

Mathematics of Infrastructure Planning

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Exercise sheet 9

Deadline: Thu, Jun 14, 2012, **23:59**, `mailto:borndoerfer@zib.de`

Exercise 30. **10 points**

Consider the following discrete stop location problems in a network $N = (S \cup T, E)$ with demand points V and covering radius r:

(DSL)	$p = \|U\|/S/\operatorname{cov}_r(U) = V/\ell_2/p$	(planning from scratch)
(DSL1)	$p = \|U\|/T/\operatorname{cov}_r(U) = V/\ell_2/p$	(closing stops)
(DSL2)	$p = \|U\|/S/\operatorname{cov}_r(U \cup T) = V/\ell_2/p$	(opening stops)
(DSL3)	$p = \|U\|/S \cup T/\operatorname{cov}_r(U) = V/\ell_2/p$	(closing and opening stops).

Prove that (DSLi) can be reduced to (DSL), $i = 1, 2, 3$.

Exercise 31. **10 points**

Consider a set covering problem (SCP) $\min c^T x,\ Ax = \mathbf{1}, x \in \{0,1\}^n$ with constraint matrix $A \in \{0,1\}^{m \times n}$ and objective $c \in \mathbb{R}^n_+$. Prove the validiaty of the following preprocessing rules:

a) $A_{i\cdot} = e_j \Longrightarrow x_j = 1$ in every solution of (SCP).
b) $A_{\cdot j} \leq A_{\cdot k}$ and $c_j < c_k \Longrightarrow x_j = 0$ in every optimal solution of (SCP).
c) $A_{i\cdot} \leq A_{k\cdot} \Longrightarrow A_{k\cdot}x \geq 1$ is redundant.
d) Find, formulate, and prove another preprocessing rule.

Exercise 32. **10 points**

Let $w^1 \geq \cdots \geq w^s > w^{s+1} = 0$ be integers and denote by $H(n) := \sum_{i=1}^{n} \frac{1}{i}$ the n-th harmonic number, $n \in \mathbb{N}$. Prove that

$$\sum_{i=1}^{s} (w^i - w^{i+1})/w^i \leq \sum_{i=1}^{s} \big(H(w^i) - H(w^{i+1})\big) \leq H(w^1).$$

Exercise 33. **(Tutorial session)**

Consider the bus stop location problem on the left of Fig. 1 with unit costs, potential stops $S = \{s_1, \ldots, s_7\}$, demand points $V = \{v_1, \ldots, v_5\}$ and covering radius r with associated covering matrix

$$A_r^{\text{cov}} = \left(\begin{array}{ccccccc|c} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & p_2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & p_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & p_4 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & p_5 \end{array}\right)$$

(cf. Schöbel [2003]). Solve this problem using set covering techniques, illustrating your reasoning in the figure.

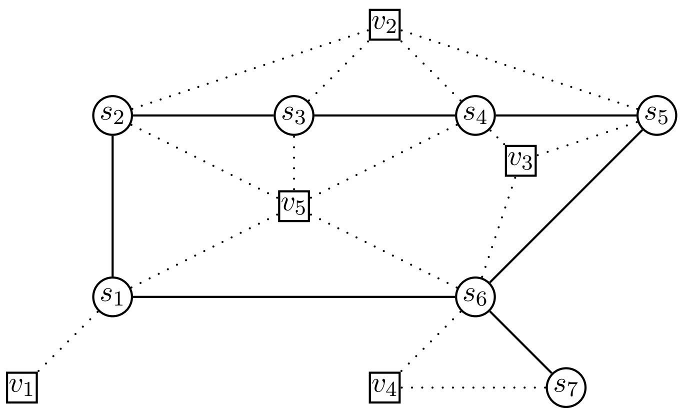

Figure 1: Bus stop location problem.

Exercise 34. **(Tutorial session)**

Consider a *discrete stop location problem on a line*, i.e., in a network $N = (S, E)$ of collinear nodes. Prove:

a) The covering matrix A_r^{cov} associated with a discrete stop location problem on a line has the *consecutive ones property*, i.e., the columns and rows of A_r^{cov} can be permuted in such a way that $a_{ri} = 1 = a_{rk}$ implies $a_{rj} = 1$, $i \leq j \leq k$, for all $r \in V$, $i, k \in S$.
b) A 0/1 matrix with the consecutive ones property is totally unimodular.
c) The set covering problem associated with a discrete stop location problem on a line can be solved by linear programming.

Exercise 35. **(Tutorial session)**

Consider a continuos stop location problem in a network $N = (S, E)$ with demand points V and covering radius r:

$$\text{(CSL)} \qquad p = |U|/N/\operatorname{cov}_r(U) = V/\ell_2/p.$$

Prove: If (CSL) is feasible, the optimum is finite.

Exercise 36. **(Tutorial session)**

Consider a continuos stop location problem in a network $N = (S, E)$ with demand points V and covering radius r

$$\text{(CSL)} \qquad p = |U|/N/\operatorname{cov}_r(U) = V/\ell_2/p$$

and denote

$$S_r := S \cup \{x \in N : \exists v \in V : ||x - v||_2 = r\}.$$

Prove that (CSL) can be reduced to (DSL):

a) S_r is finite.
b) Consider an edge $e = uv \in E$ and let $S_r^e := S_r \cap e = \{s_1, s_2, \ldots, s_k\}$. Order $\{s_1, \ldots, s_k\}$ as $u = s_1 <_e \cdots <_e s_n = v$ with respect to the natural order along the edge $e = uv$, starting from u. Let s be a point on uv between s_i and s_{i+1}, i.e., $s_i <_e s <_e s_{i+1}$. Then $\operatorname{cov}_r(s_i) \supseteq \operatorname{cov}_r(s)$ and $\operatorname{cov}_r(s_{i+1}) \supseteq \operatorname{cov}_r(s)$.
c) (CSL) has an optimal solution $S^* \subseteq S_r$.