Graph Colouring and Frequency Assignment

Martin Grötschel
Andreas Eisenblätter
Arie M. C. A. Koster

Martin Grötschel
Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)
DFG Research Center MATHEON
“Mathematics for key technologies”
Institut für Mathematik
Technische Universität Berlin

groetschel@zib.de http://www.zib.de/groetschel
Coloring Graphs

Given a graph $G = (V,E)$, color the nodes of the graph such that no two adjacent nodes have the same color.

The smallest number of colors with this property is called \textit{chromatic} or \textit{coloring number} and is denoted by $\chi(G)$. 
Coloring Graphs

A typical **theoretical question**: Given a class $C$ of graphs (e.g., planar or perfect graphs, graphs without certain minors), what can one prove about the chromatic number of all graphs in $C$?

A typical **practical question**: Given a particular graph $G$ (e.g., arising in some application), how can one determine (or approximate) the chromatic number of $G$?
Coloring Graphs: Some History

• 1852 Francis Guthrie: Can any map be colored using four colors?
  1879 “False Proof” by Kempe
  1880 “False Proof” by Tait
  etc.

• 1890 Heawood “Map Color Theorem”
  (e. g., 7 colors suffice on the torus)
  1966 Proof by Ringel & Youngs, see
  G. Ringel: Map Color Theorem”, Springer, 1974
Coloring Graphs

• The Four Color Problem
  
  Appel & Haken (1977) “Proof” (Heesch)

  Appel & Haken (1986):

  This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in 24 lemmas in the main section of the text. In addition the reader is told that certain facts have been verified with the use of about 1200 hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail.
Coloring Graphs

• The Four Color Problem

Robertson, Sanders, Seymour & Thomas (1997)
(on the run: coloring algorithm with quadratic running time)

Robin Thomas: “An update of the four-color theorem”,

http://www.math.gatech.edu/~thomas/FC/fourcolor.html
Coloring Graphs

• Random graph theory

Very precise estimates for the chromatic number of random graphs

• Variations

Edge coloring: Vizing’s Theorem (1964)

List coloring

T-coloring
Coloring Graphs

- Coloring, T-coloring, and list coloring graphs embedded on surfaces
  Carsten Thomassen
Coloring Graphs

- Coloring graphs algorithmically
  - NP-hard in theory
  - very hard in practice
  - almost impossible to find optimal colorings
  - playground for heuristics (e.g., DIMACS challenge)
Coloring in Telecommunication

- Frequency or Channel Assignment for radio-, tv-transmission, etc.
- Our Example: Mobile phone systems

Wireless Communication

<table>
<thead>
<tr>
<th>VLF</th>
<th>LF</th>
<th>MF</th>
<th>HF</th>
<th>VHF</th>
<th>UHF</th>
<th>SHF</th>
<th>EHF</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 km</td>
<td>10 km</td>
<td>1 km</td>
<td>100 m</td>
<td>10 m</td>
<td>1 m</td>
<td>10 cm</td>
<td>1 cm</td>
</tr>
</tbody>
</table>

- increasing wavelength
- increasing frequency

1996 Encyclopaedia Britannica, Inc.
Wireless Communication

Current Mobile Phone Standards:

**GSM** (Group Spéciale Mobile, General System for Mobile Communication)
- since 1992 (Europe, Australia, parts of Asia, also spreading in the US)

**CDMA** (Code Division Multiple Access)
- since 1995 (North America, parts of Asia)

**GPRS** (General Packet Radio System)
- since 2001 (in Germany)

**UMTS** not covered in this talk
Wireless Communication

GSM Frequencies: 450, 900, 1800, 1900 MHz

More than 500 million users in over 150 countries
Properties of wireless communication

Transmitter emits electromagnetic oscillations at a frequency.

Receiver detects oscillations.

Quality of the received signal:

- **Signal-to-noise ratio**
- Poor signal-to-noise ratio: interference of the signal

Objective: Frequency plan without interference or, second best, with minimum interference.
Cell Models

Hexagon Cell Model

- sites on regular grid
- isotropic propagation conditions
- no cell-overlapping

1st Tier

2nd Tier
Antennas & Interference

- cell
- antenna
- site
- backbone network
- co- & adjacent channel interference
Cell Models

Hexagon Cell Model
- sites on regular grid
- isotropic propagation conditions
- no cell-overlapping

Best Server Model
- realistic propagation conditions
- arbitrary cell shapes
- no cell-overlapping

Cell Assignment Probability Model
- realistic propagation conditions
- arbitrary cell shapes
- cell-overlapping

Source: E-Plus Mobilfunk, Germany
Cell Diagrams

Rural
Terrain Data

Metropolitan
Buildings 3D
Interference

Level of interference depends on

- distance between transmitters,
- geographical position,
- power of the signals,
- direction in which signals are transmitted,
- weather conditions
- assigned frequencies
  - co-channel interference
  - adjacent-channel interference
Separation

Frequencies assigned to the same location (site) have to be separated.

Blocked channels

Restricted spectrum at some locations:
- government regulations,
- agreements with operators in neighboring regions,
- requirements of military forces,
- etc.
Frequency Planning Problem

Find an assignment of frequencies to transmitters that satisfies
- all separation constraints
- all blocked channels requirements

and either
- avoids interference at all

or
- minimizes the (total/maximum) interference level
Modeling: the interference graph

- **Vertices** represent transmitters (TRXs)
- **Edges** represent separation constraints and co/adjacent-channel interference
  - Separation distance: $d(vw)$
  - Co-channel interference level: $c^\text{co}(vw)$
  - Adjacent-channel interference level: $c^\text{adj}(vw)$
Graph Coloring

Simplifications:
- drop adjacent-channel interference
- drop local blockings
- reduce all separation requirements to 1
- change large co-channel interference into separation distance 1 (inacceptable interference)

Result:
- FAP reduces to coloring the vertices of a graph

Coloring Radio Waves
Graph Coloring & Frequency Planning

Unlimited Spectrum  Predefined Spectrum

Vertex Coloring

T-Coloring  List Coloring

List T-Coloring

Minimum Span Frequency Assignment (MS-FAP)

k-Colorability

Min k-Partition  Set Packing

Minimum Interference Frequency Assignment (MI-FAP)

Minimum Blocking Frequency Assignment (MB-FAP)
FAP & Vertex Coloring

• Only co-channel interference
• Separation distance 1
• Minimization of
  - Number of frequencies used (chromatic number)
  - Span of frequencies used
• Objectives are equivalent: span = \#colors-1
• FAP is \textbf{NP-hard}
FAP & T-Coloring

Sets of forbidden distances $T_{vw}$

$$|f_v - f_w| \notin T_{vw} \quad T_{vw} = \{0, \ldots, d(vw)-1\}$$

Minimization of number of colors and span are not equivalent!
FAP & List- $T$-Coloring

Locally blocked channels:
Sets of forbidden colors $B_v$

No solution with span 3!
Minimum Span Frequency Assignment

- List-T-Coloring (+ multiplicity)
- Benchmarks: Philadelphia instances

Channel requirements (P1)
Optimal span = 426
Fixed Spectrum

- Is the graph span-$k$-colorable?
- Complete assignment: minimize interference
- Partial assignment without interference

License for frequencies \{1,\ldots,4\}

No solution with span 3
Hard & Soft constraints

• How to evaluate “infeasible” plans?
  – Hard constraints: separation, local blockings
  – Soft constraints: co- and adjacent-channel interference

• Measure of violation of soft constraints: penalty functions

\[ p_{vw}(f, g) = \begin{cases} 
  c^{co}(vw) & \text{if } f = g \\
  c^{ad}(vw) & \text{if } |f - g| = 1 \\
  0 & \text{otherwise}
\end{cases} \]
Evaluation of infeasible plans

- Minimizing total interference
- Minimizing maximum interference
  - Use of threshold value, binary search

Total penalty: $2 - 2\varepsilon$
Maximum penalty: $1 - \varepsilon$

Total penalty: $1 + \varepsilon$
Maximum penalty: $1 + \varepsilon$
What is a good objective?

Keep interference information!
Use the available spectrum!

Minimize max interference
- $T$-coloring (min span): Hale; Gamst; ...

Minimize sum over interference
- Duque-Anton et al.; Plehn; Smith et al.; ...

Minimize max “antenna” interference
- Fischetti et al.; Mannino, Sassano
Our Model

Carrier Network:

\[ N = (V, E, C, \{B_v\}_{v \in V}, d, c^{co}, c^{ad}) \]

- \((V, E)\) is an undirected graph
- \(C\) is an interval of integers \(\text{(spectrum)}\)
- \(B_v \subseteq C\) for all \(v \in V\) \(\text{(blocked channels)}\)
- \(d : E \rightarrow \mathbb{Z}_+\) \(\text{(separation)}\)
- \(c^{co}, c^{ad} : E \rightarrow [0, 1]\) \(\text{(interference)}\)
Minimum Interference Frequency Assignment

Integer Linear Program:

\[
\begin{align*}
\text{min} & \quad \sum_{vw \in E^c} c_{vw}^{co} z_{vw}^{co} + \sum_{vw \in E^a} c_{vw}^{ad} z_{vw}^{ad} \\
\text{s.t.} & \quad \sum_{f \in F_v} x_{vf} = 1 \quad \forall v \in V \\
& \quad x_{vf} + x_{wg} \leq 1 \quad \forall vw \in E^d, |f - g| < d(vw) \\
& \quad x_{vf} + x_{wf} \leq 1 + z_{vw}^{co} \quad \forall vw \in E^c, f \in F_v \cap F_w \\
& \quad x_{vf} + x_{wg} \leq 1 + z_{vw}^{ad} \quad \forall vw \in E^a, |f - g| = 1 \\
& \quad x_{vf}, z_{vw}^{co}, z_{vw}^{ad} \in \{0,1\}
\end{align*}
\]
Benchmarks

- Philadelphia (Minimum Span)
- CALMA (MS-FAP, MO-FAP, MI-FAP)
  100-916 transmitters, 40 frequencies, density 5%, "equality" constraints
- COST 259 (MI-FAP)
  2214 transmitters, 75 frequencies, graph density 13.5%, Maximum degree 916, Maximum clique size 93

http://fap.zib.de
## A Glance at some Instances

| Instance | | | | | | | |
|----------|--------------|----------|----------------|----------------|----------------|----------------|
|          | V            | dens. [%]| min. deg.      | avg. deg.      | max. deg.      | diam.          |
| k        | 267          | 56,8     | 2              | 151,0          | 238            | 3              |
| B-0-E-20 | 1876         | 13,7     | 40             | 257,7          | 779            | 5              |
| f        | 2786         | 4,5      | 3              | 135,0          | 453            | 12             |
| h        | 4240         | 5,9      | 11             | 249,0          | 561            | 10             |

Expected graph properties: planarity,…
Neither high quality nor feasibility are generally achievable within practical running times:

• Testing for feasibility is NP-complete.

• There exists an $\varepsilon > 0$ such that FAP cannot be “approximated” within a factor of $|V|^{\varepsilon}$ unless $P = NP$. 
Heuristic Solution Methods

• Greedy coloring algorithms,
• DSATUR,
• Improvement heuristics,
• Threshold Accepting,
• Simulated Annealing,
• Tabu Search,
• Variable Depth Search,
• Genetic Algorithms,
• Neural networks,
• etc.
Heuristics

- T-coloring
- Dual Greedy
- DSATUR with Costs
- Iterated 1-Opt
- Simulated Annealing
- Tabu-Search
- Variable Depth Search
- MCF
- B&C-based

construction heuristics

(randomized) local search

other improvement heuristics
Region with “Optimized Plan”

Instance k, a “toy case” from practice

264 cells
267 TRXs
50 channels

57% density
151 avg. deg.
238 max. deg.
69 clique size

DC5-VDS: Reduction 96.3%
co-channel C/I worst Interferer

Mobile Systems International Plc.

20 km

Commercial software

DC5-1M

Konrad-Zuse-Zentrum für Informationstechnik Berlin

Martin Grötschel
Region Berlin - Dresden

2877 carriers
50 channels
Interference reduction: 83.6%
Region Karlsruhe

2877 Carriers
75 channels
Interference Reduction: 83.9 %
Heuristics: Summary

- FAP is **NP-hard and hardly approximable - in theory.**
- Efficient heuristic optimization allows significant improvement over standard planning methods.
- Generating the **right input data is nontrivial.**
- **SA:** solution quality depends heavily on the neighborhood relation and also on the cooling schedule.
Guaranteed Quality

Optimal solutions are out of reach!

Enumeration: $50^{267} \approx 10^{197}$ combinations (for trivial instance k)

Hardness of approximation

Polyhedral investigation (IP formulation)
   Aardal et al.; Koster et al.; Jaumard et al.; ... Used for adapting to local changes in the network

Lower bounds - study of relaxed problems
Lower Bounds

How much better can we possibly be?

Instance k
Lower Bounding Technology

- LP lower bound for coloring
- TSP lower bound for $T$-coloring
- LP lower bound for minimizing interference
- Tree Decomposition approach
- Semidefinite lower bound for minimizing interference
Region with “Optimized Plan”

Instance k, the “toy case” from practice

264 cells
267 TRXs
50 channels

57% density
151 avg. deg.
238 max. deg.
69 clique size

DC5-VDS

Further Reduction:
46.3%
Lower Bounds

- **Weak Clique Bound:** add cheapest edges of clique
- **Clique Bound:** optimal assignment on clique
- **Clique-Cover Bound:** max sum clique bounds
- **Min k-Partition:** optimal assignment for simplified network

5-clique & 3 channels

- two interfering pairs
A Simplification of our Model

Simplified Carrier Network:

\[ N = (V, E, C, \{B_v\}_{v \in V}, d, c^{co}, c^{ad}) \]

- \((V,E)\) is an undirected graph
- \(C\) is an interval of integers (spectrum)
- \(B_v \subseteq C\) for all \(v \in V\) (blocked channels)
- \(d : E \rightarrow \mathbb{Z}_+ \{0, 1\}\) (separation)
- \(c^{co}, c^{ad} : E \rightarrow [0,1]\) (interference)
MIN K-Partition

- No blocked channels
- No separation constraints larger than one
- No adjacent-channel interference

**min k-partition (max k-cut)**

Chopra & Rao; Deza et al.; Karger et al.; Frieze & Jerrum

IP, LP-based B&C, SDP
MIN K-Partition

**Given:** an undirected graph \( G = (V,E) \) together with real edge weights \( w_{ij} \) and an integer \( k \).

**Find a partition** of the vertex set into (at most) \( k \) sets \( V_1, \ldots, V_k \) such that the sum of the edge weights in the induced subgraphs is minimal!

\[
\min_{V_1, \ldots, V_k} \sum_{p=1}^{k} \sum_{i,j \in V_p} w_{ij}
\]

**NP-hard** to approximate optimal solution value.
Integer Linear Programming

\[ \min \sum_{i,j \in V} w_{ij} z_{ij} \]
\[ z_{ih} + z_{hj} - z_{ij} \leq 1 \quad \forall h, i, j \in V \rightarrow \text{partition consistent} \]
\[ \sum_{i,j \in Q} z_{ij} \geq 1 \quad \forall Q \subseteq V \text{ with } |Q| = k + 1 \]
\[ z_{ij} \in \{0, 1\} \]

Number of ILP inequalities (facets)

| Instance* | |V| | k | Triangle | Clique Inequalities |
|-----------|------------|----------------|----------------|---------------------|---------------------|
| cell.k    | 69         | 50             |                | 157182              | 17231414395464984   |
| B-0-E     | 81         | 75             |                | 255960              | 25621596            |
| B-1-E     | 84         | 75             |                | 285852              | 43595145594         |
| B-2-E     | 93         | 75             |                | 389298              | 1724861095493098563 |
| B-4-E     | 120        | 75             |                | 842520              | 1334655509331585084721199905599180 |
| B-10-E    | 174        | 75             |                | 2588772             | 361499854695979558347628887341189586948364637617230 |
Lemma: For each \( k, n \) \((2 \leq k \leq n+1)\) there exist \( k \) unit vectors \( u_1, \ldots, u_k \) in \( n \)-space, such that their mutual scalar product is \(-1/(k-1)\). (This value is least possible.)

Fix \( U = \{u_1, \ldots, u_k\} \) with the above property, then the min \( k \)-partition problem is equivalent to:

\[
\min_{\phi: V \rightarrow U} \sum_{\substack{i, j \in E \\atop i \mapsto \phi_i}} \left( \frac{k - 1}{k} \langle \phi_i, \phi_j \rangle + \frac{1}{k} \right) w_{ij}
\]

\( X = [\langle \phi_i, \phi_j \rangle] \) is positive semidefinite, has 1’s on the diagonal, and the rest is either \(-1/(k-1)\) or 1.
Semidefinite Relaxation

\[ \begin{align*}
\min \quad & \sum_{ij \in E(K_n)} \frac{(k - 1) V_{ij} + 1}{k} \\
\text{s.t.} \quad & V_{ii} = 1 \quad \forall i \in V \\
\text{and} \quad & V_{ij} \geq \frac{-1}{k - 1} \quad \forall i, j \in V \\
\text{and} \quad & V \succeq 0
\end{align*} \]

(SDP) is an approximation of (ILP)

Given \( V \), let \( z_{ij} := \frac{((k-1) V_{ij} + 1)}{k} \), then:

- \( z_{ij} \in [0,1] \)
- \( z_{ih} + z_{ih} - z_{ij} < \sqrt{2} \ (<=1) \)
- \( \sum_{i,j \in Q} z_{ij} > \frac{1}{2} \ (>=1) \)

Karger et al.; Frieze & Jerrum
Computational Results

S. Burer, R.D.C Monteiro, Y. Zhang; Ch. Helmberg; J. Sturm

<table>
<thead>
<tr>
<th>Instance</th>
<th>clique cover</th>
<th>min k-part.</th>
<th>heuristic</th>
<th>clique cover</th>
<th>min k-part.</th>
<th>heuristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>cell.k</td>
<td>0.0206</td>
<td>0.0206</td>
<td>0.0211</td>
<td>0.0248</td>
<td>0.1735</td>
<td>0.4023</td>
</tr>
<tr>
<td>B-0-E</td>
<td>0.0016</td>
<td>0.0013</td>
<td>0.0016</td>
<td>0.0018</td>
<td>0.0096</td>
<td>0.8000</td>
</tr>
<tr>
<td>B-1-E</td>
<td>0.0063</td>
<td>0.0053</td>
<td>0.0064</td>
<td>0.0063</td>
<td>0.0297</td>
<td>0.8600</td>
</tr>
<tr>
<td>B-2-E</td>
<td>0.0290</td>
<td>0.0213</td>
<td>0.0242</td>
<td>0.0378</td>
<td>0.4638</td>
<td>3.1700</td>
</tr>
<tr>
<td>B-4-E</td>
<td>0.0932</td>
<td>0.2893</td>
<td>0.3481</td>
<td>0.2640</td>
<td>4.3415</td>
<td>17.7300</td>
</tr>
<tr>
<td>B-10-E</td>
<td>0.2195</td>
<td>2.7503</td>
<td>3.2985</td>
<td></td>
<td></td>
<td>146.2000</td>
</tr>
</tbody>
</table>

Lower bound on co-channel interference by a factor of 2 to 85 below co- and adjacent-channel interference of best known assignment.
Semidefinite Conclusions

Lower bounding via
Semidefinite Programming works,
at least better than LP!

• Challenging computational problems
• Bounds too far from cost of solutions to give strong quality guarantees
• How to produce good k-partitions starting from SDP solutions?
Summary: Radio Planning

- Signal Prediction
  - Cell Area
  - Handover Relations
- Channel Demand
- Interference & Separation
- Frequency Assignment
  - Operation

- site data
- terrain data
- land-use
- traffic blocking rate
- signal predictions
- handover-relations
- cell area
Mathematical Approach: Summary

• Collecting sound data is intricate.
• Minimizing the sum of interference is a compromise with practical under-pinning.
• Huge improvements are possible.
• Practice: time-savings, increased quality!

Optimization really helps!


The End

Martin Grötschel
Konrad-Zuse-Zentrum für
Informationstechnik Berlin (ZIB)
DFG Research Center MATHEON
“Mathematics for key technologies”
Institut für Mathematik
Technische Universität Berlin
groetschel@zib.de  http://www.zib.de/groetschel