect in some third face of lower dimension. The way how F_1 and F_2 intersect geometrically is mainly determined by the minimum angle between these faces. We will measure this angle relative to $C^{[G]} = \mathrm{cone}(V) \oplus \mathrm{lin}(G)$ and thus make it a measurement for G.

We recall that we calculated some interior set in $\mathcal{S}^F = \mathcal{S}(C^{[F]}, \alpha)$ depending on the measurement $\mathrm{d}(y, \mathrm{lin}(F))$ and there is some outer 'padding' for $\mathcal{S}^G$ depending on $\mathrm{d}(y, \mathrm{lin}(G))$. In this section we will learn how to linearly convert between these distances for any $y \in C^{[G]}$ using this specific angle. This will help to 'fit' $\mathcal{S}^G$ into the $\mathcal{S}^{F_i}$ according to the containment restrictions discussed in Section 5.2.

Definition 10.1.1. Let $C = \text{cone}(V) \oplus \text{lin}(U)$ be a d-dimensional polyhedral cone, where $V = \{ v_1, \ldots, v_n \}$, cone$(V)$ is pointed and $\dim(U) < d-1$.
Then we define a minimum interior angle of the boundary of C by

$$\angle_{min}(C) := \min\{\angle(F,G) | \, F,G \in \mathcal{F}(C) \setminus \text{lin}(U) \, , F \cap G = \text{lin}(U)\} \quad . \tag{10.1}$$

Remark 10.1.2. We have to restrictdim$(U) < d-1$ as otherwise C is a hyperplane with a single face, and cone(V) is a ray, and so there is no interior angle to measure.

Corollary 10.1.3. *The angle $\angle_{min}(C) > 0$ is well defined.*

Proof: We may assume that C is given in irreducible description.
As $\dim(\text{cone}(V)\,) = d - \dim(U) > 1$ we must have $n \geq 2$. For the pair $v_1, v_2 \in V$ we find each $F_i := \text{ray}(v_i) + \text{lin}(F)$, $i = 1, 2$ is a face of C as C is given in irreducible description and clearly $F_1 \cap F_2 = \text{lin}(F)$ holds.
Thus there is at least one pair of faces F_1, F_2 matching the requirements in (10.1).

Further every face $F \in \mathcal{F}(C)$, $F \neq \text{lin}(U)$ is a cone with lineal$(F) = \text{lin}(U)$ and therefore $F \cap U^\perp \neq \mathbf{0}$ is a convex polyhedral cone with lineality space $\{\mathbf{0}\}$ and thus pointed.
Recalling that the definition of $\angle(C', C'')$ for polyhedral cones C', C'' requires that $lineal(C') =$ lineal(C''), we find that any pair $F, G \in \mathcal{F}(C)$ with $F \cap G = \text{lin}(U)$ meets these requirements.
As there are finitely many such pairs of faces the minimum of all the corresponding angles is well defined.
Further for any such pair of faces we have $\angle(F,G) := \angle(F \cap U^\perp, G \cap U^\perp)$ with $F \cap G \cap U^\perp = \mathbf{0}$ which implies $\angle(F,G) > 0$ as $F \cap U^\perp, F \cap U^\perp \neq \emptyset$. $\square$

We recall that the faces F_i are represented in $C^{[G]}$ by $F_i + \text{lin}(G)$ and so we will prove that any $y \in C^{[G]}$ is 'far away' from at least one of the $F_i + \text{lin}(G)$, where the distance is measured by $\text{d}(y, \text{lin}(G))$ and $\angle_{min}(C^{[G]}\,)$ and thus is determined by the relation of y and G only.

Later, in Section 12.2 (p. 75) we will then prove that for $y \imath C^{[G]}$ the distances $\text{d}(y, F + \text{lin}(G))$ and $\text{d}(y, \text{lin}(F))$ are linked linearly.

Lemma 10.1.4. *Let C be a pointed, d-dimensional, polyhedral cone in $\mathbb{R}^d$, $d \geq 2$*
Let further $F_1, F_2 \in \mathcal{F}_k(C)$ where $0 < k \leq d-1$ and such $G := F_1 \cap F_2 \in \mathcal{F}(C)$.
Define $\omega := \angle_{min}(C^{[G]}\,)$, then for any $y \in C^{[G]}$ we have

$$\max_{i=1,2} \text{d}(y, F_i + \text{lin}(G)\,) \geq \, \sin(\omega/2) \, \text{d}(y, \text{lin}(G)) \quad , \tag{10.2}$$

where $\sin(\omega/2) > 0$.

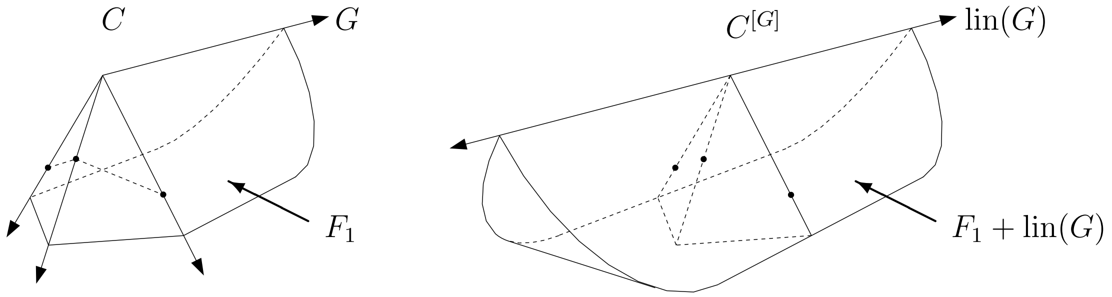

Fig. 10.1: The distance $d(y, F + \text{lin}(G))$

Remark 10.1.5. Here we have to restrict d by $d \geq 2$ as for $d < 2$ the cone C has less than three proper faces.

The proof is motivated by the sketch in Figure 10.1.

Proof for Lemma 10.1.4:
First we check that $\sin(\omega/2) > 0$:
By choice of $G = F_1 \cap F_2$ we must have $\dim(G) < k \leq d-1$.
As we know $C^{[G]}$ is a full-dimensional cone with lineality space $\text{lin}(G)$. Therefore we find that $C^{[G]} \cap \text{lin}(G)^{\perp}$ is a cone of dimension $d - \dim(G) > d - (d-1) = 1$ with lineality space $\{\mathbf{0}\}$, and thus pointed.
Checking the definition of $\angle_{min}(\cdot)$ this leads to $0 < \angle_{min}(C^{[G]}) < \pi$.

Through the lineality space $\text{lin}(G)$ of $C^{[G]}$ we know that for every $y \in C^{[G]}$ there is a point $y' \in C^{[G]} \cap \text{lin}(G)^{\perp}$ with $y = y' + u_G$ where $u_G \in \text{lin}(G)$.
Thus we find $\text{d}(y, F_i + \text{lin}(G)) = \text{d}(y', F_i + \text{lin}(G))$ as well as $\text{d}(y', \text{lin}(G)) = \text{d}(y, \text{lin}(G))$.
Therefore it suffices to prove the lemma for $C^{[G]} \cap G^{\perp}$ as $G^{\perp} = \text{lin}(G)^{\perp}$.

Let $y \in C^{[G]} \cap G^{\perp}$ and let further $v_i \in F_i + \text{lin}(G)$ be the closest point of $F_i + \text{lin}(G)$ to y for each $i = 1, 2$.
Then obviously for each $i = 1, 2$ we find $y - v_i \in G^{\perp}$, as v_i is the point closest to y of the affine subspace $v_i + \text{lin}(G)$. This implies $v_i \in G^{\perp}$ as $y \in G^{\perp}$.

If $v_1 = \mathbf{0}$ or $v_2 = \mathbf{0}$ then we might assume w.l.o.g. $v_1 = \mathbf{0}$ and thus we have $\text{d}(y, F_1 + \text{lin}(G)) = \text{d}(y, \mathbf{0})$ by the definition of v_1.
Further we find $\text{d}(y, \mathbf{0}) = \text{d}(y, \text{lin}(G))$ as both $\mathbf{0}, y \in G^{\perp}$ and $\mathbf{0} \in \text{lin}(G)$. Thus we conclude

$$\begin{array}{rcrl} \text{d}(y, F_1 + \text{lin}(G)) & = & & \text{d}(y, \text{lin}(G)) \\ & \geq & \sin(\omega/2) & \text{d}(y, \text{lin}(G)) \quad . \end{array} \tag{10.3}$$

Else if both $v_1 \neq \mathbf{0}$ and $v_2 \neq \mathbf{0}$, then we show $\angle(v_1, v_2) \geq \omega$ as follows:
We know that by choice of G, the faces $\widetilde{F}_i := F_i + \text{lin}(G) \in \mathcal{F}(C^{[G]})$ intersect only in $\text{lin}(G)$ and thus we have $\angle(\widetilde{F}_1, \widetilde{F}_2) \geq \angle_{min}(C^{[G]}) = \omega$.

By definition we have

$$\angle(\widetilde{F}_1, \widetilde{F}_2) := \min\{\angle(w_1, w_2) : w_i \in \widetilde{F}_i \cap G^{\perp},\ w_i \neq \mathbf{0}\}$$

And so as $v_1, v_2 \in G^{\perp}$ are non-zero vectors with $v_i \in \widetilde{F}_i$ we find $\angle(v_1, v_2) \geq \omega$.
Using the triangle inequality (cf. [Cho70]) for angles we know

$$\omega \leq \angle(v_1, v_2) \ \leq \ \angle(v_1, y) + \angle(y, v_2) \quad .$$

and therefore we may assume w.l.o.g. $\frac{1}{2}\omega \leq \angle(v_1, y)$.
This implies $y \neq v_1$ as otherwise we had $0 = \angle(v_1, y) \geq \omega > 0$.

We remember $v_1 \in F_1 + \mathrm{lin}(G)$, $v_1 \neq 0$ and thus $\mathrm{ray}(v_1) \subset F_1 + \mathrm{lin}(G)$ must hold as F_1 is a cone. As v_1 is the point of $F_1 + \mathrm{lin}(G)$ closest to y, the same holds for $\mathrm{ray}(v_1)$.
This shows $y – v_1 \ \perp \ v_1$ and such the triangle $\triangle(\mathbf{0}, v_F, y)$ is perpendicular with hypothenuse $[\mathbf{0}, y]$. We thus obtain $\mathrm{d}(y, v_1) \geq \sin(\omega/2)\mathrm{d}(0, y)$, and by the definition of v_1 we conclude

$$\begin{aligned} \mathrm{d}(y, F + \mathrm{lin}(G)) &\geq \sin(\omega/2)\,\mathrm{d}(y, \mathbf{0}) \\ &= \sin(\omega/2)\,\mathrm{d}(y, \mathrm{lin}(G)) \quad , \end{aligned}$$

where the last equation holds as $y \in \mathrm{lin}(G)^{\perp}$.

So in either case we have proven (10.2). $\square$

11. Bounds to Measurements Necessary for the Construction

Here in the first section we give estimations for the values needed for the construction of $\mathcal{S}(C^{[F]}, \alpha)$, namely r, R and $\gamma = \angle_{max}(C^{[F]} \cap F^{\perp}, -c(F))$, defined in the construction in Theorem 8.3.1 "Construction" (p. 51).

In the second section we will provide a lower bound on the interior angle $\angle_{min}(C^{[F]})$ of a cone $C^{[F]}$, which we defined in Section 8.3.1 (p. 51).

With these estimations made, in the next chapter we will provide a construction of a set $\mathcal{S}^F = \mathcal{S}(C^{[F]}, \alpha^F)$ for each proper face $F \in \mathcal{F}(C)$ using a special α^F, such that all $\mathcal{S}^F$, $F \in \mathcal{F}(C)$ fulfill the restrictions postulated in Proposition 5.2.1 "Target of Construction" . This proves the Main Theorem using Lemma 5.3.2 which was directly derived from this proposition.

11.1 The Maximum Relative Angle and the relative Diameter of an Apex-Figure

11.1.1 Upper Bound

Given a pointed polyhedral cone $C = \text{cone}(W)$ with a proper face F, $\dim(F) < d-1$ we will give an upper bound R to $\text{diam}(C^{[F]} \cap H^c \cap F^{\perp})$, the diameter of the apex figure of $C^{[F]}$, where $H^c := H(c(F), -1)$.

For $\gamma = \angle_{max}(C^{[F]} \cap F^{\perp}, -c(F))$ an upper bound is provided, as for the construction of $\mathcal{S}(C^{[F]}, \alpha)$ any used α is bound by $\alpha \leq \pi/4 - \gamma/2$.

Both bounds depend on the following constant, defined relative to a choice of support vectors.

Definition 11.1.1. Let $C = \text{cone}(W) \subset \mathbb{R}^d$, $d \geq 2$ be a d-dimensional pointed cone where $W = \{ w_1, \ldots, w_n \}$. Further let $c(\mathcal{F})$ be any choice of support vectors of C, then define

$$M_c(C) := \min_{F \in \mathcal{F}(C)} \min\{-\frac{c(F)^T w}{\|w\|} : w \in W,\ c(F)^T w \neq 0\} \quad . \tag{11.1}$$

When clear which polyhedron is meant, $M_c(C)$ is abbreviated to M_c.

Corollary 11.1.2. *The restriction $0 < M_c(C) < 1$ must hold as by $n \geq d \geq 2$ there must be at least $w_1, w_2 \in W$, $w_i \neq \mathbf{0}$ and one vertex v in C, where $c(v)^T w_i < 0$.*

Lemma 11.1.3. *Let* $C = \operatorname{cone}(W) \subset \mathbb{R}^d$, $d > 0$ *be a* d*-dimensional pointed cone with a proper face* $F \in \mathcal{F}_k(C)$, $k \le d-2$. *Further let* $c(\mathcal{F})$ *be any choice of support vectors of* C. *Define* $M_c := M_c(C)$ *and*

$$\begin{aligned}\gamma_1 &:= \max\{\angle(-c(F), v) :\ v \neq 0,\ v \in C^{[F]} \cap \operatorname{lineal}(C^{[F]})^{\perp}\}\\ \gamma_2 &:= \max\{\angle(-c(F), w) :\ w \neq 0,\ w \in \operatorname{cone}(W \backslash W^F)\}\end{aligned}$$

where $W^F \subset W$ *is such that* $F = \operatorname{cone}(W^F)$. *Then the following holds:*

$$\gamma_1 \le \gamma_2 \le \arccos(M_c) < \pi/2. \tag{11.2}$$

Further with $H^c : H(c(F), -1)$ *and* $R := 2/M_c$ *we find*

$$\operatorname{diam}(C^{[F]} \cap F^{\perp} \cap H^c) \le R \quad . \tag{11.3}$$

Remark 11.1.4. Although here we make use of the function arccos, we would like to stress that the final construction does not require the use of arccos.
Please notice further that the value of M_c is intrinsic to the principle of construction we are following, which will become clear in the proof of Lemma 12.2.1 where the interior containment restriction on $\mathcal{S}(C^{[F]}, \alpha)$ is recalculated.

Proof: At first we prove $\gamma_1 \le \gamma_2$, from which we then derive (11.2).
Abbreviating the notation:
We know by previous discussion that $C^{[F]} \cap F^{\perp}$ is a pointed polyhedral cone, as $\dim(F) \le d-2$. So there is some set $V = \{ v_1, \dots, v_k \}$, $\|v_i\| = 1$ such that $\operatorname{cone}(V) = C^{[F]} \cap F^{\perp}$. Further we abbreviate $\bar{c} := -c(F)$, as in the proof mainly $-c(F)$ is used. So by definition

$$\gamma_1 \ = \ max\{\angle(\bar{c}, v) :\ v \neq 0,\, v \in \operatorname{cone}(V)\} \tag{11.4}$$

which is well defined and bounded by $0 \le \gamma_1 < \pi/2$ as $\operatorname{cone}(V)$ is pointed.

Proof for $\gamma_1 \le \gamma_2$:
At first we recall that $W \backslash W^F = \{w \in W : \bar{c}^T w \neq 0\}$: By choice of W^F for any $w \in W^F$ holds $\bar{c}^T w = 0$. Since C is given in irreducible description $w' \in W \backslash W^F$ implies $w' \notin F$ and thus $\bar{c}^T w' \neq 0$ must hold.

Further as $-\bar{c}$ is a support vector for the apex of $C^{[F]} = \operatorname{cone}(W \backslash W^F) + \operatorname{lin}(F)$, for any $w \in \operatorname{cone}(W \backslash W^F)$ we conclude $\angle(\bar{c}, w) < \pi/2$ which proves $\gamma_2 < \pi/2$.

Let $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be the linear orthogonal projection of $\mathbb{R}^d$ to $\operatorname{lin}(F)^{\perp}$.
Then we have $\operatorname{cone}(V) = \pi(\operatorname{cone}(W \backslash W^F))$:
As $\operatorname{lineal}(C^{[F]}) = \operatorname{lin}(F)$ we find $\pi(C^{[F]}) = C^{[F]} \cap F^{\perp} = \operatorname{cone}(V)$. Furthermore we have $C^{[F]} = \operatorname{cone}(W \backslash W^F) + \operatorname{lin}(F)$ and thus $\pi(C^{[F]}) = \pi(\operatorname{cone}(W \backslash W^F))$ as π operates linear and $\pi(\operatorname{lin}(F)) = \{\mathbf{0}\}$.

So let $v \in \text{cone}(V) \setminus \{\mathbf{0}\}$ then we know there is $w \in \text{cone}(W \setminus W^F) \setminus \{\mathbf{0}\}$ such that $v = \pi(w)$ and thus $w = v + u$ with $u \in \text{lin}(F)$ and therefore $v \perp u$. This implies $\|v\| \leq \|w\|$. Further we conclude $\bar{c}^T v = \bar{c}^T w$, knowing that $\bar{c} \in \text{lin}(F)^\perp$. Therefore we find

$$\cos(\angle(\bar{c}, v)\,) = \frac{\bar{c}^T v}{\|v\|} \geq \frac{\bar{c}^T w}{\|w\|} = \cos(\angle(\bar{c}, w)\,) \quad .$$

By recalling $\angle(\bar{c}, v), \angle(\bar{c}, w) \in [0, \pi/2)$ we find $\angle(\bar{c}, v) \leq \angle(\bar{c}, w)$ as the function cos is monotonous in this area. This directly proves $\gamma_1 \leq \gamma_2$:

$$\begin{aligned} \gamma_1 \;&:=\; \max\{\angle(\bar{c}, v) :\; v \neq \mathbf{0},\, v \in \text{cone}(V)\} \\ &\leq\; \max\{\angle(\bar{c}, w) :\; w \neq \mathbf{0},\, w \in \text{cone}(W \setminus W^F)\} \;=\; \gamma_2 \quad . \end{aligned}$$

Proof for (11.2):
As $-\bar{c}$ is a support vector of $\text{cone}(W \setminus W^F)$, we use Lemma 7.3.5, stating that the maximum angle we are looking for is assumed in the extremal rays of the cone:

$$\gamma_2 = \max\{\angle(\bar{c}, w) :\; w \neq 0,\; w \in W \setminus W^F\} \quad .$$

Recalling $\gamma_2 \in [0, \pi/2)$ using the monotony of the function cos in this interval, leads to the following

$$\begin{aligned} \cos(\gamma_2) \;&\geq\; \min\{\bar{c}^T w : w \in W \setminus W^F\} \\ &=\; \min\{-c(F)^T w : w \in W,\; c(F)^T w \neq 0\} \\ &\geq\; \min_{G \in \mathcal{F}(C)} \min\{-c(G)^T w : w \in W,\; c(G)^T w \neq 0\} \quad > 0 \quad , \end{aligned}$$

which directly proves (11.2).

Finally we have to prove the upper-bound to $\text{diam}(\text{cone}(V) \cap H^c)$ in (11.3).
In the brief notation used here the set $H^c = \{x \in \mathbb{R}^d \,:\, c(F)^T x = -1\}$ becomes $H^c = \{x \in \mathbb{R}^d \,:\, \bar{c}^T x = -1\}$

Clearly for any pair $v_1, v_2 \in \text{cone}(V) \cap H^c$ we find that $\|v_1 - v_2\| \leq \|v_1 - \bar{c}\| + \|\bar{c} - v_2\|$. Further we know that by definition of γ_2 we find $\angle(\bar{c}, v_i) \leq \gamma_2$, $i = 1, 2$. Further $(\bar{c} - v_i) \perp \bar{c}$ holds for both $i = 1, 2$ as $\bar{c}, v_1, v_2 \in H^c$. So by basic trigonometry we have $\|v_1 - v_2\| \leq 2\tan(\gamma) < 2\cos(\gamma_2)^{-1} \leq 2/M_c$, which proves (11.3). □

11.1.2 Lower Bound on the Relative Diameter of the Apex-Figure

Here we provide a lower bound on the relative diameter of the apex figure of a cone $C^{[F]}$ and introduce a way to calculate the values of the support functions used in the construction of $\mathcal{S}(C^{[F]},\alpha)$. Notice that the lower bound on the diameter is purposely chosen lower than necessary, in order to make it independent of the chosen face F.

Please recall from Lemma 8.2.1 (p. 50) that given a normalized support vector $\bar{c}$ of the apex of $C^{[F]} = \{x \in \mathbb{R}^d \,:\, a_i^T x \le 0,\, i \in [F]\}$, then the apex figure P can be described as

$$\begin{aligned} P &:= \left(C^{[F]} \cap H^c\right) \oplus \operatorname{lin}(\{\bar{c}\}) && (11.5)\\ &= \{x \in \mathbb{R}^d \,:\, \bar{a}_i^T x \le \bar{c}^T a_i \,, i \in [F]\}, \quad \text{where } \bar{a}_i := a_i - (\bar{c}^T a) c && (11.6)\end{aligned}$$

and $H^c = H(\bar{c},-1)$. Therefore please mind the difference of a_i and $\bar{a}_i$ in the following, especially in (11.7) and (11.8).

Lemma 11.1.5. *Let $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \le 0 \,, i=1,\ldots,m\} = \operatorname{cone}(W)$, $W = \{\, w_1,\ldots,w_n \,\}$ be a d-dimensional pointed cone and $F \in \mathcal{F}_k(C)$, $k \le d-2$.*
Let further $c(\mathcal{F}(P))$ be any choice of normalized support vectors for C, and $H^c := H(c(F),-1)$. Then for the apex figure of $C^{[F]}$,

$$P := \{x \in \mathbb{R}^d \,:\, \bar{a}_i^T x \le \bar{c}^T a_i \,, i \in [F]\}, \quad \textit{where } \bar{a}_i := a_i - (\bar{c}^T a) c,$$

the following holds for every $i \in [F]$:

$$\begin{aligned} h_P(\bar{a}_i) &= -c(F)^T a_i \quad , && (11.7)\\ h_P(-\bar{a}_i) &= c(F)^T a_i + \max\{\frac{a_i^T w}{c(F)^T w} \,:\, c(F)^T w \neq 0,\, w \in W\} \quad . && (11.8)\end{aligned}$$

Further we obtain a lower bound for the relative diameter of the apex figure by

$$0 < M_c(C) \;\le\; \operatorname{diam}(C^{[F]} \cap H^c \cap F^{\perp}) \quad . \qquad (11.9)$$

Proof: Let $F \in \mathcal{F}_k(C)$ with $k \le d-2$, then by previous discussion on $C^{[F]}$ from $\dim(F) \le d-2$ directly follows that $C^{[F]} \cap F^{\perp}$ is a pointed cone, and thus $C^{[F]}$ meets the restrictions of Lemma 8.2.1 stating $h(\bar{a}_i), h(\bar{a}_i)$ and $\operatorname{diam}(C^{[F]} \cap H^c \cap F^{\perp})$ are finite.

To shorten notation let in the following $\bar{c} := c(F)$.
Proof for (11.7):
Clearly $h(\bar{a}_i) \le \bar{c}^T a_i$ holds for every $i \in \{1,\ldots,m\}$, therefore (11.7) must hold as the equation $\bar{a}_i$ induces a facet of $C^{[F]} \cap H^c$, the set $C^{[F]} \cap H^c \cap \{x \in \mathbb{R}^d \,:\, \bar{a}_i^T x = c^T a_i\}$ is a facet of $C^{[F]} \cap H^c$ and thus non-empty.
Proof for (11.8):
Let $W^F \subset W$ be such that $F = \operatorname{cone}(W^F)$ then as $C = \operatorname{cone}(W)$ is given in irreducible description, for $F = \operatorname{cone}(W^F)$ holds the same and thus

$$\{w \in W :\; \bar{c}^T w \neq 0\} = \{w \in W :\; w \notin F\} = W \setminus W^F \quad . \qquad (11.10)$$

We will now show that

$$h(-\bar{a}_i) = max\{-\bar{a}_i^T w : \ w \in \text{cone}(W \backslash W^F) \cap H^c\} \quad . \tag{11.11}$$

So let $y \in C^{[F]} \cap H^c$, then recalling that $C^{[F]} = C + \text{lin}(F)$, there are $w \in \text{cone}(W \backslash W^F)$ and $u \in \text{lin}(F)$ such that $y = w + u$. Since $\text{lin}(F) \subset \{a_i\}^\perp$ holds for every $i \in \{1, \ldots, m\}$, we find $-\bar{a}_i^T y = -\bar{a}_i^T w$. Further we have $\text{lin}(F) \subset \{\bar{c}\}^\perp$ and thus by $\bar{c}^T y = \bar{c}^T w$ we find $w \in H^c$, which proves (11.11).

The maximum in (11.11) is a linear optimization problem, so the solution is found in one of the vertices of the polytope $\text{cone}(W \backslash W^F) \cap H^c$. These vertices are the intersection of the extremal rays of $\text{cone}(W \backslash W^F)$ with H^c which have the form $-w/(\bar{c}^T w)$, $w \in W \setminus W^F$, and thus they are contained in the set

$$\{\text{ray}(w) \cap H^c : \ w \in W \backslash W^F\} = \{-w/(\bar{c}^T w) : \ \bar{c}^T w \neq 0 \, , w \in W\} \quad ,$$

where we used (11.10) on the right-hand side. Using this we conclude

$$h(-\bar{a}_i) = max\{\frac{\bar{a}_i^T w}{\bar{c}^T w} : \ \bar{c}^T w \neq 0 \, , w \in W\} \quad , \tag{11.12}$$

which proves (11.8) as $\bar{a}_i^T w = a_i^T w - \bar{c}^T a_i \, \bar{c}^T w$.

Proof for (11.9):
For every $k \in \{1, \ldots, m\}$ the values of $h(\bar{a}_k)$ and $h(-\bar{a}_k)$, measured for $C^{[F]} \cap H^c$, apply as upper and lower bounds to the according support functions on $C^{[F]} \cap H^c \cap F^\perp$ as this set is just the orthogonal projection of $C^{[F]} \cap H^c$ to $F^\perp$ and $\bar{a}_k \in F^\perp$. So for every $k \in [F]$ the value

$$r_k := h(\bar{a}_k) + h(-\bar{a}_k) = \max\{\frac{a_k^T w}{\bar{c}^T w} : \ w \in W \backslash W^F\}$$

is a lower bound for $\text{diam}(C^{[F]} \cap H^c \cap F^\perp)$.

Let $w \in W \backslash W^F$ then $0 < -\bar{c}^T w \leq 1$ holds by choice of $\bar{c}$ as a normalized support vector of F, and $w \in C \setminus F$ implies that $-a_i^T w > 0$ for any $i \in [F]$. Hence $(a_i^T w)/(-\bar{c}^T w) \geq (a_i^T w)$ holds for any such w. Therefore for any $k \in [F]$ the following holds:

$$\begin{aligned} r_k \ &\geq \ \min\{(-a_k^T w) : \ w \in W, \, a_k^T w \neq 0\} && (11.13) \\ &\geq \ \min_{i=1,..,m} \min\{(-a_i^T w) : \ w \in W, \, a_i^T w \neq 0\} && (11.14) \\ &\geq \ \min_{F \in \mathcal{F}(C)} \min\{(-c(F)^T w) : \ w \in W, \, c(F)^T w \neq 0\} \quad . && (11.15) \end{aligned}$$

Here the last inequality holds, due to the fact that for any facet $F_i := C \cap \{x \in \mathbb{R}^d \, : \, a_i^T x = 0\}$, $i \in \{1, \ldots, m\}$, the only possible normalized support vector is $c(F_i) := a_i$, since C is full-dimensional.

This proves (11.9) as any r_k, $k \in [F]$ is a lower bound to $\text{diam}(C^{[F]} \cap H^c \cap F^\perp)$ and $M_c(C) > 0$ holds, as proven already. $\square$

11.2 A lower bound for the Minimum Interior Angle

Lemma 11.2.1. *Let* $C = \mathrm{cone}(W)$ *be a* d*-dimensional polyhedral cone where* $d > 1$ *and* $\dim(U) < d-1$.
For every proper face $F \in \mathcal{F}(C)$ *let* $c(F)$ *be a support vector of* F *and let* $G \in \mathcal{F}_k(C)$ *be any proper face of dimension* $k < d-1$. *Then defining*

$$\delta := \min_{F\in\mathcal{F}(C)} \min\{-c(F)^T w : w \in W,\ c(F)^T w \neq 0\} \tag{11.16}$$

we find that for interior minimum angle $\omega := \angle_{min}(C^{[G]})$ *of* $C^{[G]}$ *we find*

$$\sin(\frac{\omega}{2}) \ \geq \ \frac{\delta}{2} \tag{11.17}$$

The geometrical idea is fairly simple:
When looking for the ray in some face F with the smallest angle to the rays in some other face G, we might simply give a relaxation to this by choosing the ray in F closest to the hyperplane $\left\{ x\in\mathbb{R}^d \,:\, c(G)^T x = 0 \right\}$ in which G is contained.
This equivalent to the search for the ray in F with maximum angle to $-c(G)$, thus the bounds for this maximum angle found in Lemma 11.1.3 hold here, too.

Proof:
By the definition of $\angle_{min}(C)$ there is a pair of different faces $F_1, F_2 \in \mathcal{F}(C)$ with $\angle_{min}(F_1, F_2) = \omega < \frac{\pi}{2}$, which by definition are different from $\mathrm{lin}(U)$ and fulfill with $F_1 \cap F_2 = \mathrm{lin}(U)$.
Please recall that as $\dim(U) < d$–1 there is at least one pair of proper faces of $C^{[G]}$ fulfilling these constraints.

The value of $\angle_{min}(F_1, F_2)$ is measured relative to $\mathrm{lineal}(F_i)^\perp = \mathrm{lin}(G)^\perp$. Thus let $v_i \in F_i \cap G^\perp$, $i = 1, 2$ be the pair of non-zero vectors such that $\angle_{min}(F_1, F_2) = \angle(v_1, v_2)$. Then as both sets are cones we can assume w.l.o.g. $\|v_i\| = 1$, $i = 1, 2$.

From the definition of $\angle_{min}(\cdot)$ we know that $0 < \omega < \pi$.
In case $\omega > \pi/2$ we define $v_1' := v_1$ such that we have $\angle(v_1', v_2) = \pi - \omega < \pi/2$. As the function sin is symmetric on $[0, \pi]$ we find $\sin(\omega) = \sin(\angle(v_1', v_2))$.
When $\omega \leq \pi/2$ then we define $v_1' = v_1$.

In any case we have $\angle(c(F_1), v_1') = \pi/2$ as by choice $c(F_1)$ is a support vector of F_1. Further as $v_2 \notin F_1$ we find $0 \leq \angle(-c(F_1), v_1') < \pi/2$ which is equivalent to

$$0 < \pi/2 - \angle(-c(F_1), v_1') \leq \pi/2 \quad . \tag{11.18}$$

Using the triangle inequality (cf.[Cho70]) for angles we find

$$\begin{array}{rcrcll} & & \angle(-c(F_1), v_1') & \leq & \angle(-c(F_1), v_2) \ + & \angle(v_2, v_1') \\ \Leftrightarrow \ 0 & < & \pi/2 - \angle(-c(F_1), v_2) & \leq & & \angle(v_2, v_1') < \pi/2 \quad . \end{array} \tag{11.19}$$

Therefore we know

$$\begin{array}{lllll} 0 & < & -c(F_1)^T v_2 & = & \cos(\angle(-c(F_1), v_2)) \\ & & & = & \sin(\pi/2 - \angle(-c(F_1), v_2)) \\ & & & \leq & \sin(\angle(v_1', v_2)) \qquad = \sin(\omega) \end{array} \tag{11.20}$$

where the inequality derives from the bounds in (11.19) on the used angles and from the fact that the function sin operates monotonous on $[0, \pi/2]$.

We know that $C^{[G]}$ is of the form $\mathrm{cone}(V) \oplus \mathrm{lin}(U)$ thus we may we define V^1 such that $\mathrm{cone}(V^1) = F^1 \cap U^\perp$ as $F^1 \cap F_2 \cap U^\perp = \mathbf{0}$ we find that $F_2 \subset \mathrm{conv}(V) \setminus V^1)$. Thus we find using Lemma 11.1.3 (p. 66), that $\cos(\angle(-c(F_1), v_2)\,) \geq \delta$.

Further by $0 \leq \omega < \pi$ we know $\cos(\omega/2) > 0$ and therefore

$$\sin(\omega/2) \geq \sin(\omega/2)\cos(\omega/2) = \frac{1}{2}\sin(\omega) > 0 \quad ,$$

which proves the Lemma. □

12. The Intersection of Approximated Cones

12.1 Feasible Sets

In this section we will for the first time switch the form of presentation from an angle-oriented description to a primarily distance-oriented description.

When building $\mathcal{S}(C^{[F]}, \alpha)$ we have α restricted by '$\alpha \le \pi/4 - \gamma/2$'. In this section this turns to '$\sin(\alpha) \le (1/2)\cos(\gamma)$'. For the object '$\cos(\gamma)$' we have found a lower bound $M_c(C)$ in the previous chapter, Lemma 11.1.3 (p. 66) which we will use to restrict the feasible α in the following way:

Lemma 12.1.1. *Let $C = \operatorname{cone}(W) \subset \mathbb{R}^d$ be a d-dimensional pointed cone with a proper face $F \in \mathcal{F}(C)$ with $\dim(F) < d-1$. Further let $c(\mathcal{F})$ be any choice of support vectors for C, and $\varepsilon \in (0, (1/2)M_c(C)]$. Define*

$\alpha := \arcsin(\varepsilon)$, $r := M_c(C)$ and $R := (2/M_c(C))$.

Then for every face $F \in \mathcal{F}_k(C)$, $k \le d-2$ the set $\{C^{[F]}, c(F), \alpha, r, R\}$ is feasible for the construction of $\mathcal{S}(C^{[F]}, \alpha)$.

Remark 12.1.2. We state here that using $c(F)$ (as support vector of the apex of $C^{[F]}$) and r, R (as bounds to the relative diameter of the apex figure) the construction $\mathcal{S}(C^{[F]}, \alpha)$ according to Theorem 8.3.1 (p. 51) is possible.

Proof: Let ε be in $(0, (1/2)M_c(C)]$ and define $M_c := M_c(C)$. As we have proved in lemma 11.1.3 we find $0 < M_c < 1$ and thus $\bar{\gamma} := \arccos(M_c)$ is well defined with $0 < \bar{\gamma} < \pi/2$. Therefore defining $\beta := \pi/4 - \bar{\gamma}/2$ we find $0 < \beta < \pi/4$. Thus as $\sin(\beta) > 0$ and $\cos(\beta) > 0$ we find

$$\begin{array}{rcccl} \sin(\beta) & \ge & \sin(\beta)\cos(\beta) & = & (1/2)\sin(2\beta) \\ & & & = & (1/2)\sin(\pi/2 - \bar{\gamma}) \quad = \quad (1/2)\cos(\bar{\gamma}) > 0 \end{array}$$

Thus defining $0 < \varepsilon \le (1/2)M_c$ using the definition of $\bar{\gamma}$leads to $0 < \varepsilon \le \sin(\beta) < 1$ and thus as arcsin is monotonous we find $0 < \arcsin(\varepsilon) \le \pi/4 - \bar{\gamma}/2$ by definition of β.

Further we proved in Lemma 11.1.3 (p. 66) that $\gamma(F) \le \bar{\gamma}$, where

$$\gamma(F) := \max\{\angle(-c(F), w) : \ w \ne 0, \ w \in C^{[F]} \cap \operatorname{lineal}(C^{[F]})^{\perp} \cap H^c\}$$

and thus $\alpha \le \pi/4 - \gamma(F)/2$ which is the restriction to α given in Theorem 8.3.1.

Further the construction of $\mathcal{S}(C^{[F]}, \alpha)$ requires both a lower and upper bound to the diameter of the apex figure $P := C^{[F]} \cap \operatorname{lineal}(C^{[F]})^{\perp} \cap H^c$. In Lemma 11.1.5 (p. 73) it is proved that $0 < r \leq \operatorname{diam}(P) \leq R$.

Thus the values α, r, R and the vector $c(F)$ form a set fulfilling the requirements given in Theorem 8.3.1. $\square$

12.2 Interior Containment Restriction

To understand the following Lemma, it is necessary to recall that if we have faces $F_1, F_2, G \in \mathcal{F}(C)$ of a pointed cone C with $G = F_1 \cap F_2$ then $F_i + \mathrm{lin}(G)$ are special faces of $C^{[G]}$. Discussing $\angle_{min}(C^{[G]})$, we proved in Lemma 10.1.4 (p. 62) for faces of this type, that any $y \in C^{[G]}$ is 'far' from at least one of the faces $F_i + \mathrm{lin}(G)$, where the distance is measured relative to $\mathrm{d}(y, \mathrm{lin}(G))$.

We know from the construction of the $\mathcal{S}^{F_i} := \mathcal{S}(C^{[F_i]}, \alpha)$ that each set provides some space $B(y, \Delta_i(y))$ inside of $\mathcal{S}^{F_i}$ for every y in $C^{[F_i]}$ where the size of this space depends on the distance $\mathrm{d}(y, F_i)$.Knowing that $C^{[G]} \subset C^{[F_i]}$ we here give the 'translation' from $\mathrm{d}(y, \mathrm{lin}(F_i))$ to $\mathrm{d}(y, F_i + \mathrm{lin}(G))$ for all those y in $C^{[G]}$.

Lemma 12.2.1. *Let $C = \mathrm{cone}(W) \subset \mathbb{R}^d$ be a d-dimensional pointed cone.*
Let further $F, G \in \mathcal{F}(C)$ be two proper faces of C such that $\dim(F) \leq d-2$ and $G \subset F$.
For some choice of support vectors $c(\mathcal{F})$ let $0 < \varepsilon < (1/2)M_c(C)$ and define $\alpha := \arcsin(\varepsilon)$.
Using $r := M_c(C)$, $R := 2/(M_c(C))$ and $\bar{c} := c(F)$ construct $\mathcal{S}^F := \mathcal{S}(C^{[F]}, \alpha)$ according to theorem 8.3.1.
Then for any $y \in C^{[G]}$ we find

$$B(y, \Delta(y, F)) \subset \mathcal{S}^F \quad \textit{where} \;\; \Delta(y, F) = \varepsilon\, M\, \mathrm{d}(y, F + \mathrm{lin}(G)) \;\; \textit{and} \tag{12.1}$$

$$M := \frac{1}{4(d+1)^3} \left(\sqrt{1-\varepsilon^2}\right) (M_c(C))^5 \quad .$$

Remark 12.2.2. Recall that for the construction of $\mathcal{S}(C^{[F]}, \alpha)$ besides α a support vector $\bar{c}$ and the bounds $r \leq \mathrm{diam}(C^{[F]} \cap F^{\perp} \cap H^{\bar{c}}) \leq R$ need to be known.
Please notice that we use $\Delta(y)$ and $\Delta(y, F)$ in the following proof which are not equal.

Proof: Define by $\gamma := \max\{\angle(-c(F), w) : \; w \in C^{[F]} \cap \mathrm{lineal}(C^{[F]})^{\perp}\}$ the maximum angle between $-c(F)$ and the cross-section of $C^{[F]}$ perpendicular to its lineality space. Then from the construction of $\mathcal{S}(C, \alpha)$ in Theorem 8.3.1 (p. 51) it is known that for every $y \in C^{[F]}$ there is $B(y, \Delta(y)) \subset \mathcal{S}(C^{[F]}, \alpha)$, where $\Delta(y)$ is defined as follows:

$$\Delta(y) = \sin(\alpha)\, \mathrm{d}(y, \mathrm{lin}(F))\, \frac{r}{2R(d+1)^3} \cos(\alpha) \cos^2(\gamma).$$

In the previous chapter, Lemma 11.1.3 (p. 66) it is proven that $\cos(\gamma) \geq M_c(C)$, using the definitions of r, R and α this leads to the following lower bound to $\Delta(y)$:

$$\Delta(y) \;\; \geq \;\; \varepsilon\, \mathrm{d}(y, \mathrm{lin}(F))\, \frac{1}{4(d+1)^3} \left(\sqrt{1-\varepsilon^2}\right) (M_c(C))^4 \tag{12.2}$$

The only difference between the last term in (12.2) to the 'target' given in (12.1) is that there we have $\mathrm{d}(y, F + \mathrm{lin}(G))$ and $(M_c(C))^5$. Thus prove the lemma it suffices to show

$$\mathrm{d}(y, \mathrm{lin}(F)) \geq M_c(C)\, \mathrm{d}(y, F + \mathrm{lin}(G)) \tag{12.3}$$

which used in (12.2) leads to $\Delta(y) \geq \Delta(y, F)$, implying finally

$$B(y, \Delta(y, F)) \subset B(y, \Delta(y)) \subset \mathcal{S}(C^{[F]}, \alpha) \quad .$$

Let $W^F \subset W$ such that $F = \mathrm{cone}(W^F)$. As $\mathrm{cone}(W)$ is given in irreducible description, so is F, and thus $\mathrm{cone}(W \setminus W^F) \setminus F = \{\mathbf{0}\}$.

Let $y \in C^{[G]}$, then from $G \subset F$ it is clear that $\mathrm{lin}(G) \subset \mathrm{lin}(W^F)$ and thus we know by $C^{[G]} = \mathrm{cone}(W) + \mathrm{lin}(G)$ that we may split y into three parts $y = y' + u_F + u_G$, where $y' \in \mathrm{cone}(W \setminus W^F)$, $u_G \in \mathrm{lin}(G)$ and $u_F \in \mathrm{cone}(W^F)$.

If $y' = 0$ then $y = u_F + u_G \in F + \mathrm{lin}(G)$ and thus as $\mathrm{lin}(G) \subset \mathrm{lin}(F)$ we find

$$\mathrm{d}(y, \mathrm{lin}(F)) = M_c(C)\, \mathrm{d}(y, F + \mathrm{lin}(G)) = 0 \quad .$$

Else if $y' \neq 0$ we know from Lemma 11.1.3 (p. 66) that for $y' \in \mathrm{cone}(W \setminus W^F)$ we have $\cos(\angle(-c(F), y')) \geq M_c(C) > 0$. From this we conclude $c(F)^T y' \geq M_c(C)\, \|y'\|$, which is greater zero as $M_c(C) > 0$ and $y' \neq 0$.

Using this together with $\mathrm{lin}(F) \subset \{c(F)\}^\perp$ we find

$$\mathrm{d}(y, F) \geq \mathrm{d}(y', \{c(F)\}^\perp) = -c(F)^T y' \geq M_c(C)\, \|y'\| \quad .$$

As $u_F + u_G \in F + \mathrm{lin}(G)$ the value $\|y'\| = \|y - (u_F + u_G)\|$ is an upper bound to $\mathrm{d}(y, F + \mathrm{lin}(G))$ and thus

$$\mathrm{d}(y, F) \geq M_c(C)\, \mathrm{d}(y, F + \mathrm{lin}(G))$$

holds. In either case $y' = 0$, $y' \neq 0$ we have proven (12.3) which proves the lemma. □

Lemma 12.2.3. *Let $C = \mathrm{cone}(W) \subset \mathbb{R}^d$ be a d-dimensional pointed cone with any choice of support vectors $c(\mathcal{F})$. Let further $F, G \in \mathcal{F}(C)$ be two proper faces of C such that $\dim(F) \leq d-2$ and $G \subset F$, and define $\mathcal{S}^F := \{x \in \mathbb{R}^d : -c(F)^T x \geq 0\}$.*
Then for any $y \in C^{[G]}$ we find

$$B(y, \Delta(y, F)) \subset \mathcal{S}^F \quad \textit{where} \;\; \Delta(y, F) = M_c(C)\, \mathrm{d}(y, F + \mathrm{lin}(G)) \tag{12.4}$$

Proof: Let W^F be such that $F = \mathrm{cone}(W^F)$. Then by the restrictions on F and C any $y \in C^{[G]}$ can be decomposed into three parts $y = y' + u_g + u_F$ where $y' \in \mathrm{cone}(W \setminus W^F)$, $u_G \in \mathrm{lin}(G)$ and $u_F \in \mathrm{cone}(W^F)$.

If $y' = 0$ then $\mathrm{d}(y, F + \mathrm{lin}(G)) = \mathrm{d}(y, F) = 0$

Define the hyperplane bounding $\mathcal{S}^F$ by $H := \{x \in \mathbb{R}^d : c(F)^T x = 0\}$ then

$$\mathrm{d}(y', H) = \left|c(F)^T y'\right| = -c(F)^T \frac{y'}{\|y'\|}\, \|y'\| \geq M_c(C)\, \|y'\| \,.$$

where the last inequality follows from Lemma 11.1.3 (p. 66) as $y \in \mathrm{cone}(W \setminus W^F)$. With $\|y'\| = \|y' - u_G - u_F\|$ we find that $\|y'\|$ is an upper bound to $\mathrm{d}(y', F + \mathrm{lin}(G))$.

Further as $\mathcal{S}^F$ is a halfspace we know that for $y' \in \mathcal{S}^F$ there is $B(y', \Delta(y')) \subset \mathcal{S}^F$ where $\Delta(y') := \mathrm{d}(y', H)$.

And finally since $\mathrm{lin}(G) \subset \mathrm{lin}(F) \subset \{c\}^\perp$ we have $\mathrm{lin}(G) \subset \mathrm{lin}(F) \subset \mathcal{S}^F$. Therefore we find

$$B(y, \Delta(y)) = B(y', \Delta(y')) + u_G + u_F \subset \mathcal{S}^F.$$

$\square$

12.3 Intersection

The following Lemma is the key to the whole construction, so let's recall what is claimed here:
We state, that if α_k is chosen correctly, then constructing $\mathcal{S}^F := \mathcal{S}(C^{[F]}, \alpha_k)$ for every $F \in \mathcal{F}_k(C)$, $k \leq d{-}3$, these sets fulfill the containment restrictions (5.13) postulated in Proposition 5.2.1 (these restrictions are literarily the same as in (12.5), see below).

From Proposition 5.2.1 the Main Theorem 1.0.1 directly follows by the use of Lemma 5.3.2 (p. 28). There defining further $\mathcal{S}^F$ for any $F \in \mathcal{F}_{d-1}(C)$ it is proven that for all fixed $0 < k \leq d{-}1$ it is possible to multiply all the polynomials defining the $\mathcal{S}^F$ with $\dim(F) = k$ into (maximally) two polynomials, without changing the set

$$C = \bigcap_{i=0}^{d-1} \bigcap_{F \in \mathcal{F}_k(C)} \mathcal{S}^F.$$

Thus here we finally prove the Main Theorem.

Lemma 12.3.1 (Main Lemma). *Let $C \subset \mathbb{R}^d$, $d \geq 2$ be a d-dimensional pointed polyhedral cone, and let $c(\mathcal{F})$ be any choice of support vectors of C. With $M_c := M_c(C)$ define the constants*

$$M_1 := \frac{\left(\sqrt{1-(M_c)^2}\right)\,(M_c)^5}{4(d+1)^3} \qquad M_2 := \frac{M_c}{2}$$

and for every $i \in \{0, \ldots, d-3\}$ define $\alpha_i := \arcsin(\varepsilon_i)$ where ε_i is defined by

$$\varepsilon_i := (M_2)^2\ (M_1\, M_2)^{(d-3-i)} \quad .$$

With these let

$$\begin{array}{lcll} \mathcal{S}^F & := & \{x \in \mathbb{R}^d : -c(F)^T x \geq 0\} & \text{for every } F \in \mathcal{F}_{d-2}(C), \\ \mathcal{S}^F & := & \mathcal{S}(C^{[F]}, \alpha_k) & \text{for every } F \in \mathcal{F}_k(C), \quad k \leq d{-}3, \end{array}$$

where $\mathcal{S}(C^{[F]}, \alpha_k)$ is constructed as done in Lemma 12.1.1.
Then the sets $\mathcal{S}^F$ have the following properties: Let $F_1, F_2, G \in \mathcal{F}(C)$ with $0 < \dim(F_1) = \dim(F_2) \leq d{-}2$ and $G := F_1 \cap F_2$. Then

$$\mathcal{S}^G \subset \operatorname{int}_Z(\mathcal{S}^{F_1}) \cup \operatorname{int}_Z(\mathcal{S}^{F_2}) \cup \left(\operatorname{clos}_Z(\mathcal{S}^{F_1}) \cap \operatorname{clos}_Z(\mathcal{S}^{F_2})\right) \quad . \tag{12.5}$$

Corollary 12.3.2. *The Main Theorem 1.0.1 holds, since Lemma 12.3.1 directly proves Proposition 5.2.1 ("Target of Construction") from which the Main Theorem was concluded in Lemma 5.3.2.*

Proof: Let $\alpha_0, \ldots, \alpha_{d-3}$ be defined as above. A first we show that the α_i are well defined, and that they meet the restrictions on the construction of $\mathcal{S}(C^{[F]}, \alpha_k)$ given in Lemma 12.1.1 (p. 73).

Feasibility

By construction in Definition 11.1.1 (p. 65), $M_c(C)$ is restricted by $M_c(C) \in (0,1)$. Therefore $M_i \in (0,1)$ holds for both $i = 1,2$ implying that the defined ε_i are rising: $0 < \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_{d-3} < 1$. Accordingly the same holds for α_i as the function arcsin operates monotonously: $0 < \alpha_0 < \ldots < \alpha_{d-3}$.

Since $\varepsilon_i < (M_2)^2 < (1/2)M_c$ holds for all $i = 0,\ldots,d-3$ using Lemma 12.1.1 and defining $r := M_c$ and $R := 2/M_c$, we conclude that for every $F \in \mathcal{F}_k(P)$ with $k \le d-2$ the set $\{C^{[F]}, c(F), \alpha_k, r, R\}$ is feasible for construction of $\mathcal{S}(C^{[F]}, \alpha_k)$. Thus let

$$\mathcal{S}^F \;\; := \;\; \mathcal{S}(C^{[F]}, \alpha_k) \quad \text{for every} \quad F \in \mathcal{F}_k(C),\; k \le d-3$$

be constructed using r, R and $c(F)$. And define

$$\mathcal{S}^F \;\; := \;\; \left\{ -c(F)^T x \ge 0 \right\} \quad \text{for every} \quad F \in \mathcal{F}_{d-2}(C).$$

We show that the restrictions on the $\mathcal{S}^F$ given in the Main Lemma hold for these sets: Let $0 < k \le d-2$ and F_1, F_2 any two faces in $\mathcal{F}_k(C)$, $F_1 \ne F_2$. Then as C is a pointed cone, F_1 and F_2 intersect in some face $G \in \mathcal{F}_j(C)$, $0 \le j < k$. We will show that from the interior containment restrictions on $\mathcal{S}^{F_1}$, $\mathcal{S}^{F_2}$ and from the exterior restrictions on $\mathcal{S}^G$ we might directly prove Lemma 12.3.1.

Containment Restrictions on $\mathcal{S}^G$

We find a set containing $\mathcal{S}^G$, given by the exterior containment restrictions found in the construction of $\mathcal{S}(C^{[G]}, \alpha_j)$ in Theorem 8.1.1 (p. 49) as follows:

$$\mathcal{S}^G \subset \bigcup_{x \in C^{[G]}} B(x, \varepsilon(x)) \quad \text{where } \varepsilon(x) := \sin(\alpha_j)\; \mathrm{d}(x, \mathrm{lin}(G)) \quad .$$

In order to prove this lemma we show that this set is a subset of

$$\mathrm{int}_Z(\mathcal{S}^{F_1}) \cup \mathrm{int}_Z(\mathcal{S}^{F_2}) \cup \left(\mathrm{clos}_Z(\mathcal{S}^{F_1}) \cap \mathrm{clos}_Z(\mathcal{S}^{F_2})\right).$$

For any $x \in \mathrm{lin}(G)$ clearly $\varepsilon(x, G) = 0$ holds and therefore

$$\bigcup_{x \in \mathrm{lin}(G)} B(x, \varepsilon(x)) \;=\; \mathrm{lin}(G) \;\subset\; \mathrm{clos}_Z(\mathcal{S}^{F_1}) \cap \mathrm{clos}_Z(\mathcal{S}^{F_2})$$

holds, since by choice of G we have $\mathrm{lin}(G) = \mathrm{lin}(F_1 \cap F_2) \subset \mathrm{lin}(F_1) \cap \mathrm{lin}(F_2)$ and as proven in Lemma 8.4.1 $\mathrm{clos}_Z(\mathcal{S}^{F_i}) = \mathrm{lin}(F_i)$ for both $i = 1,2$.

So to prove the lemma we will show that for every $y \in C^{[G]} \setminus \mathrm{lin}(G)$ we have

$$B(y, \varepsilon(y)) \subset \mathrm{int}_Z(\mathcal{S}^{F_1}) \cup \mathrm{int}_Z(\mathcal{S}^{F_2}) \quad . \tag{12.6}$$

Such assume $y \in C^{[G]} \setminus \mathrm{lin}(G)$.
By the choice of $G = F_1 \cap F_2$ it is clear that $\mathrm{lin}(G) = \mathrm{lin}(F_1) \cap \mathrm{lin}(F_2) \cap C^{[G]}$. Thus y can not be both in $F_1 + \mathrm{lin}(G)$ and $F_2 + \mathrm{lin}(G)$. Otherwise $y \in \mathrm{lin}(F_1) \cap \mathrm{lin}(F_2)$ leads to the contradiction $y \in \mathrm{lin}(G)$.

$$k = d-2$$

Assume $\dim(F_i) = k = d-2$ then the sets $\mathcal{S}^{F_i} := \{x \in \mathbb{R}^d : -c(F_i) \geq 0\}$ are halfspaces and so Lemma 12.2.3 proves that for both F_i, $i = 1, 2$ for any $y \in C^{[G]}$ we have

$$B(y, \Delta(y, F_i)) \subset \mathcal{S}^{F_i} \quad \text{where } \Delta(y, F_i) := M_c \, \mathrm{d}(y, F_i + \mathrm{lin}(G)) \quad . \tag{12.7}$$

Further, independently of the dimension of the F_i, from Lemma 10.1.4 we know that by choice of $G = F_1 \cap F_2$ the points in $C^{[G]}$ are 'far' from at least one of its faces $F_i + \mathrm{lin}(G)$, specifically

$$\max_{i=1,2} \mathrm{d}(y, F_i + \mathrm{lin}(G)) \quad \geq \quad \sin\left(\frac{\omega}{2}\right) \, \mathrm{d}(y, \mathrm{lin}(G)), \tag{12.8}$$

where $\omega := \angle_{min}(C^{[G]})$. Lemma 11.2.1 provides a lower bound for this by proving $\sin(\omega/2) \geq M_c/2$. Using this with the restrictions on the $\mathcal{S}^{F_i}$ in (12.7), yields

$$\begin{array}{rcll} \max_{i=1,2} \Delta(y, F_i) & \geq & M_c \, (M_c/2) & \mathrm{d}(y, \mathrm{lin}(G)) \\ & > & \varepsilon_{d-3} & \mathrm{d}(y, \mathrm{lin}(G)) \quad . \end{array}$$

Here the last strict inequality follows since $\mathrm{d}(y, \mathrm{lin}(G)) \neq 0$ by choice of $y \notin \mathrm{lin}(G)$ and as $(M_c)^2/2 > \varepsilon_{d-3}$ since by definition $\varepsilon_{d-3} = (M_c)^2/4$.

We assume w.l.o.g. that F_1 is such that $\Delta(y, F_1) > \varepsilon_{d-3}\mathrm{d}(y, \mathrm{lin}(G))$. Since $G = F_1 \cap F_2$ we must have $j = \dim(G) \leq d-3$, implying $\varepsilon_j \leq \varepsilon_{d-3}$ and thus

$$\varepsilon(y) = \varepsilon_j \mathrm{d}(y, G) \leq \varepsilon_{d-3}\mathrm{d}(y, G) < \Delta(y, F_1)$$

must hold. This proves

$$B(y, \varepsilon(y)) \subsetneq \overset{\circ}{B} (y, \Delta(y, F_1)) \subset \mathrm{int}_Z(\mathcal{S}^{F_1}) \quad .$$

Where $\overset{\circ}{B} (x, \delta)$ denotes the open ball with radius δ wrapping x. The last inclusion holds due to the special situation that the topological interior of any set $\mathcal{S}(C, \alpha)$ is equal to its algebraical interior $\mathrm{int}_Z(\mathcal{S}(C, \alpha))$, proved in Lemma 8.4.1.

The proof for $k \leq d-3$ follows the same structure as the proof for $k = d-2$.

$$0 < k \leq d-3$$

Assume $0 \leq k \leq d$–3 then for the the sets $\mathcal{S}^{F_i} := \mathcal{S}(C^{[F_i]}, \alpha_k)$ Lemma 12.2.1 (p. 75) proves that for both F_i, $i = 1, 2$ and for any $x \in C^{[F]}$ we have

$$B(x, \Delta(x, F_i)) \subset \mathcal{S}^{F_i} \quad \text{where } \Delta(x, F_i) := \varepsilon_{d-2} \, L_k \, \mathrm{d}(x, F_i + \mathrm{lin}(G)) \tag{12.9}$$

$$\text{and} \qquad L_k := \frac{\left(\sqrt{1 - (\varepsilon_k)^2}\right) \, (M_c)^5}{4(d+1)^3}. \tag{12.10}$$

Further, as discussed above in (12.8) and (12.9), independently of the dimension of the F_i, $i = 1, 2$, from Lemma 10.1.4 and Lemma 11.2.1 follows, that by choice of $G = F_1 \cap F_2$ we have

$$\max_{i=1,2} \Delta(y, F_i) \quad \geq \quad \varepsilon_k \, L_k \, (M_c/2) \, \mathrm{d}(y, \mathrm{lin}(G)) \quad . \tag{12.11}$$

Since as proven $\varepsilon_i < M_c$, holds for all $i = 0, \dots, d-2$, L_k and M_1 differ only in the two following terms $\sqrt{1-(\varepsilon_k)^2} > \sqrt{1-(M_c)^2}$ and thus $L_k > M_1$. This implies

$$\varepsilon_k \, L_k \, (M_c/2) > \varepsilon_k \, M_1 \, (M_c)/2 = \varepsilon_{k-1} \quad .$$

Let w.l.o.g. F_1 be such that

$$\mathrm{d}(y, F_1 + \mathrm{lin}(G)) = \max_{i=1,2} \{ \mathrm{d}(y, F_i + \mathrm{lin}(G)) \} \, ,$$

Then $\mathrm{d}(y, \mathrm{lin}(G)) \neq 0$ by choice of $y \notin \mathrm{lin}(G)$, and since from $j = \dim(G) < \dim(F_i) = k$ follows $\varepsilon_j \leq \varepsilon_{k-1}$, we find that

$$\varepsilon(y) = \varepsilon_j \mathrm{d}(y, G) \leq \varepsilon_{k-1} \mathrm{d}(y, G) < \Delta(y, F_1)$$

and thus

$$B(y, \varepsilon(y)) \subsetneq \overset{\circ}{B} \, (y, \Delta(y, F_1)) \subset \mathrm{int}_Z(\mathcal{S}^{F_1}) \quad .$$

where the last inclusion holds again as the topological interior of any set $\mathcal{S}(C, \alpha)$ is equal to its algebraical interior $\mathrm{int}_Z(\mathcal{S}(C, \alpha))$, proved in Lemma 8.4.1. $\square$

13. Construction

In the previous chapter we have proved the Main Theorem by showing that the objects postulated in Proposition 5.2.1 (p. 28) exist. There we claimed that there is a semi-algebraic set $\mathcal{S}^F$ for every face $F \in \mathcal{F}(C)$, $k \leq d$–3 such that the intersection of all $\mathcal{S}^F$ yields C, and further the polynomials used to describe the $\mathcal{S}^F$ can be multiplied into a constant number of polynomials without changing the semi-algebraic set C they describe.

Up to now we proved that these objects are sets $\mathcal{S}^F = \mathcal{S}(C^{[F]}, \alpha_k)$ for every face $F \in \mathcal{F}_k(C)$, $k \leq d-3$ and $\mathcal{S}^F = \{x \in \mathbb{R}^d \,:\, -c(F)^T x \geq 0\}$ for any $F \in \mathcal{F}_k(C)$, $k \geq d-2$.

In this chapter we first provide the necessary constructions for the polynomials of the sets $\mathcal{S}(C^{[F]}, \alpha_k)$. Then we will merge the descriptions of these sets $\mathcal{S}^F$ into a single description for a pointed polyhedral cone.

Finally we will provide an explicit construction for the necessary $2d-1$ polynomials needed to describe a d-dimensional polytope.

13.1 Explicit Construction for an Approximation of a Relaxation Cone

In this section we provide an explicit version of Theorem 8.3.1 (p. 51), where we constructed $\mathcal{S}(C, \alpha)$ implicitly, based on the construction of $\mathcal{K}(P, \varepsilon)$ found in Theorem 6.3.3 (p. 35). We have spared this calculation up to now, to be able to use the bounds on $h_P(\bar{a}_i)$ and $h_P(-\bar{a}_i)$ calculated in the previous chapter. To understand the proof given, it will thus be necessary to review the constructions previously made.

In this construction we make use of the number $\#[F]$, denoting the number of elements in $[F]$.

Theorem 13.1.1. *Let $d \geq 2$ and $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i = 1, \ldots, m\} = \operatorname{cone}(W)$ be a d-dimensional polyhedral cone given in irreducible double description and $F \in \mathcal{F}_k(P)$ where $k \leq d-2$. Let further $c(\mathcal{F}(C))$ be any choice of normalized support vectors of C and let $0 < \varepsilon \leq (1/2) M_c(C)$.*

Define $\delta := \frac{1}{2(d+1)}\, \varepsilon\, M_c(C)$, $m := \#[F]$ and $k \in \mathbb{N}$ such that $k > m\left(2\ln(1 + \frac{\delta}{1+\delta})\right)^{-1}$. Then *with the following definitions*

$$h_i^F \;:=\; \max\{\frac{a_i^T w}{c(F)^T w} \,:\, c(F)^T w \neq 0,\ w \in W\} \qquad \textit{for all } i \in [F] \quad , \tag{13.1}$$

$$\mathfrak{p}_1^F(x) \;:=\; (2c^T x)^{2k} - \sum_{i \in [F]} \frac{1}{m\,(1+\delta)^{2k}} \left(\frac{2a_i^T x - (c(F)^T x)\, h_i^F}{h_i^F} \right)^{2k} \tag{13.2}$$

$$\mathfrak{p}_2^F(x) \;:=\; -c(F)^T x \quad , \tag{13.3}$$

the set $\mathcal{S}(C^{[F]}, \alpha)$ *constructed using the values* $\{C^{[F]}, c(F), \alpha, r, R\}$ *where and* $\alpha := \arcsin(\varepsilon)$, $r := M_c(C)$ *and* $R := 2/M_c(C)$ *is given by*

$$\mathcal{S}(C^{[F]}, \alpha) = \{\mathfrak{p}_i^F(x) \geq 0,\ i = 1, 2\} \quad .$$

Remark 13.1.2. In the following proof mind the difference between a_i as 'original' vectors used to define $C^{[F]}$ and $\bar{a}_i$ defining the apex figure of $C^{[F]}$.

Proof: In the following we construct $\mathcal{S}(C^{[F]}, \alpha)$ based on the set $\{C^{[F]}, c(F), \alpha, r, R\}$. We recall from Lemma 12.1.1, that such construction is possible due to the restriction of ε.

Further since $\mathcal{S}(C^{[F]}, \alpha)$ is of the form $\{x \in \mathbb{R}^d \,:\, \mathfrak{p}_\alpha(x) \geq 0,\ -c(F)^T x \geq 0\}$, to prove the theorem it suffices to show that $\mathfrak{p}_\alpha(x) = \mathfrak{p}_1^F(x)$. To condense notation in the following let $\bar{c} := c(F)$.

Brief Recapitulation

The construction for $\mathfrak{p}_\alpha$, given in Theorem 8.3.1 (p. 51) follows three steps: First the apex figure P of $C^{[F]}$ is calculated, this polyhedron then is approximated by a set $\mathcal{K}(P, \varepsilon) = \{x \in \mathbb{R}^d \,:\, \mathfrak{p}_\varepsilon(x) \geq 0\}$ where

$$\mathfrak{p}_\varepsilon(x) \;=\; 1 - \sum_{i=1}^{m} \mu_i \left(\bar{a}_i^T x - d_i\right)^{2k} \quad \text{where } \mu_i > 0 \text{ and } k \in \mathbb{N} \quad .$$

Finally the polynomial $\mathfrak{p}_\alpha(x)$ is derived from $\mathfrak{p}_\varepsilon(x)$ by the following:

$$\mathfrak{p}_\alpha(x) \;=\; \mathfrak{p}_\varepsilon\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x)^{2k} \tag{13.4}$$

$$\;=\; (\bar{c}^T x)^{2k} - \sum_{i=1}^{m} \mu_i \left(\bar{a}_i^T x + (\bar{c}^T x) d_i\right)^{2k} \quad . \tag{13.5}$$

Construction of $\mathcal{K}(P, \varepsilon)$

Recall that the apex figure of $C^{[F]}$ (see Lemma 8.2.1) is defined by

$$P \;:=\; \left\{\, x \in \mathbb{R}^d \,:\, \bar{a}_i^T x \leq \bar{c}^T a_i \,, i \in [F] \,\right\}, \quad \text{where } \bar{a}_i := a_i - (\bar{c}^T a)\bar{c}, \tag{13.6}$$

According to the construction given in Theorem 6.3.3 (p. 35) we use δ and k as defined in the given theorem to construct $\mathfrak{p}_\varepsilon$, since they fit the restrictions given there:

$$\delta := \frac{1}{2(d+1)}\varepsilon M_c(C) \stackrel{!}{=} \frac{1}{R(d+1)}\varepsilon \qquad \text{and } k \stackrel{!}{>} m\left(2\ln(1+\frac{\delta}{1+\delta})\right)^{-1} .$$

With these values The polynomial $\mathfrak{p}_\varepsilon$ is constructed as follows: For every $i \in [F]$ let

$$\mathfrak{v}_i(x) := \frac{2\bar{a}_i^T x - h_P(\bar{a}_i) + h_P(-\bar{a}_i)}{h_P(\bar{a}_i) + h_P(-\bar{a}_i)} \quad .$$

Then $\mathfrak{p}_\varepsilon$ is given by

$$\mathfrak{p}_\varepsilon(x) := 1 - \sum_{i\in[F]} \frac{1}{m} \left(\frac{\mathfrak{v}_i(x)}{1+\delta}\right)^{2k} \quad .$$

With this definition we find that $\mathfrak{p}_\alpha$ is given by

$$\begin{aligned} \mathfrak{p}_\alpha(x) &= \mathfrak{p}_\varepsilon\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x)^{2k} \\ &= (\bar{c}^T x)^{2k} - \sum_{i\in[F]} \frac{1}{m(1+\delta)^{2k}} \left(\mathfrak{v}_i\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x)\right)^{2k} \quad . \end{aligned}$$

and thus similar to the polynomial $\mathfrak{p}_2^F$ in (13.3), to prove they are equal it suffices to prove

$$\mathfrak{v}_i\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x) = \frac{-2a_i^T x + (\bar{c}^T x)\, h_i^F}{h_i^F} \quad , \tag{13.7}$$

where the change of sign on the right-hand side relative to the definition of $\mathfrak{p}_1^F$ given in (13.2) is compensated by the power $2k$.

In Lemma 11.1.5 (p. 68) we provided a calculation for the values of $h_P(\bar{a}_i)$ and $h_P(\bar{a}_i)$, using the constant h_i^F defined here in (13.1). We proved the following there for any $i \in [F]$:

$$\begin{aligned} h_P(\bar{a}_i) &= c(F)^T a_i \quad , \\ h_P(-\bar{a}_i) &= -c(F)^T a_i + h_i^F \quad , \end{aligned}$$

mind $a_i \neq \bar{a}_i$ here. Using these values we examine the polynomial

$$\mathfrak{v}_i\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x) = \frac{-2\bar{a}_i^T x + (\bar{c}^T x)\,(-h_P(\bar{a}_i) + h_P(-\bar{a}_i))}{h_P(\bar{a}_i) + h_P(-\bar{a}_i)} \quad . \tag{13.8}$$

Numerator:
We find that $-h_P(\bar{a}_i) + h_P(-\bar{a}_i) = -2\bar{c}^T a_i + h_i^F$ and thus

$$(\bar{c}^T x)\,(-h_P(\bar{a}_i) + h_P(-\bar{a}_i)) = (\bar{c}^T x)\left(-2(\bar{c}^T a_i) + (h_i^F)\right) \quad .$$

Further $\bar{a}_i^T x = a_i^T x - (\bar{c}^T a_i)(\bar{c}^T x)$ due to the definition of $\bar{a}_i = a_i - (\bar{c}^T a_i)\bar{c}$. Therefore the numerator of the polynomial in (13.8) is given by

$$\begin{array}{rlcccccccl} & -2\bar{a}_i^T x & + & (\bar{c}^T x) & (& -h_P(\bar{a}_i) & + & h_P(-\bar{a}_i) &) & \\ = & -2a_i^T x + 2(\bar{c}^T a_i)(\bar{c}^T x) & + & (\bar{c}^T x) & (& -2(\bar{c}^T a_i) & + & h_i^F &) & = \ -2a_i^T x + (\bar{c}^T x)\, h_i^F \end{array}$$

Denominator:
We find that $h_P(\bar{a}_i) + h_P(-\bar{a}_i) = h_i^F$. Thus the the expression in (13.7) holds, as

$$\mathfrak{v}_\varepsilon\left(\frac{x}{-\bar{c}^T x}\right)(\bar{c}^T x) = \frac{-2a_i^T x + (\bar{c}^T x) h_i^F}{h_i^F} \quad .$$

This proves the lemma, as from (13.7) we directly conclude
$\mathcal{S}(C^{[F]}, \alpha) = \{x \in \mathbb{R}^d \,:\, \mathfrak{p}_i^F(x) \geq 0,\ i = 1, 2\}$. □

13.2 Construction for Pointed Cones

Theorem 13.2.1. *Let $d \geq 2$ and $C = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0 \,, i=1,\ldots,m\} = \mathrm{cone}(W)$ be a d-dimensional polyhedral cone given in irreducible double description.*
And let $c(\mathcal{F}(C))$ be any choice of normalized support vectors of C and with these define

$$\begin{aligned} M_c &= \min_{G\in\mathcal{F}(C)} \min\{-c(G)^T w : w \in W,\; c(G)^T w \neq 0\} \\ M &:= \frac{1}{8(d+1)^3}\,(M_c)^6\,\sqrt{1-(M_c)^2} \end{aligned}$$

For each $n \in \{0,\ldots,d-3\}$ define $\delta_n := (1/4)\,(M_c)^2\,(M)^{(d-3-n)}$ and $k_n \in \mathbb{N}$ such that

$$k_n > m\left(2\ln(1+\frac{\delta_n}{1+\delta_n})\right)^{-1} \quad ,$$

for every $F \in \mathcal{F}_n(C)$ let $\#[F]$ denote the number of elements in $[F]$ and define

$$h_i^F := \max\{\frac{a_i^T w}{c(F)^T w} \,:\, c(F)^T w \neq 0,\; w \in W\} \qquad \text{for all } i \in [F] \tag{13.9}$$

$$\mathfrak{p}_1^F(x) := (2c^T x)^{2k_n} - \sum_{i\in[F]} \frac{1}{\#[F]}\left(\frac{2a_i^T x - c^T x\, h_i^F}{(1+\delta_n)\, h_i^F}\right)^{2k_n} \tag{13.10}$$

$$\mathfrak{p}_2^F(x) := -c(F)^T x \tag{13.11}$$

Define $\mathfrak{P}_1^k(x) := \prod_{F\in\mathcal{F}_k(C)}(\mathfrak{p}_1^F(x))$ for every $k \in \{0,\ldots,d-3\}$.

Define $\mathfrak{P}_2^k(x) := \prod_{F\in\mathcal{F}_k(C)}(\mathfrak{p}_2^F(x))$ for every $k \in \{0,\ldots,d-1\}$.

Then with $\mathfrak{P}_1^k = 0$ for $k = (d-1),(d-2)$ we obtain that

$$C = \left\{\, x\in\mathbb{R}^d \,:\, \mathfrak{P}_i^k(x) \geq 0,\; k \in \{0,\ldots,d-1\},\; i = 1,2 \,\right\}.$$

Proof: For every $F \in \mathcal{F}_k(C)$, $k \leq d-3$ let

$$\mathcal{S}^F := \{x \in \mathbb{R}^d \,:\, \mathfrak{p}_i^F(x) \geq 0,\ i = 1,2\} \quad , \tag{13.12}$$

with the $\mathfrak{p}_i^F$ given in the construction. For every $F \in \mathcal{F}_k(C)$ with $k \geq d-2$ define $\mathcal{S}^F := \{x \in \mathbb{R}^d \,:\, -c(F)^T x \geq 0\}$.

With these definitions and recalling the results of Proposition 5.2.1 and Lemma 12.3.1, it suffices to prove $\mathcal{S}^F = \mathcal{S}(C^{[F]}, \alpha_k)$ for any $F \in \mathcal{F}_k(C)$, $k \leq d-3$ where $\alpha_k := \arcsin(\varepsilon_k)$ is defined as in Lemma 12.3.1,

$$\varepsilon_k := (1/2)\,(M_c)^2\,(M)^{d-2-k} \quad .$$

Using these values with the explicit construction of $\mathcal{S}(C^{[F]}, \alpha_k)$ given in Lemma 13.1.1, leads to the fact that the values of h_i^F and $\delta = \delta_k$ defined here in this theorem are equal to those used in the construction there, and thus the Lemma states that defining

$$\mathfrak{q}_1(x) \;:=\; (2c^T x)^{2k} - \sum_{i \in [F]} \frac{1}{m\,(1+\delta)^{2k}} \left(\frac{2a_i^T x - (c(F)^T x)\,h_i^F}{h_i^F} \right)^{2k} , \tag{13.13}$$

$$\mathfrak{q}_2(x) \;:=\; -c(F)^T x \quad , \tag{13.14}$$

the following holds:

$$\mathcal{S}(C^{[F]}, \alpha) = \{\mathfrak{q}_i(x) \geq 0,\ i = 1,2\} \quad .$$

Clearly $\mathfrak{p}_1^F = \mathfrak{q}_1$ and $\mathfrak{p}_2^F = \mathfrak{q}_2$, thus $\mathcal{S}^F = \mathcal{S}(C^{[F]}, \alpha_k)$. $\square$

13.3 Linear Basic Open Semi-algebraic Sets

Here we finally prove that any basic *open* semi-algebraic set in $\mathbb{R}^d$ can be described by at most $2d$-many strict polynomial inequalities.

Theorem 13.3.1. *Every basic open semi-algebraic set given by*

$$\mathcal{S} := \{x \in \mathbb{R}^d \,:\, \mathfrak{q}_1(x) > 0, \dots, \mathfrak{q}_n(x) > 0\},$$

where for all $i \in \{1, \dots, n\}$ the polynomial $\mathfrak{q}_i \in \mathbb{R}[X_1, \dots, X_d]$ is of degree 1, can be described by at most $2d$-many polynomial inequalities.

Proof: Clearly the topological closure of $\mathcal{S}$ is a polyhedron

$$P := \{x \in \mathbb{R}^d \,:\, \mathfrak{q}_1(x) \geq 0, \dots, \mathfrak{p}_n(x) \geq 0\},$$

which can be described using $2d$ polynomials as proven. Let P be given by the polynomials constructed for its homogenization cone according to the previous construction:

$$P := \{x \in \mathbb{R}^d \,:\, \mathfrak{P}_1(x) \geq 0, \dots, \mathfrak{P}_{2d}(x) \geq 0\},$$

Then since every point y in the boundary of P is represented by a point $y' = (y^T, 1)^T$ in the boundary of $\mathrm{homog}(P)$ we find that through the according face $F \in \mathcal{F}(\mathrm{homog}(P))$ in which y' is contained, there is a polynomial $\mathfrak{p}_2^F(x) = -c(F)^T x$ such that $\mathfrak{p}_2^F(y') = \mathfrak{p}_2^F\left((y^T, 1)^T\right) = 0$.

Further every point y in the interior of P is represented by a point $y' = (y^T, 1)^T$ in the interior of $\mathrm{homog}(P)$. Therefore by Lemma 8.4.1 we find $\mathfrak{p}_i^F(y') > 0$ for every $F \in \mathcal{F}(\mathrm{homog}(P))$ and every $i = 1, 2$. Therefore $\mathfrak{P}_i^k(y) > 0$ must hold for any $i = 1, 2$ and $k \in \{0, \dots d-1\}$.

This proves

$$\mathring{P} := \{x \in \mathbb{R}^d \,:\, \mathfrak{P}_1(x) > 0, \dots, \mathfrak{P}_2 d(x) > 0\},$$

which proves the Theorem. $\square$

14. Review

14.1 Further Approaches

14.1.1 Colorings

In this thesis we follow the most intrinsic approach when trying to reduce the inequalities a polyhedron $P = \left\{ x \in \mathbb{R}^d \,:\, a_i^T x \leq b_i \,, i = 1, \ldots, m \right\}$ is given in:
The First idea will certainly be to multiply the polynomials $b_i - a_i^T x$ which describe the polyhedron into a single polynomial. This seems reasonable since if $P = \{x \in \mathbb{R}^d \,:\, \mathfrak{p}_1 \geq 0, \ldots, \mathfrak{p}_n \geq 0\}$ then each polynomial $b_i - a_i^T x$ is a factor of at least one of the polynomials $\mathfrak{p}_i$ (cf. [GH]).

This might not always be the best solution, which is motivated by Example 4.2.1 where a description of the crosspolytope in $\mathbb{R}^3$ is given, involving three polynomial inequalities. There the facet-defining inequalities where first separated into two groups, and then multiplied.

This separation implicitly used a 2-coloring of the facets: Every facet is colored by a color $c \in \{0, 1\}$ such that each two faces intersecting in a ridge (edge) are colored differently. Such a 2-coloring is possible for those polyhedra in $\mathbb{R}^3$ where each vertex is the intersection of an even number of facets, since the according graph of adjacent faces is bipartite [GLS88].

Further in Lemma 5.1.1 it shows that the ridges of a polyhedron are important to any approach of multiplying the polynomials defining P. There we proved:
For every $y \in \mathbb{R}^d$ which which is part of the positive product of all inequalities of P but not in P, there is a ridge G of P, whose defining inequalities both are violated by y, at least one of them strictly.

Thus it seems to be a good approach to first separate the polynomials $b_i - a_i^T x$ inducing a facet $F_i = P \cap \{x \in \mathbb{R}^d \,:\, a_i^T x = b_i\}$, such that each two i, j are separated, if the facets F_i, F_j intersect in a ridge. This works well in $\mathbb{R}^3$ but must fail in higher dimensions as a general approach. Still this could work well with special classes of polyhedra.

14.1.2 Cones close to the Object

In this thesis we make use of cones $\mathcal{S}(C, \alpha)$ which are not 'close' to the approximated object C. This is done to avoid that the hulls of two cones $\mathcal{S}^F, \mathcal{S}^G$ intersect on the surface of C elsewhere as in the intersection of their apexes.

Nevertheless there is a way of using cones whose hulls intersect with the given object. A simple example (which we do not prove) is the 'upper cone' of the crosspolytope in $\mathbb{R}^3$ given by $C := \{x \in \mathbb{R}^3 : (1, \pm1, \pm1)\, x \leq 1\}$ where $(1, \pm1, \pm1)$ denotes all four possible vectors $(1, 1, 1), \ldots, (1, -1, -1)$.

This object can be described as result set of three polynomial inequalities where one polynomial is the product of all polynomials $1-(1, \pm1, \pm1)\, x$ and the other two polynomial inequalities are $(1 - x_1)^2 - (x_2)^2 - (x_3)^2 \geq 0$ and $1 - x_3 \geq 0$. The cone $\{x \in \mathbb{R}^d : (1-x_1)^2-(x_2)^2-(x_3)^2 \geq 0,\ 1-x_1 \geq 0\}$ contains the extremal rays of C and is thus 'close' to C.

Using such cones it should be possible to find a general construction for a description in few polynomials at least for the d-crosspolytope .

14.2 Implicitly used Concepts

14.2.1 Using Cones

The main construction in this work is an appropriate approximation of a cone. The reason why cones are used is based on a property of semi-algebraic sets. While for disjoint closed convex objects there always is a separating hyper-plane, given two disjoint semi-algebraic sets A, B in $\mathbb{R}^d$ it is not always possible to separate them, using only a *single* polynomial inequality. Nevertheless it is possible to separate A and B by a function of the form

$$f = \sum_{i=1}^{m} \mathfrak{p}_i \sqrt{1 + \sum_{j=1}^{l} \mathfrak{q}_{i,j}}$$

where $\mathfrak{p}_i, \mathfrak{q}_{i,j} \in \mathbb{R}[X_1, \ldots, X_d]$ are polynomials [BCR98]. These functions meet the form of the polynomials we are using in this thesis.

Further from the theory on manifolds it is known that the result sets of polynomial equalities are local manifolds, where the singularities are excepted. If for example we intend to separate the Triangle in Figure 14.2.1 (A) from the two additional segments, then any polynomial without a singularity at the vertex v of the triangle, has a well defined tangential space T_vM at the point v, alike the one in (C). So the set chosen in (B) is either not the result set of a polynomial, or the according polynomial must have a singularity in v.

14.2.2 Using the Concept of Homogenization on Semi-Algebraic Sets

In this thesis we use the concept of homogenization for *both* convex polyhedra *and* convex semi-algebraic sets, the class of semi-algebraic sets which the whole theorem is based on are merely homogenizations of other well-known semi-algebraic sets.

Let $P = \left\{ x\in\mathbb{R}^d : a_i^T x \leq b_i\ , i=1,\ldots,m \right\}$ be a polytope, then the homogenization of P is a convex cone in $\mathbb{R}^{d+1}$ which can be described by

$$\operatorname{homog}(P) = \left\{ \left(x^T, t\right)^T \in \mathbb{R}^d \times \mathbb{R} :\ t \geq 0,\ \ a_i^T x \leq t\, b_i,\ i=1,\ldots,m \right\} \quad .$$

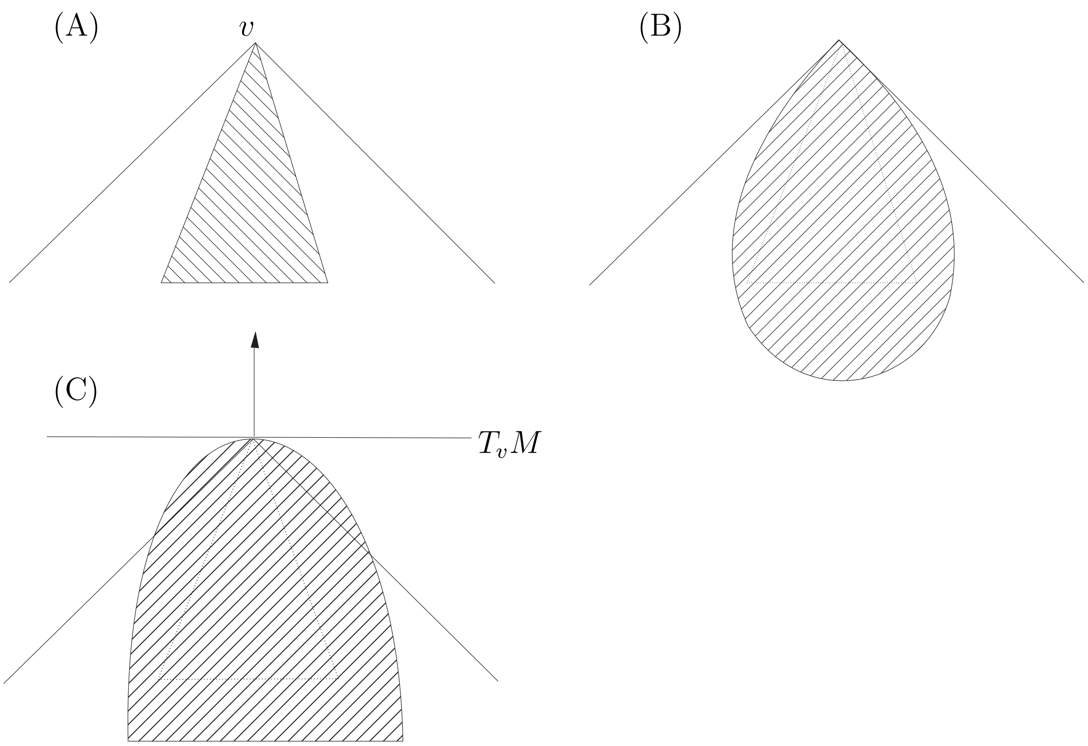

Fig. 14.1: Separation of Closed Semi-algebraic Sets

The same is possible for semi-algebraic sets of a special structure:

From the disc $\{x \in \mathbb{R}^2 \,:\, 4 - x_1^2 - x_2^2 \geq 0\}$ the cone $\{x \in \mathbb{R}^3 \,:\, 4x_3^2 - x_1^2 - x_2^2 \geq 0,\ x_3 \geq 0\}$ is derived in the very same way as homog(P) is derived from P: For semi-algebraic sets $\mathcal{S} = \{x \in \mathbb{R}^d \,:\, \mathfrak{p}(x) \geq 0\}$ where $\mathfrak{p} \in \mathbb{R}[X_1, \ldots, X_d]$ is of special structure with degree k, the set

$$\mathcal{S}' = \{(x^T, t) \in \mathbb{R}^d \times \mathbb{R} :\ \mathfrak{p}(x/t)\, t^k \geq 0,\ t \geq 0\}$$

is exactly the homogenization of $\mathcal{S}$.

This property is not proven in general in this thesis, but it is used implicitly when constructing $\mathcal{S}(C, \alpha)$.

15. Appendix

15.1 Proof for Lemma 7.3.1, Section 7.3.1

Here we will prove Lemma 7.3.1, found in Section 7.3.1, page 46, which we repeat in full here:

Lemma 15.1.1. *Let* $H^c := \left\{ x \in \mathbb{R}^d \,:\, c^T x = 1 \right\}$ *be a hyperplane defined by* $c \in \mathbb{R}^d$, $\|c\| = 1$. *Let further* $y \in H^c$ *and define* $\gamma := \angle(c, y)$.
Then if we restrict α *by* $0 \leq \alpha \leq (\pi/4 - \gamma/2)$ *we find*

$$H^c \cap \mathrm{K}(y, \beta) \;\subset\; H^c \cap B(y, \sin(\alpha)) \;\subset\; H^c \cap \mathrm{K}(y, \alpha) \tag{15.1}$$

where $\beta \in [0, \alpha]$ *is defined by sin*$(\beta) := \sin(\alpha)\ \left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right)$.
Further any ray of $\mathrm{K}(y, \alpha)$ *intersects* H^c, *we have*

$$\mathrm{K}(y, \alpha) = \mathrm{cone}(H^c \cap \mathrm{K}(y, \alpha)\,) \quad , \tag{15.2}$$

where for β *holds accordingly the same as* $\beta \leq \alpha$.

Remark 15.1.2. For a motivation of the notation and geometrical structure we refer to the original Lemma (p. 46).

All objects are well defined

As $y \in H^c$ we know that $\angle(c, y) < \pi/2$, thus we have $\pi/4 - \gamma/2 > 0$ and thus the restrictions on α are feasible.

We find $0 \leq \alpha \leq \pi/4$ and thus $0 \leq \sin(\alpha)\ (1/2)\cos^2(\gamma)\cos(\alpha) \leq \sin(\alpha)$. Therefore β is well defined and we have $0 \leq \beta \leq \alpha < \pi/4$ by as the function sin is monotonous on $[0, \pi/4]$.
Further for any $v \in \mathrm{K}(y, \alpha)$, $v \neq \mathbf{0}$ we find $\angle(c, v) \leq \angle(c, y) + \angle(v, y) \leq \gamma + \alpha < \pi/2$ and thus $\mathrm{ray}(v)$ must intersect H^c. The same clearly holds for β as $0 \leq \beta \leq \alpha$, which proves (15.2).

Intersection of H^c and $\mathrm{K}(y, \alpha)$

Let $y \in H^c$. Then we know $\|y\| \geq c^T y = 1$ and thus we find $H^c \cap B(y, \sin(\alpha)) \subset B(y, \|y\| \sin(\alpha))$. We know that for $y \neq 0$ we have $\mathrm{K}(y, \alpha) = \mathrm{cone}(B(y, \|y\| \sin(\alpha))\,)$ by it's basic properties, and thus we find $H^c \cap B(y, \sin(\alpha)) \subset H^c \cap \mathrm{K}(y, \alpha)$, proving (15.1) for $\mathrm{K}(\alpha, \alpha)$

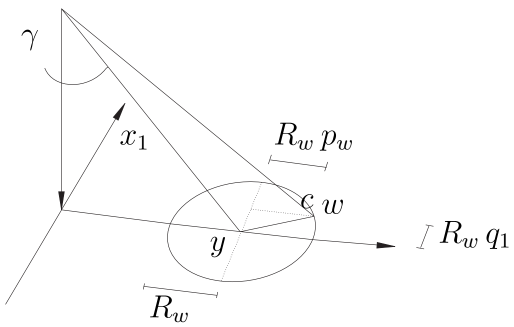

Fig. 15.1: The Choice of a Relative Basis

Intersection of H^c and $\mathrm{K}(y, \beta)$

The proof for this inclusion will be a bit more complicated.

We will prove that $H^c \cap \mathrm{K}(w, \beta) \subset H^c \cap B(y, \sin(\alpha))$ by proving that for any $w \in H^c$ with $\|y - w\| > \sin(\alpha)$ we find $\angle(y, w) > \beta$.

To do so we will prove that for any $w \in H^c$ the angle $\angle(y, w)$ grows with the distance between y and w and thus for every $w \in H^c$ with $\|y - w\| > \sin(\alpha)$ we find $\angle(y, w) > min\{\angle(y, w')|\, w' \in H^c\ , \|y - w'\| = \sin(\alpha)\} \geq \beta$. These two inequalities are proved by basic analysis of a function, using two separate Lemmata.

The main proof will be to construct a function calculating $\cos(\angle(y, w)\,)$ where $w \in B(y, \sin(\alpha)) \cap H^c$. To do so we describe y and w by 'relative polar coordinates'. As $y \in H^c$ we may describe it by $y = c + M\, x_0$, $M \geq 0$, $x_0 \in \mathbb{R}^d$, $\|x_0\| = 1$ such that $c^T x_0 = 0$. Relative to c, x_0 we can build an ortho-normal basis $\{c, x_0, x_1, \ldots, x_{d-2}\}$ of $\mathbb{R}^d$ (see Figure 15.1). Let $w \in H^c$, then clearly we may describe it as $w = y + R_w v_w$, where $v_w = y - w / \|y - w\|$ and $R_w := \|y - w\| \geq 0$.

As $c^T v_w = 0$ and $\|v_w\| = 1$ we find v_w described in the new basis by $v_w = p_w x_0 + \sum_{i=1}^{d-2} q_i x_i$ with $(p_w)^2 + \sum_{i=1}^{d-2} q_i^2 = 1$. Therefore we obtain

$$cos(\angle(y, w)\,) = \frac{y^T w}{\|y\|\,\|w\|} = \frac{1}{\|y\|} \frac{1 + M^2 + R_w p_w M}{(1 + M^2 + 2R_w p_w M + R^2)^{\frac{1}{2}}}$$

which for fixed y (this fixes M) purely depends on $p_w \in [-1, 1]$ and $R_w \geq 0$.

Thus we define a function $f : [-1, 1] \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ where $\mathbb{R}_0^+ = \{x \in \mathbb{R}|\, x \geq 0\}$ by

$$f(p, r) := \frac{1 + M^2 + rpM}{(1 + M^2 + 2rpM + r^2)^{\frac{1}{2}}} \quad .$$

By basic analysis of $f(p, r)$ we will prove in the following (see lemma 15.1.3) that for any $p \in [-1, 1]$ and any pair $0 \leq R_1 < R_2$ we find $f(p, R_1) > f(p, R_2)$, and that for any fixed $R_3 \geq 0$ the function $f(p, R_3)$ reaches it's minimum on the compact set $[-1, 1]$ at $p = 1$.

So if the $w \in H^c$ we chose, is not in $H^c \cap B(y, \sin(\beta))$ then $R_w > \sin(\beta) \geq 0$ and thus

$$\cos(\angle(y,w)) = \|y\|^{-1} f(p_w, R_w) < \|y\|^{-1} f(p_w, \sin(\beta)) \leq \|y\|^{-1} f(1, \sin(\beta))$$

For the special point $v := y + \sin(\beta)x_0 \in H^c$ with we will prove $\cos(\angle(y,v)) \leq \cos(\beta)$ (see lemma 15.1.4) and thus

$$\cos(\angle(y,v)) = \|y\|^{-1} f(1, \sin(\beta)) \leq \cos(\beta) \quad .$$

So for any $w \in H^c$, $w \notin B(y, \sin(\beta))$ we have $\cos(\angle(y,w)) < cos(\beta)$. Further by knowing that $y, w \in H^c$ we find $\angle(y,w) \in [0, \pi)$ and thus $\angle(y,w) > \beta$ as the function cos is monotone in this area.

So what is left is to prove that $\angle(y,v) \geq \beta$ and the properties of $f(p,r)$.

Lemma 15.1.3. *Let $M \geq 0$, then the function $f : [-1,1] \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ defined by*

$$f(p,r) \quad := \quad \frac{1 + M^2 + rpM}{(1 + M^2 + 2rpM + r^2)^{\frac{1}{2}}}$$

reaches its maximum on the compact set $[-1,1] \times R$ in $(1,R)$ for any fixed $R \geq 0$. Further for any $p \in [-1,1]$ and any pair $0 \leq R_1 < R_2$ we find

$$f(p, R_1) > f(p, R_2) \quad . \tag{15.3}$$

Proof: Please notice that by construction the denominator of $f(p,r)$ is greater zero for any pair $(p,r) \in [-1,1] \times \mathbb{R}_0^+$.

Proof for (15.3)

For the partial derivative

$$\frac{\partial f(p,r)}{\partial r} = \frac{-r}{(1 + M^2 + 2RpM + r^2)^{\frac{3}{2}}} \, (1 + M^2(1 - p^2))$$

we find $\frac{\partial f(p,r)}{\partial r} < 0$ for any $r > 0$ and $\frac{\partial f(p,r)}{\partial r} = 0$ for $r = 0$.
Thus when $0 \leq R_1 < R_2$ we find that $f(p, R_1) < f(p, R_2)$ for any fixed $p \in [-1,1]$. This proves (15.3).

Rest of the Proof

Let $R \geq 0$ be fixed. For the sign of the derivative

$$\frac{\partial f(p,R)}{\partial p} = \frac{R^2 M}{(1 + M^2 + 2RpM + R^2)^{\frac{3}{2}}} \, (R + pM)$$

of $f(p,R)$ restricted to $p \in [-1,1]$ we find two cases:

If $R \geq M$ then as $p \in [-1,1]$ we find $\frac{\partial f(p,R)}{\partial p} \geq 0$ and thus $f(p,R)$ is monotonously growing. Thus for fixed R we get a global minimum for $f(p,R)$ at $p=-1$ and a global maximum at $p=1$.

If $R < M$ then $f(p,R)$ is falling if $p \in [-1,-R/M)$ and rising if $p \in [-R/M,1]$. Thus with R fixed, $f(p,R)$ has a global minimum at $p=-R/M$ and two relative maxima at $p=-1$ and $p=1$.

We will now prove that $f(-1,R) \leq f(1,R)$ if $0 \leq R < M$. Please note that both terms are greater zero, as we have $0 \leq R < M$:

$$\begin{array}{lcccl} f(-1,R) & = & \frac{1+M^2-RM}{(1+(M-R)^2)^{\frac{1}{2}}} & > & \frac{1}{(1+(M-R)^2)^{\frac{1}{2}}} > 0 \\ f(1,R) & = & \frac{1+M^2+RM}{(1+(M+R)^2)^{\frac{1}{2}}} & > & 0 \end{array}$$

thus to prove $f(-1,R) \leq f(1,R)$ it suffices to prove $f(-1,R)^2 \leq f(1,R)^2$ which -as the denominator are greater zero- is equivalent to

$$\begin{array}{rcl} f(-1,R)^2 & \leq & f(1,R)^2 \\ \dfrac{(1+M^2-RM)^2}{1+(M-R)^2} & \leq & \dfrac{(1+M^2+RM)^2}{1+(M+R)^2} \\ \left[1+M^2-RM\right]^2\left[1+(M+R)^2\right] & \leq & \left[1+M^2+RM\right]^2\left[1+(M-R)^2\right] \quad . \end{array}$$

Unfortunately this inequality can only be shown by direct calculation. Subtracting the left hand side from the right, we find the last inequality equivalent to

$$\begin{array}{rcll} 0 & \leq & 4RM(1+M^2)\left[1+M^2+R^2\right] & -4RM[(1+M^2)^2+R^2M^2] \\ 0 & \leq & (1+M^2)^2+(1+M^2)R^2 & -(1+M^2)^2-R^2M^2 \\ 0 & \leq & R^2 \quad . & \end{array}$$

Thus we find for fixed R with $R < M$ that $f(1,R) \geq f(p,R)$ for any $p \in [-1,1]$. □

Lemma 15.1.4. *Let $c \in \mathbb{R}^d$, with $\|c\| = 1$ and $H^c := \{x \in \mathbb{R}^d \,:\, c^T x = 1\}$.*
Let further $y \in H^c$ where $y \neq c$ and define $\gamma := \angle(c,y)$.
Further let α be restricted by $\alpha \in [0, \pi/4 - \gamma/2]$.
Then with $w := y + sin(\alpha)\frac{(y-c)}{\|(y-c)\|}$ and $\beta \in [0,\alpha]$ defined by

$$sin(\beta) := \sin(\alpha)\ \left(\frac{1}{2}\cos^2(\gamma)\cos(\alpha)\right) \tag{15.4}$$

we find $\angle(y,w) \geq \beta$.

Proof: We will at first prove that the object β is well defined: By the choice of α and γ we know that $0 \leq \gamma < \pi/2$ and $0 \leq \alpha \leq \pi/4$.

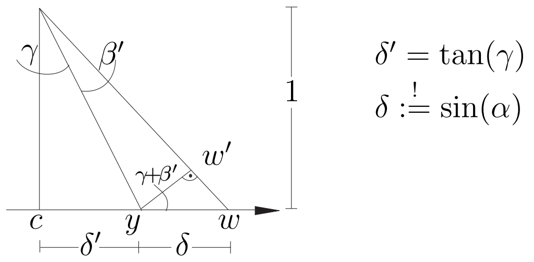

Fig. 15.2: The planar angle between y and w.

Thus we find $0 \leq \sin(\alpha)\ (1/2)\cos^2(\gamma)\cos(\alpha) \leq \sin(\alpha) \leq 1$. Thus there clearly is $\beta \in [0, \pi/2]$ fulfilling (15.4) and as the function sin is monotonous in this area we find $\beta \leq \alpha$.

If $y \in \mathrm{ray}(w)$ the lemma clearly holds, as then we must have $y = w$ by $y = \mathrm{ray}(w) \cap H^c$ and such $\sin(\alpha) = 0$ and thus $\alpha = \beta = 0$. So we assume $y \notin \mathrm{ray}(w)$.

As all four points discussed, namely 0, c, y and w are in the plane spanned by c and y, we might employ standard planar trigonometry (see Figure 15.1).

Please notice that in the following discussion $y = c$ makes no difference to $y \neq c$ until we mention other.

First of all we find that using $\beta' := \angle(y, w)$ and $\delta = \sin(\alpha)$ (see Figure 15.1) leads to

$$\tan(\gamma) + \sin(\alpha) = \tan(\gamma + \beta')$$

and therefore with $\tan(\gamma + \beta') \geq \tan(\gamma) + \tan(\beta')$ we conclude $\tan(\beta') \leq \tan(\alpha)$ and thus $\beta' \leq \alpha$.

Let $w' \in \mathrm{ray}(w)$ be the point in $\mathrm{ray}(w)$ closest to y, then the triangle $\triangle(y, w', 0)$ is perpendicular and we derive

$$sin(\beta') = \frac{\|y - w'\|}{\|y\|} = \|y - w'\| \cos(\gamma) \tag{15.5}$$

from knowing that for $y \in H^c$ we find $\|y\| = \cos(\angle(c, y)\,)^{-1}$.

Further we find the triangle $\triangle(w, w', y)$ is perpendicular where the length of the hypothenuse is $\|y - w\| = \sin(\alpha) = \delta$. And for it's angle at y we have $\angle(y - w', y - w) = \gamma + \beta'$ (see Figure 15.1), by checking that the triangle $\triangle(w, w', y)$ is congruent to $\triangle(\mathbf{0}, y, w)$.

Thus we derive that $\|y - w'\| = \delta \cos(\gamma + \beta')$ and with this and (15.5) we have

$$\begin{aligned} \sin(\beta') &= \delta \cos(\gamma) \cos(\gamma + \beta') && (15.6) \\ &\geq \delta \cos(\gamma) \cos(\gamma + \alpha) && (15.7) \end{aligned}$$

The last inequality must hold as we have $0 \leq \beta' \leq \alpha$ and thus by $\alpha \leq \pi/4 - \gamma/2$ we find $\gamma + \beta' \leq \pi/4 + \gamma/2 < \pi/2$. Thus as the function cos is monotonous on $[0, \pi/2]$ we have $\cos(\gamma + \beta') \geq \cos(\gamma + \alpha) > 0$.

If $y = c$ then we know that $\gamma := \angle(c, y) = 0$ and thus we find $\cos(\gamma) = 1$, $\sin(\gamma) = 0$ which by (15.7) implies

$$\begin{array}{rcll} \sin(\beta') & \geq & \delta & \cos(\alpha) \\ & > & \delta & (1/2)\cos(\gamma)^2\cos(\alpha) \quad , \end{array}$$

as by $\alpha \in [0, \pi/4]$ we have $\delta = \sin(\alpha) \geq 0$ and $\cos(\alpha) > 0$.

If $y \neq c$ then we know that $\gamma \neq 0$ and thus we find the following:
We know that by choice of $0 \leq \alpha \leq \frac{1}{2}(\frac{\pi}{2} - \gamma) \leq \frac{\pi}{4}$ and so by using that the function tan is monotonous on $[0, \pi/2)$ we derive

$$\tan(\alpha) \leq \tan(\frac{1}{2}(\frac{\pi}{2} - \gamma)\,) \leq \frac{1}{2\tan(\gamma)} \quad ,$$

which is well defined as $0 < \gamma < \pi/2$. Further by these restrictions on α we find $\cos(\alpha) \neq 0$ and thus $\sin(\alpha) = \cos(\alpha)\tan(\alpha)$.
With these properties in mind we finally we conclude for $y \neq c$ the following from (15.7):

$$\begin{array}{rclll} \sin(\beta') & \geq & \delta & \cos(\gamma) & \cos(\gamma + \alpha) \\ & = & \delta & \cos(\gamma) & [\cos(\gamma)\cos(\alpha) \,-\, \sin(\gamma)\sin(\alpha)] \\ & = & \delta & \cos(\gamma)\cos(\alpha) & [\cos(\gamma) \,-\, \sin(\gamma)\tan(\alpha)] \\ & \geq & \delta & \cos(\gamma)\cos(\alpha) & [\cos(\gamma) \,-\, \sin(\gamma)\frac{1}{2\tan(\gamma)}] \\ & \geq & \frac{1}{2}\delta & \cos^2(\gamma)\cos(\alpha) & . \end{array}$$

So in either case $y = c$ or $y \neq c$ we find a lower bound to $\sin(\beta')$ in

$$\begin{array}{rcl} \sin(\beta') & \geq & \sin(\alpha)(1/2)\cos^2(\gamma)\cos(\alpha) \\ & = & \sin(\beta) \quad , \end{array}$$

recalling $\delta = \sin(\alpha) \geq 0$. From $\beta, \beta' \in [0, \pi/4]$ we derive $\beta' \geq \beta$ as the function sin is monotonous on this interval.

Such the claim $\angle(y, w) \geq \beta$ is proven, as $\beta' := \angle(y, w)$. $\square$

15.2 Proof for Lemma 8.2.1, Section 8.2

Here we will prove Lemma 8.2.1, found in section 8.2, page 50.
Which we repeat in full here:

Lemma 15.2.1. *Let* $C := \left\{ x \in \mathbb{R}^d \, : \, a_i^T x \leq 0 \, , i = 1, \ldots, m \right\} = \operatorname{cone}(W) \oplus \operatorname{lin}(U)$ *be a d-dimensional cone, where* $\operatorname{cone}(W)$ *is pointed and* $\dim(U) \leq d-2$.
Let further $\bar{c} \in \mathbb{R}^d$ *be a normalized support vector of the apex of* C*, define* $H^c := H(\bar{c}, -1)$ *and* $V := \{-w/(c^T w) : \, w \in W\}$.

Then the following definitions for an apex figure P *of* C *are equal:*

$$\begin{aligned} P &:= (C \cap H^{\bar{c}}) \oplus \operatorname{lin}(\{c\}) && (15.8)\\ &= \operatorname{conv}(V) \oplus \operatorname{lin}(U) \oplus \operatorname{lin}(\{c\}) && (15.9)\\ &= \left\{ x \in \mathbb{R}^d \, : \, \bar{a}_i^T x \leq \bar{c}^T a_i \, , i = 1, \ldots, m \right\}, \quad \textit{where } \bar{a}_i := a_i - (\bar{c}^T a)\bar{c}, && (15.10) \end{aligned}$$

where the description in (15.10) is irreducible. Further the relative diameter of P*, given by*

$$\operatorname{diam}(P \cap \operatorname{lineal}(P)^{\perp}) = \operatorname{diam}(\operatorname{cone}(W) \cap H^{\bar{c}}) \quad , \tag{15.11}$$

is finite as well as $h_P(\bar{a}_i)$ *and* $h_P(-\bar{a}_i)$ *are finite for every* $i \in \{1, \ldots, m\}$.

Proof: By choice of $\bar{c}$ such that $\operatorname{lin}(U) \subset \{\bar{c}\}^{\perp}$ the cone C shares the same lineality space with $C \cap H^{\bar{c}}$, namely $\operatorname{lin}(U)$ (cf: Lemma 6.1.4). For the polyhedron $P' := (C \cap H^{\bar{c}}) \oplus \operatorname{lin}(\{\bar{c}\})$ this implies

$$\begin{aligned} P' &= (C \cap U^{\perp} \cap H^{\bar{c}}) \oplus \operatorname{lin}(U) \oplus \operatorname{lin}(\{\bar{c}\})\\ &= (\operatorname{cone}(W) \cap H^{c}) \oplus \operatorname{lin}(U) \oplus \operatorname{lin}(\{\bar{c}\}). \end{aligned}$$

Further clearly $\bar{c}$ is a support vector for the vertex of the pointed cone $\operatorname{cone}(W) = C \cap U^{\perp}$, and thus we find that $\operatorname{cone}(W) \cap H^{\bar{c}} = \operatorname{conv}(V)$. Therefore (15.8) and (15.9) are equal, implying that $\operatorname{lineal}(P') = \operatorname{lin}(U) \oplus \operatorname{lin}(\{c\})$ and (15.11) must hold (which proves $\operatorname{diam}(P \cap \operatorname{lineal}(P)^{\perp})$ is finite).

Properties of the Half Space description

Recall that for any $u \in \mathbb{R}^d$ and any polyhedron P, by the statement u *induces a facet of* P we state that $P \cap \{x \in \mathbb{R}^d \, : \, u^T x = h_P(u)\}$ is a facet of P.

Let $P'' := \left\{ x \in \mathbb{R}^d \, : \, \bar{a}_i^T x \leq \bar{c}^T a_i \, , i = 1, \ldots, m \right\}$, then $P' \cap H^{\bar{c}} = C \cap H^{\bar{c}}$ as

$$\begin{aligned} C \cap H^{\bar{c}} &= \left\{ x \in \mathbb{R}^d \, : \, a_i^T x \leq 0 \, , i \in \{1, \ldots, m\} \right\} \cap H^{\bar{c}}\\ &= \left\{ x \in \mathbb{R}^d \, : \, \bar{a}_i^T x + \bar{c}^T a_i c^T x \leq 0, \, i \in \{1, \ldots, m\} \, , c^T x = -1 \right\}\\ &= \left\{ x \in \mathbb{R}^d \, : \, \bar{a}_i^T x \leq \bar{c}^T a_i \, , i \in \{1, \ldots, m\} \right\} \cap H^{\bar{c}} \qquad = P'' \cap H^{\bar{c}} \quad . \end{aligned}$$

Further every $\bar{a}_i$ induces the same facette of $C \cap U^\perp \cap H^{\bar{c}}$ as a_i does:
By using Fourier-Motzkin elimination to project C to $C \cap U^\perp$ (see e.g. [Zie98] and [Sch89b]) we find that each a_i induces a separate facet F_i of $C \cap U^\perp$, since $a_i \in U^\perp$. Therefore by construction each $\bar{a}_i$ induces a separate facet $\widetilde{F}_i$ of the vertex figure $C \cap U^\perp \cap H^{\bar{c}}$ of $C \cap U^\perp$, namely $\widetilde{F}_i := F_i \cap H^{\bar{c}}$ [Zie98]. This proves that P'' is given in non-redundant description.

Further $\mathrm{lineal}(P'') = \{\bar{a}_1, \dots, \bar{a}_m\}^\perp$ which is equal to $\mathrm{lin}(U) \oplus \mathrm{lin}(\{c\})$, since $\{a_1, \dots, a_m\}^\perp = \mathrm{lin}(U)$ and every $\bar{a}_i$ is the projection of $a_i \in U^\perp$ to $\{c\}^\perp$ and $\bar{c} \in U^\perp$.
This implies $\mathrm{lineal}(P'') = \mathrm{lineal}(P')$, proving $P'' = P'$ as they both share the same cross-section $C \cap U^\perp \cap H^{\bar{c}}$ perpendicular to their lineality spaces. So the definitions in (15.8) and (15.10) are equal.

Thus we have proven all the restrictions on P, by proving them either for P' or P''. $\square$

15.3 Proof for Lemma 9.1.2, Section 9.1

Here we will prove Lemma 9.1.2, found in Section 9.1, page 59, which we repeat in full here:

Lemma 15.3.1. *Let* $C = \{x \in \mathbb{R}^d \ : \ a_i^T x \leq 0 \ , i{=}1,..,m\} = \mathrm{cone}(W)$ *be a d-dimensional pointed cone and let* $F \in \mathcal{F}(C)$ *be some proper face of* C*.*
Let further $W^F \subset W$ *such that* $F = \mathrm{cone}(W^F)$*.*

Then for $C^{[F]}$ *we find*

$$C^{[F]} = \mathrm{cone}(W \backslash W^F) + \mathrm{lin}(W^F) = C + \mathrm{lin}(F) \quad , \tag{15.12}$$

and thus $\mathrm{lineal}(C^{[F]}) = \mathrm{lin}(F)$*.*
Further the faces $\mathcal{F}(C^{[F]})$ *of* $C^{[F]}$ *are given by*

$$\mathcal{F}(C^{[F]}) = \{G + \mathrm{lin}(F) : \ G \in \mathcal{F}(C) \ , F \subset G\} \quad . \tag{15.13}$$

Proof: If F is the apex of C then clearly $[F] = \{1, \dots, m\}$ holds, and thus there is noting to prove as $C = C^{[F]}$ and $\mathrm{lin}(F) = \{\mathbf{0}\}$.

For any other case we will first prove $\mathrm{lineal}(C^{[F]}) = \mathrm{lin}(F)$ and $C^{[F]}{+}\mathrm{lin}(F) = C{+}\mathrm{lin}(F)$ from which we conclude (15.13) by the use of the Fourier Motzkin elimination method as done in [Zie98], [Sch89b] or [Ros00]. In the following by stating that a vector $a \in \mathbb{R}^d$ *defines a facet of* a cone C' we express that $C' \cap \{x \in \mathbb{R}^d \ : \ a^T x = 0\}$ is a facet of C'.

Assume $\dim(F) > 0$, then the lineality space $\mathrm{lineal}(C^{[F]})$ is given by $\{a_i : \ i \in [F]\}^\perp$ which as C is full-dimensional is equal to $\mathrm{lin}(F)$.

By appropriate numeration of the set $W = \{w_1, \dots, w_n\}$ we may assume w.l.o.g. that $\{w_1, \dots, w_k\} = W^F$. Thus C can be projected to $F^\perp$ in k-steps – from which some are redundant – by successively projecting C to $C_k := \{w_i\}^\perp$, $i{=}\,1,..,k$: Let $C_0 = C$ and for $j{=}1,..,k$ let C_j be the projection of C_{j-1} to $\{w_j\}^\perp$ given by

$$C_j := (C_{j-1} + \mathrm{lin}(\{w_j\})\,) \cap \{w_j\}^\perp.$$

This clearly results in $C_k = (C + \mathrm{lin}(F)) \cap F^\perp$.

To project $C_{j-1} = \{x \in \mathbb{R}^d \ : \ (\tilde{a}_i)^T x \leq 0, i{=}\,1,..,\widetilde{m}\}$ to $\{w_j\}^\perp$ it is possible to use the Fourier-Motzkin elimination method, separating $\{1, \dots, m\}$ into three sets

$$\begin{aligned} I_0 &:= \{i \in \{1, \dots, \tilde{m}\} : w_{j+1}^T \tilde{a}_i = 0\} \\ I_+ &:= \{i \in \{1, \dots, \tilde{m}\} : w_{j+1}^T \tilde{a}_i > 0\} \\ I_- &:= \{i \in \{1, \dots, \tilde{m}\} : w_{j+1}^T \tilde{a}_i < 0\}. \end{aligned}$$

Knowing from [Zie98] that the facets of C_j are defined by all a_i with $i \in I_0$ and some newly generated $a_{i,j}$, $(i,j) \in I_+ \times I_-$, we find the following:

By induction $I_+ = \emptyset$ and thus $I_+ \times I_- = \emptyset$ hold in the jth-step of the elimination for any $j \in \{1, \ldots, k\}$:

We find in the first step that $I_+ =$ because $a_i^T w_j \leq 0$ must hold for any $i = 1,\ldots, m$, due to $w_j \in F \subset C$. This proves, that the facets of C_1 are defined only by vectors a_i with $i \in \{1, \ldots, m\}$. The same then holds according for any $j = 1,\ldots, m$, proving the assumption.

Let $i \in [F]$ then $a_i^T w_j = 0$ holds for any $j \in \{1, \ldots, k\}$, due to $w_j \in F$. Therefore in any step of the elimination $i \in I_0$ holds and thus the set $C_j \cap \{x \in \mathbb{R}^d \,:\, a_i^T x = 0\}$ is a facet of C_j for any $j \in \{1, \ldots, k\}$. Hence a_i defines a facet of C_k.

Finally for any $i \in \{1, \ldots, m\} \setminus [F]$ there must be $l \in \{1, \ldots, k\}$ with $a_i^T w_l < 0$ as otherwise $a_i^T x = 0$ for all $x \in F$ leading to the contradiction $i \in [F]$. Hence we have $i \in I_-$ in the l-th step of the elimination. Therefore no a_i with $i \in i \in \{1, \ldots, m\} \setminus [F]$ defines a facet of C_k.

From the above we conclude that any facet of C_k is defined by a_i with $i \in [F]$ proving

$$(C + \mathrm{lin}(F)) \cap F^\perp = \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0,\ i \in [F]\} \cap F^\perp \quad ,$$

which is equal to $C + \mathrm{lin}(F) = C^{[F]}$.

Finally we find that the same holds for any face $G \in \mathcal{F}(C)$ with $F \subset G$, as through $F \subset G$ we know $[G] \subset [F]$ and thus $a_i \in F^\perp$ for every $i \in [G]$, thus $G + \mathrm{lin}(F)$ is a face of $C^{[F]}$ for such G. For any other face $G' \in \mathcal{F}(C)$ we must have $[G'] \setminus [F] \neq \emptyset$ and therefore $G' + \mathrm{lin}(F)$ is not a face of $C^{[F]} := \{x \in \mathbb{R}^d \,:\, a_i^T x \leq 0, i \in [F]\}$, this proves (15.13). $\square$

Index

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16. Summary – Zusammenfassung

Diese Arbeit fußt auf einem überraschenden Ergebnis von Bröcker und Scheiderer auf dem Gebiet der algebraischen Geometrie ([Sch89a], [Brö91] und [BCR98]). Sie bewiesen, dass jede einfache, geschlossene semi-algebraische Menge der Form

$$\mathcal{V} = \{x \in R^n : \mathfrak{p}_1(x) \geq 0, \ldots, \mathfrak{p}_n(x) \geq 0\},$$

wobei $\mathfrak{p}_i \in R[X_1, \ldots, X_d]$ Polynome über einem geschlossenen reellen Körper R sind, stets durch maximal $d(d+1)/2$-viele polynomielle Ungleichungen beschrieben werden können, hierbei gibt d die algebraische Dimension der Menge $\mathcal{V}$ an.

Hieraus ergibt sich, dass jedes polyeder in $\mathbb{R}^d$ durch maximal $d(d+1)/2$-viele polynomielle Ungleichungen beschreibbar ist. Der Beweis für diese obere Schranke ist nicht konstruktiv, deshalb existieren bis jetzt nur einige Ansätzte zur Lösung dieses Problems.

In der vorliegenden Diplomarbeit wird ein Verfahren präsentiert, mittels dem konvexe Polyeder gegeben als Durchschnitt endlich vieler Halbräume $P := \{x \in \mathbb{R}^d : a_i^T x \leq b_i\}$ in eine Darstellung in $2d$-vielen polynomiellen Ungleichungen überführt werden können.

Dies ist zum einen ein neues Ergebnis in der Polyedertheorie, zum anderen ein Konstruktionsbeispiel für die Ergebnisse von Bröcker und Scheiderer im Falle der Beschränkung von semi-algebraischen Mengen auf jene die durch lineare Polynome erzeugt werden.