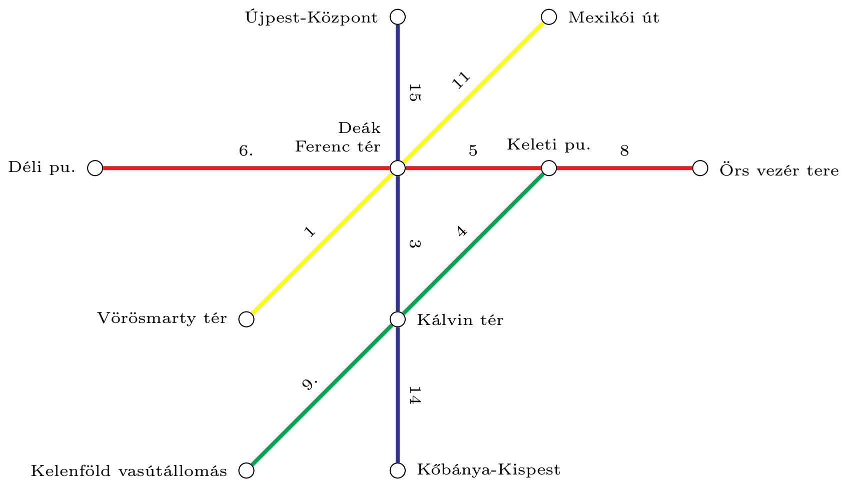
Exercise 1 **10 points**

Consider the metro network $\mathcal{N}$ of Budapest:



There are four lines. The travel times are indicated on the edges.

(a) Solve the Chinese Postman Problem: Find the shortest closed walk through $\mathcal{N}$ w.r.t. travel time visiting each edge at least once. How long does this walk take?

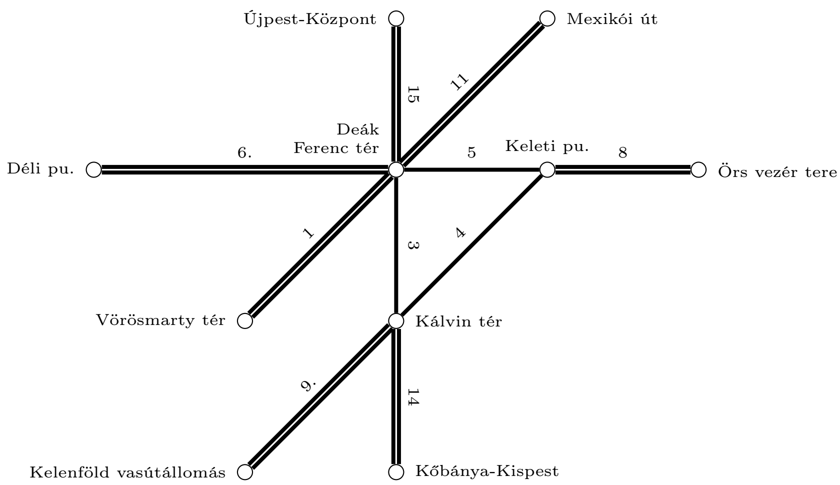
(b) Consider the periodic expansion $\mathcal{E}$ of $\mathcal{N}$ for the period time $T = 10$ minutes. Each line runs every 10 minutes, in both directions. At every station, transfers are only possible between *distinct* lines. For example, there are exactly 4 transfer activities at the station Keleti pu. Construct a periodic timetable $\pi : V(\mathcal{E}) \to \{0, 1, \dots, 9\}$ such that

- driving activities take precisely as long as indicated on the edges,
- waiting activities have duration 0 minutes,
- the minimum transfer time is 3 minutes,
- the maximum transfer time is 7 minutes,
- in both directions, the blue line departs at minute 0 at Deák Ferenc tér.

(c) Prove that there is no periodic timetable that satisfies all conditions of (b), and additionally provides a maximum transfer time of 6 minutes at each station.

Solution:

(a) In a Chinese Postman tour, every vertices must be entered and left. In particular, edges leading to vertices of degree one must be visited twice. Therefore, we need to solve the Chinese Postman Problem on the following graph:



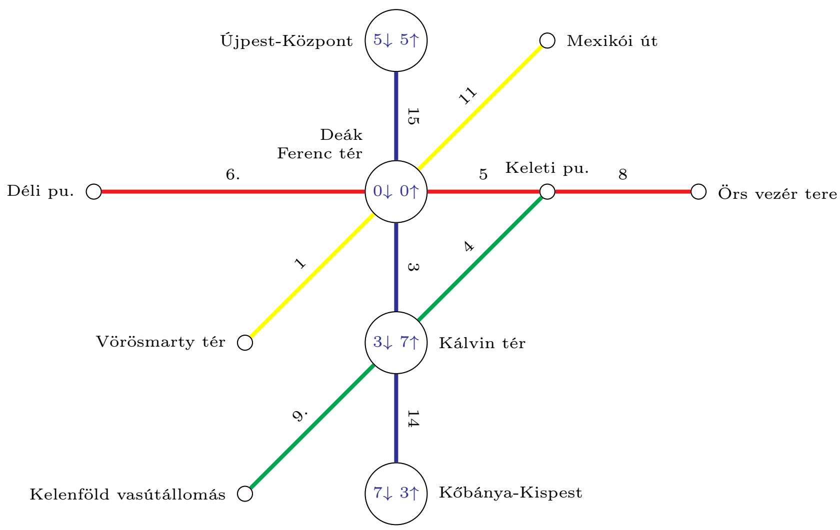
Now all vertex degrees have become even so that this graph is Eulerian. In particular, the optimal Chinese Postman tour is simply an Euler tour. The length of any Euler tour is

$$2 \cdot (15 + 11 + 8 + 14 + 9 + 1 + 6) + 5 + 4 + 3 = 140.$$

Points:

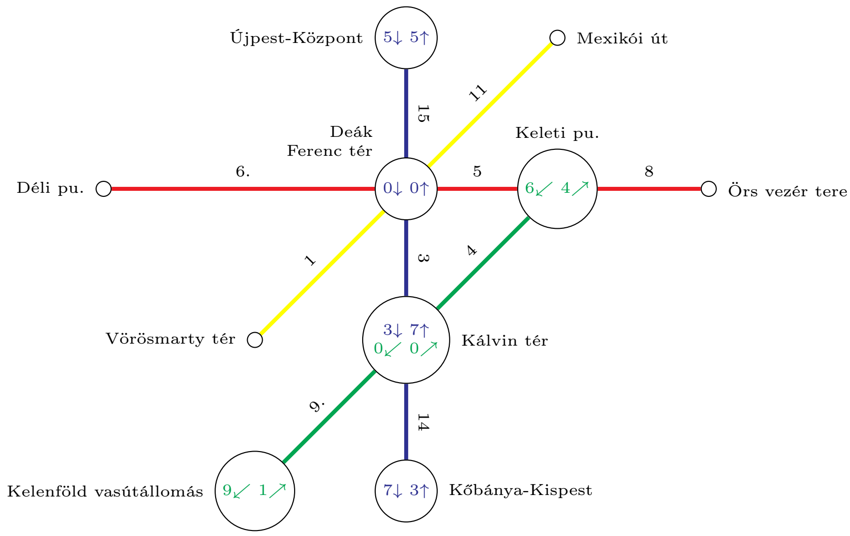
explanation of solution method	1
description of optimal tour	1
length of optimal tour	1
total	**3**

(b) Since both directions of the blue line are fixed at Deák Ferenc tér, waiting activites take 0 minutes and there are no turnarounds, the timetable for the blue line is fixed:

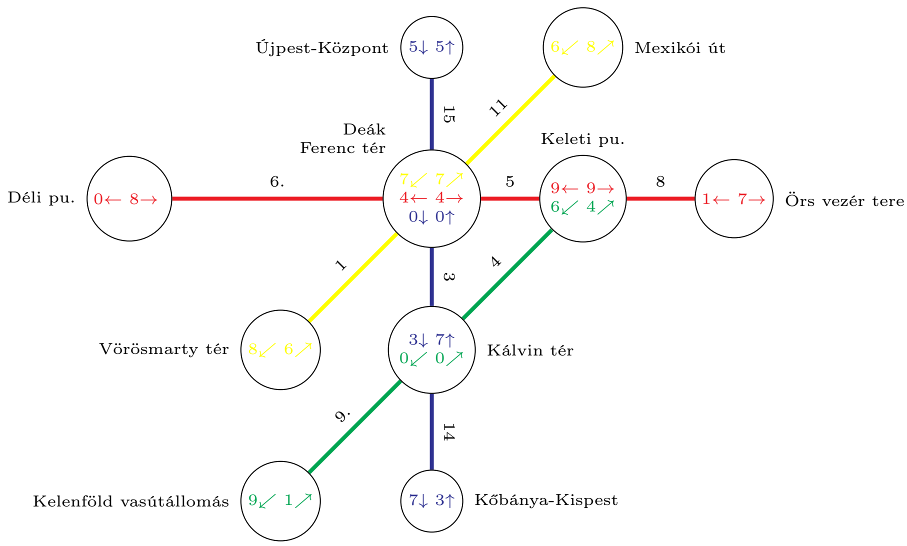


Note that since waiting times are 0 minutes, there is no need to distinguish arrival and departure times of the same direction of a line at a station.

At Kálvin tér, the transfer from the blue line downward to the green line towards Keleti pu. must take between 3 and 7 minutes, so that the possible departures of the green line are $6, 7, 8, 9, 0$. On the other hand, the transfer from the blue line upward to the green line towards Keleti pu. takes also between 3 and 7 minutes, so that the possible departures of the green line are $0, 1, 2, 3, 4$. In particular, the green line has to depart at minute 0 towards Keleti pu. The same holds for the departure in the opposite direction. This fixes also the green line:



Consider now the red line. At Keleti pu. departures must be in $\{7, 8, 9, 0, 1\} \cap \{9, 0, 1, 2, 3\} = \{9, 0, 1\}$ (transfer from and to the green line). We can choose 9 as departure time for both directions of the red line at Keleti pu. This implies minute 4 for both directions of the red line at Deák Ferenc tér. Putting the yellow line at minute 7 in both directions fixes the whole timetable:



Finally observe that the possible transfer relations $0 \to 4$, $0 \to 7$, $4 \to 0$, $4 \to 7$, $7 \to 0$, $7 \to 4$ at Deák Ferenc tér all lie between 3 and 7 minutes.

Points:

visualization or description of a timetable	1
correct timetable at the three transfer hubs	2
feasibility of remaining timetable	2
total	**5**

(c) As observed in (b), the conditions already fix the entire blue and green line. In particular, there is always a 7-minute-transfer between blue and green at Kálvin tér.

Points:

proof of infeasibility	2
total	**2**

Exercise 2 **5 points**

Consider the following list of elementary connections:

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departure stop, arrival stop, departure time, arrival time, trip number
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1 Berlin,Erfurt,10:30,12:29,1509
2 Berlin,Hannover,10:36,12:36,146
3 Berlin,Hannover,10:51,12:28,640
4 Berlin,Erfurt,11:04,12:50,1632
5 Berlin,Erfurt,11:28,13:30,599
6 Berlin,Hannover,11:49,13:28,940
7 Erfurt,Frankfurt,12:17,14:36,1650
8 Hannover,Kassel,12:41,13:37,577
9 Erfurt,Frankfurt,12:50,14:56,1632
10 Kassel,Fulda,13:15,13:47,371
11 Kassel,Fulda,13:23,13:54,537
12 Erfurt,Fulda,13:30,14:50,599
13 Kassel,Frankfurt,13:37,15:00,577
14 Hannover,Kassel,13:41,14:37,77
15 Fulda,Frankfurt,13:47,14:44,371
16 Fulda,Frankfurt,14:07,15:28,4521
17 Kassel,Frankfurt,14:37,16:00,77
18 Fulda,Frankfurt,14:50,15:44,599
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The minimum change time is 0 minutes. Find the earliest arrival times for all reachable stops when departing in Berlin at 11:00. Use one of the following algorithms: Connection Scan, Time-Expanded Dijkstra, Time-Dependent Dijkstra.

Solution:

Connection Scan Algorithm: The list of elementary connections is already sorted by departure time in increasing order. Start scanning the connections:

Connection no.	Reachable	Used trips	Earliest arrival times
–	–	–	Berlin 11:00
1	no	–	–
2	no	–	–
3	no	–	–
4	yes	1632	Erfurt 12:50
5	yes	1632, 599	–
6	yes	1632, 599, 940	Hannover 13:28
7	no	1632, 599, 940	–
8	no	1632, 599, 940	–
9	yes	1632, 599, 940	Frankfurt 14:56
10	no	1632, 599, 940	–
11	no	1632, 599, 940	–
12	yes	1632, 599, 940	Fulda 14:50
13	no	1632, 599, 940	–
14	yes	1632, 599, 940, 77	Kassel 14:37
15	no	1632, 599, 940, 77	–
16	no	1632, 599, 940, 77	–
17	yes	1632, 599, 940, 77	–
18	yes	1632, 599, 940, 77	–

Points:

CSA:

sorting of connections	1
marking used trips	1
execution of algorithm	1
earliest arrival times	2
total	**5**

Dijkstra:

description of graph	1
marking permanent labels	1
execution of algorithm	1
earliest arrival times	2
total	**5**

Exercise 3 **5 points**

Let $\mathcal{E} = (V, E)$ be an event-activity network, $T \in \mathbb{N}$ a period time, and let $\ell, u \in \mathbb{R}^E$ be vectors such that $0 \leq \ell < T$ and $0 \leq u - \ell < T$.

Further let $\gamma \in \{0,1\}^E$ be an incidence vector of an oriented cycle in $\mathcal{E}$ containing *only forward* activities. Suppose that $y \in \mathbb{R}^E$ with $0 \leq y \leq u - \ell$ is a feasible periodic slack for the periodic event scheduling instance given by $\mathcal{E}$, T, ℓ, u and an arbitrary weight vector.

Prove that

$$[-\gamma^t \ell]_T \leq \gamma^t y \leq \gamma^t u - \gamma^t \ell - [\gamma^t u]_T.$$

Solution:

Direct solution: Since y is a feasible slack for the periodic event scheduling instance given by $\mathcal{E}, T, \ell, u$, the cycle periodicity condition

$$\gamma^t(y + \ell) \equiv 0 \mod T$$

is satisfied. In particular

$$[-\gamma^t \ell]_T = [\gamma^t y]_T \text{ and } [\gamma^t u]_T = [\gamma^t(u - \ell - y)]_T.$$

Since $\gamma, y, u - \ell - y$ are all non-negative vectors, and $[a]_T$ is the smallest non-negative number b with $a \equiv b \bmod T$, we have $\gamma^t y \geq [\gamma^t y]_T$ and analogously $\gamma^t(u - \ell - y) \geq [\gamma^t(u - \ell - y)]_T$. In total,

$$\gamma^t y \geq [\gamma^t y]_T = [-\gamma^t \ell]_T \text{ and } \gamma^t y \leq \gamma^t(u - \ell) - [\gamma^t(u - \ell - y)]_T = \gamma^t u - \gamma^t \ell - [\gamma^t u]_T.$$

Using the cycle inequality: Since y is a feasible slack for the periodic event scheduling instance given by $\mathcal{E}, T, \ell, u$, the cycle inequality

$$\left\lceil \frac{\gamma_+^t \ell - \gamma_-^t u}{T} \right\rceil \leq \frac{\gamma^t(y + \ell)}{T} \leq \left\lfloor \frac{\gamma_+^t u - \gamma_-^t \ell}{T} \right\rfloor$$

holds. As γ contains only forward activites, $\gamma = \gamma_+$ and $\gamma_- = 0$, and we obtain

$$\left\lceil \frac{\gamma^t \ell}{T} \right\rceil \leq \frac{\gamma^t(y + \ell)}{T} \leq \left\lfloor \frac{\gamma^t u}{T} \right\rfloor$$

and hence

$$T \left\lceil \frac{\gamma^t \ell}{T} \right\rceil - \gamma^t \ell \leq \gamma^t y \leq T \left\lfloor \frac{\gamma^t u}{T} \right\rfloor - \gamma^t \ell.$$

Since $[a]_T = a - T \left\lfloor \frac{a}{T} \right\rfloor = a + T \left\lceil -\frac{a}{T} \right\rceil$, we obtain

$$[-\gamma^t \ell]_T \leq \gamma^t y \leq \gamma^t u - \gamma^t \ell - [\gamma^t u]_T.$$

Points:

cycle periodicity condition $\gamma^t(y+\ell) \equiv 0 \bmod T$	1
first inequality	2
second inequality	2
total	**5**

Exercise 4 **5 points**

The *minimum vertex cover problem* is the following: Given an undirected graph H, find a subset $W \subseteq V(H)$ of minimum cardinality such that each edge of H is incident to a vertex in W.

Construct a polynomial-time reduction of the minimum vertex cover problem to the following line planning problem: Given an undirected graph G, a line pool $\mathcal{L}_0$, lower frequency bounds $f^{\min} : E(G) \to \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$ with the minimum number of lines subject to

$$\forall e \in E(G) : f_e^{\min} \leq \sum_{\ell \in \mathcal{L}\colon e \in E(\ell)} f_\ell \quad \text{and} \quad \forall \ell \in \mathcal{L}_0 : f_\ell \in \{0, 1\}.$$

Hint: Construct a complete graph on $2|E(H)|$ vertices.

Solution:

Let G be the complete graph on the vertices $\{e^+, e^- \mid e \in E(H)\}$. For each vertex $v \in V(H)$, fix an enumeration $e_{v,1}, e_{v,2}, \ldots, e_{v,\deg v}$ of its incident edges. Define the line pool $\mathcal{L}_0$ via

$$\mathcal{L}_0 := \{(e^+_{v,1}, e^-_{v,1}, e^+_{v,2}, e^-_{v,2}, \ldots, e^+_{v,\deg(v)}, e^-_{v,\deg(v)}) \mid v \in V(H)\}$$

and set the lower frequency bound $f^{\min}$ to 1 on all edges of the form $\{e^+, e^-\}$ for some $e \in E(H)$ and to 0 otherwise.

Claim: Let $k \in \mathbb{N}$. Then there is a vertex cover W of H with $|W| = k$ if and only if there is a feasible line plan $(\mathcal{L}, f)$ for $(G, \mathcal{L}_0, f^{\min})$ with k lines.

Proof: Any subset $W \subseteq V(H)$ with $|W| = k$ naturally gives rise to a collection of k lines. This results in a line plan covering each edge $\{e^+, e^-\} \in E(G)$ with frequency at least one, where $e \in E(H)$ is incident to some vertex in W . In particular, a vertex cover W covers all such edges. Conversely, if a feasible line plan uses k lines, then these correspond to the vertices of a vertex cover using k vertices. □

Points:

construction of G	1
construction of line pool	1
construction of lower frequency bound	1
proof of reduction	2
total	**5**