Geometry of Cuts and Metrics (ADM III)

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Brief overview of topics

Cuts in graphs, metrics in vector spaces, cycles in matroids are mathematical objects that appear in many areas of mathematics and its applications and in quite diverse contexts. Students will learn in this class to study and approach these objects from a geometric point of view, in particular, via cones and polyhedra. One special aim is to obtain a better understanding of these polyhedra in order to be able to solve related optimization problems. The applications range from statistical mechanics, via VLSI design and measure theory to embeddability problems of finite metric spaces.

Contents

Chapter 1. Königsberg bridges, Chinese postmen, T-joins, and the max-cut problem in planar graphs

Review of Euler's paper of 1736: Solutio Problematis ad Geometriam Situs Pertinentis, *Commentarii Academiae Scientiarium, Imperialis Petropolitanae*, 8(1736)128-140 that solved the Königsberg bridges problem and founded the theory of graphs. Some applications of Eulerian graphs.

Introduction of Eulerian graphs, odd joins and T-joins, and the Chinese postman problem. Transformation of the minimum weighted T-join problem to the nonnegative case. Solution of the weighted T-join problem by reduction to a sequence of shortest paths problems with nonnegative weights and a minimum weighted perfect matching problem. Reduction of the weighted max-cut problem in planar graphs to the Chinese postman problem in the dual planar graph.

Chapter 2. Minimum Weight Cuts

Brief review of the well-known max-flow min-cut theorem (with integrality result) and the Ford-Fulkerson augmenting path algorithm.

Literature: Grötschel (2010), and Schrijver (1986, 2003).

Chapter 3. Cut Inequalities, Path and Flow Polyhedra and Total Unimodularity

Introduction of the s-t path polytope of a directed graph and its dominant $P^{\uparrow}_{s\text{-}t\ path}$ (D), the max-potential min-work theorem and the proof of the complete characterization of the dominant of the s-t path polytope. Brief review of the theory of blocking polyhedra (see Schrijver (1986, Chapter 9)). Introduction of the s-t cut polytope of a directed graph and its dominant $P^{\uparrow}_{s\text{-}t\ cut}$ (D), complete description of the dominant via blocking theory. Total dual integrality. Adjacency and facets of the dominants of the s-t path and the s-t cut polytopes. The s-t connector polytope. Total unimodularity: various characterizations and examples.

Literature: Schrijver (2003), Chapter 13.

Chapter 4. Cut Cones and Polytopes, Correlation Cones and Polytopes

Definition of the cut cone CUT(G) and the cut polytope $CUT^{\square}(G)$ of a general undirected graph G=(V,E), the cut cone CUT_n and the cut polytope $CUT^{\square}_n$ as well as the correlation cone COR_n and correlation polytope $COR^{\square}_n$ of the complete graph $K_n = (V_n,E_n)$with n nodes. The max-cut problem, the covariance mapping between the cut and the covariance polyhedra, quadratic {0,1}-programming and its relation to the max-cut problem. Introduction of the triangle inequalities, the semimetric polytope $MET^{\square}_n$, an IP formulation of the max-cut problem. Proof that the triangle inequalities define facets of $CUT^{\square}_n$ and that the cut polytope has diameter 1.

Literature: Deza, Laurent (1997), pages 8-10, 14-17, 53-58, 421-426.

Chapter 5. Ground States of Spin Glasses and Via Minimization

This chapter closely follows the article
F. Barahona, M. Grötschel, M. Jünger, and G. Reinelt, *An Application of Combinatorial Optimization to Statistical Physics and Circuit Layout Design*, Operations Research 36(1988)493-513.
A didactical example (created for high school students) of the via minimization problem can be found in the paper
Martin Grötschel, Thorsten Koch, Nam Dũng Hoàng, *Lagenwechsel minimieren - oder das Bohren von Löchern in Leiterplatten*, in Katja Biermann, Martin Grötschel, Brigitte Lutz-Westphal (eds.), *Besser als Mathe*,Vieweg+Teubner, 2010, 161-174.

At the University of Cologne (M. Jünger's research group) a server is offered to which instances of spin glass problems can be submitted for which exact ground states will be computed, see http://www.informatik.uni-koeln.de/spinglass/ .

Chapter 6. Approximation Algorithms for Max-Cut Problems

This chapter was presented by David P. Williamson on May 24, 2011 who reported about his research on this topic, in particular about the paper
Michel X. Goemans and David P. Williamson, *Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming*, Journal of the ACM, 42, 1115-1145, 1995,
and subsequent work.

Chapter 7. Distances, Semimetrics, Metrics and Embeddability

Definitions and examples of distances, semimetric, and metric spaces are given. In particular examples are provided where such metrics appear in the real world (such as in archaeology, coding and decoding, path metrics in graphs, hypercube metric, Manhattan or Hamming distances, l_p-metrics). The triangle inequalities and the semimetric cone MET_n are introduced. (Isometric) l_p-embeddability is defined, as well as measure and measure semimetric spaces and L_p-spaces and related embeddability issues. It is mentioned that the L_p- and the l_p^m-embeddability of a distance space (X,d) can be reduced to the emdeddability of all finite subspaces of (X,d).

Literature: Chapter 3 of Deza, Laurent (1997).

Chapter 8. The Cut Cone and l_1-Metrics

The focus of this chapter is the presentation and proof of Propositions 4.2.1, 4.2.2, 4.2.4, and 4.2.5 of section 4.2 of the Deza-Laurent book which characterize various embeddability properties of elements of the cut cone. These results are summarized in Theorem 4.2.6. Finally, several complexity results related to cuts and embeddability are mentioned (everything is hard).

Literature: Sections 4.2 and 4.4 of Chapter 4 of Deza, Laurent (1997).

Chapter 9. Conditions for L_1-Embeddability

The main issue of this chapter is the introduction of hypermetric inequalities, inequalities of negative type, and k-gonal inequalities and their relation to the cut cone. The cones HYP_n and NEG_n are introduced and it is shown that $CUT_n \subseteq HYP_n \subseteq NEG_n$ and that $HYP_n \subseteq MET_n$.

Literature: Sections 6.1.1, 6.1.2. and Theorems 6.2.2 (relation of negative type distance spaces and L_2-embeddability) and 6.3.1 (embeddability implication chain) of Chapter 6 of Deza, Laurent (1997).

Chapter 10. Facets of the Cut Cone and Polytope

The cut polytope $CUT^{\square}_n$ is investigated. Operations on valid and facet defining inequalities are introduced, such as *permutation*, *switching*, *lifting, projecting* and *collapsing*. Switching is considered for more general polyhedra, e.g., polytopes defined as the convex hull of the incidence vectors of sets belonging to a set system that is closed under taking symmetric differences. Various results on the transformation of facet defining inequalities into new facet defining inequalities are shown.

It is shown that, once one has proved that one triangle inequality defines a facet of $\mathrm{CUT}^{\square}_3$, then one can derive that triangle inequalities are facet defining for all cut polytopes and cones (using permutation, switching and 0-lifting).

The symmetry group of the cut polytope is mentioned and it is shown that a complete description of the cut polytope can be derived from a complete description of the cut cone.

The IP formulation of the max-cut problem on a complete graph (using just triangle inequalities) is introduced.

The cut polytope CUT(G) on a general graph is re-introduced and the odd cycle inequalities which generalize the triangle inequalities and provide an IP formulation of the max-cut problem in general graphs. These inequalities yield the metric polytope $\mathrm{MET}^{\square}(G)$, and it is shown that the separation problem for odd cycle inequalities can be solved in polynomial time from which one can infer that linear programs over $\mathrm{MET}^{\square}(G)$ can be solved in polynomial time.

Finally, various other classes of inequalities defining facets of CUT(G) are introduced (e.g., hypermetric inequalities), some example proofs are shown and the complexity status of the associated separation problem is mentioned.

Literature: Chapters 26, 27 and 28 of Deza, Laurent (1997),

Chapter 10 “Cut Problems” generalized to Matroids

This chapter addresses “more general cut problems”, in particular, cut problems in graphs that can be stated in the framework of matroid theory.

Matroids are introduced in general, and especially, matroids that can be represented as “matrix matroids” over some field. The most important class of matroids in this context are *binary matroids*, where a set I is defined as independent if the columns indexed by I of the associated matrix are linearly independent over GF(2). Various examples are mentioned, in particular graphic and cographic matroids such as the graphic and cographic matroid of the complete graph on 5 nodes denoted by $M(K_5)$ and $M(K_5)^*$, respectively. Further special matroids are introduced such as the Fano matroid F_7, and its dual F^*_7, $U_{2,4}$, as well as R_{10}. Minors are introduced and the construction of minors of representable matroids by column deletion and pivoting (contraction of an element).

As important results of matroid theory the characterization of binary, ternary, graphic, cographic and regular matroids via forbidden minors is mentioned. Matroids with the “max-flow min-cut property” as well as matroids with the “sum of circuits property” are introduced, their relation to cut problems in graphs is mentioned, and forbidden minor characterizations are given.

Circuits, cocircuits, cycles and cocycles of binary matroids are introduced and it is shown that these objects naturally generalize Eulerian subgraphs (cycles of a graphic matroid) and cuts of a graph (cycles of a cographic matroid). The problem of finding a cycle of maximum weight in a binary matroid M is introduced and the associated *cycle polytope* $\mathrm{CYC}^{\square}(M)$. The polytope $\mathrm{MET}^{\square}(M)$, defined by the system of odd cocycle inequalities, is a natural generalization of $\mathrm{MET}^{\square}(G)$ and yields an IP formulation of the weighted cycle problem in binary matroids. It is mentioned that $\mathrm{CYC}^{\square}(M) = \mathrm{MET}^{\square}(M)$ if and only if the binary matroid M does not have F^*_7, $M^*(K_5)$, or R_{10} as a minor (i.e., has the sum of circuits property) and that the weighted cycle problem can be solved in polynomial time for such matroids. This result implies that, in a general graph, Eulerian subgraphs of maximum weight and, for planar graphs, a maximum weight cut can be found in polynomial time.

A characterization of the odd cycle inequalities which define facets of $\mathrm{CYC}^{\square}(M)$ is given. This result yields complete and non-redundant characterizations of the polytope of Eulerian subgraphs for general graphs and of the cut polytope of graphs not contractible to K_5.

Finally, master polytopes for cycles of binary matroids are briefly mentioned.

Literature:
Francisco Barahona, Martin Grötschel, *On the Cycle Polytope of a Binary Matroid,* Journal of Combinatorial Theory, Series B, 40 (1986) 40-62,
Martin Grötschel, Klaus Truemper, *Master Polytopes for Cycles of Binary Matroids*, Journal Linear Algebra and its Applications, 114/115 (1989) 523-540,
Martin Grötschel, Klaus Truemper, *Decomposition and Optimization over Cycles in Binary Matroids*, Journal of Combinatorial Theory, Series B, 46:3 (1989) 306-337.

General References:

M. M. Deza, M. Laurent (1997), Geometry of Cuts and Metrics, Algorithms and Combinatorics 15, Berlin, Springer. (elektronische Version auf M. Deza's Homepage verfügbar: http://www.liga.ens.fr/~deza/BOOK1/cutbook.pdf)

Martin Grötschel (2010), Lineare und ganzzahlige Optimierung (ADM II), Vorlesungsmanuskript WS 2009/10,
http://www.zib.de/groetschel/teaching/WS0910/skriptADMII-WS0910neu.pdf

Martin Grötschel, László Lovász, Alexander Schrijver (1993), Geometric algorithms and combinatorial optimization. 2. corr. ed., Algorithms and Combinatorics 2, Berlin, Springer

Alexander Schrijver (1986), Theory of linear and integer programming, Chichester, Wiley.

Alexander Schrijver (2003), Combinatorial optimization. Polyhedra and efficiency (3 vol.), Algorithms and Combinatorics 24, Berlin, Springer. (CD-ROM-Version verfügbar)

All books/articles with M. Grötschel as coauthor are electronically available and downloadable from http://www.zib.de/groetschel/research/Musterbiblio.html. Lecture notes can be found on http://www.zib.de/groetschel/teaching/materials.html.