Technische Universität Berlin
Institut für Mathematik

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Mathematics of Infrastructure Planning

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Exercise sheet 12

Deadline: Thu, July 5, 2012, **23:59**, `mailto:hiller@zib.de`

Exercise 43. **10 points**

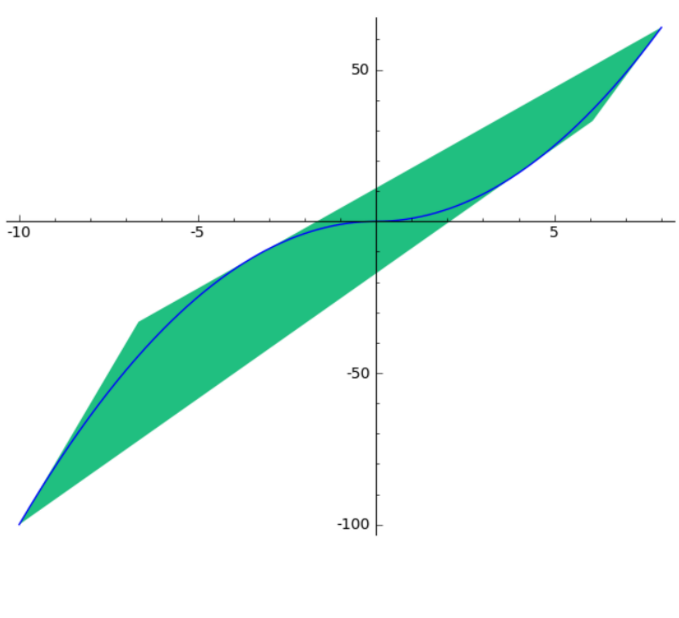
A pipe in a gas network can be modelled as an arc (u, v) from node u to node v in a directed graph. The direction of the arc determines the flow direction of the gas in the pipe: If the flow q_{uv} is non-negative, the gas flows from u to v and in the opposite direction otherwise. The pressures at the end nodes of the pipe are denoted by p_u and p_v, respectively; the pressure difference or more precisely the difference of the squared pressures determines (under suitable simplifying assumptions) the flow q_{uv} according to the equation

$$p_u^2 - p_v^2 = \alpha q_{uv}|q_{uv}|. \tag{1}$$

To handle the solution set of this nonlinear equation in a (linear) mixed-integer global solver, it has to be linearized. One possibility to do this is to approximate the solution set by a "tight" approximation of its convex hull. To this end, it is sufficient to find approximations for the function

$$y = f(x) = \alpha x|x|. \tag{2}$$

Derive formulas for a "good" linear approximation of the convex hull of the graph $F = \{(x, f(x)) \mid \underline{x} \leq x \leq \overline{x}\}$ of $f(x)$ in the interval $[\underline{x}, \overline{x}]$ using at most four faces (see picture)!



Exercise 44. **10 points**

To improve the linear approximation of the solution set F of Equation (2), one can add cutting planes. In this way, the convex hull of F may be approximated to any degree of precision. To refine the approximation further, one eventually has to resort to "branching". It is reasonable to branch on $x = 0$ first, since then we have a concave function in one branch ($x \leq 0$), whereas the function in the other branch ($x > 0$) is convex. Let us assume that the convex domain of $f(x)$ has already been reduced to the interval $[\underline{x}, \overline{x}]$ with $\underline{x} \geq 0$. How should the branching process proceed in order to achieve a "good" resulting approximation?

Exercise 45. **2 points for each of $(a), \ldots, (e)$, 10 total points**

We consider a compressor station of a gas pipeline system. This station consists of many devices (such as pumps, switches, valves,...) which have to satisfy various side constraints. Let us consider four such devices, named 1, 2, 3, and 4, and let the 0/1-variable x_i indicate whether device i is switched on ($x_i = 1$) or off ($x_i = 0$). The devices have to satisfy the following conditions:

(1) If device 1 is switched on, then device 2 must be switched off.

(2) If devices 3 and 4 are switched on, then device 2 must be switched on as well.

(3) If devices 2 and 3 are switched on, then one of the devices 1 or 4 must be switched on.

(a) Formulate an integer program so that every feasible 0/1-solution of this IP corresponds to a feasible on/off-setting of the four devices by "turning" the requirements (1)–(3) into "natural inequalities".

(b) Determine all feasible 0/1-solutions of the IP of (a).

Let P be the convex hull of the feasible solutions determined in (b).

(c) Determine a system of linear inequalities that defines P.

(d) Find, for each facet of P, a linear inequality defining it.

(e) Which of the inequalities that you came up with in (a) define facets of P? What is the dimension of the faces determined by the inequalities you provided in (a)?

Exercise 46. **10 points**

We consider a mixed-integer program (MIP) in just two variables x and y. The real variable x is required to be nonnegative, the integral variable y is unbounded. Consider, for some real number b, the set

$$P_I := \{(x, y) \in \mathbb{R} \times \mathbb{Z} | x + y \geq b, x \geq 0\}$$

of feasible MIP-solutions and its convex hull

$$P := conv(P_I).$$

Let $\lfloor b \rfloor$ denote the largest integer not larger than b, and set

$$r := r(b) := b - \lfloor b \rfloor.$$

Prove that

$$x + ry \geq r(\lfloor b \rfloor + 1)$$

defines a facet of P and that

$$P := \{(x, y) \in \mathbb{R}^2 | x + y \geq b, x \geq 0, x + ry \geq r(\lfloor b \rfloor + 1)\}.$$