Exercise 1 **6 points**

Decide whether the following statements are true or false. Justify your decision.

(a) Let G be an undirected graph with positive cost function $c : E(G) \to \mathbb{R}_{>0}$. Then any optimal solution to the Chinese Postman problem on (G, c) visits each edge at most twice.

(b) Let $\mathcal{N}$ be a line network with an aperiodic timetable π. If the length of any activity is positive, then the aperiodic time expansion of $\mathcal{N}$ w.r.t. π contains no cycles.

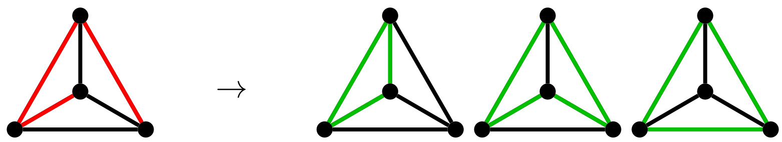
(c) Any cycle basis of the complete graph K_4 on 4 vertices is a 2-basis, i.e., each edge is contained in at most two cycles of the basis.

Solution:

(a) *True:* Let T be an arbitrary optimal Chinese Postman tour of cost $c(T)$. Let G' be the graph where each edge is multiplied according to the number of visits by T. Then G' is Eulerian, and T can be seen as an Euler tour in G'. Suppose that T visits some edge $e \in E(G)$ at least three times. Removing two copies of e from G' does not change the evenness of the vertex degrees in G', so G' remains Eulerian and an Euler tour has cost $c(T) - 2c_e < c(T)$. In particular, T was not optimal – contradiction.

(b) *True:* Let $\{v_1, \ldots, v_n\}$ be a cycle in the aperiodic time expansion. Since every activity has positive length, $\pi(v_1) < \cdots < \pi(v_n) < \pi(v_1)$ – contradiction.

(c) *False:* Consider the fundamental cycle basis induced by the following spanning tree of K_4:



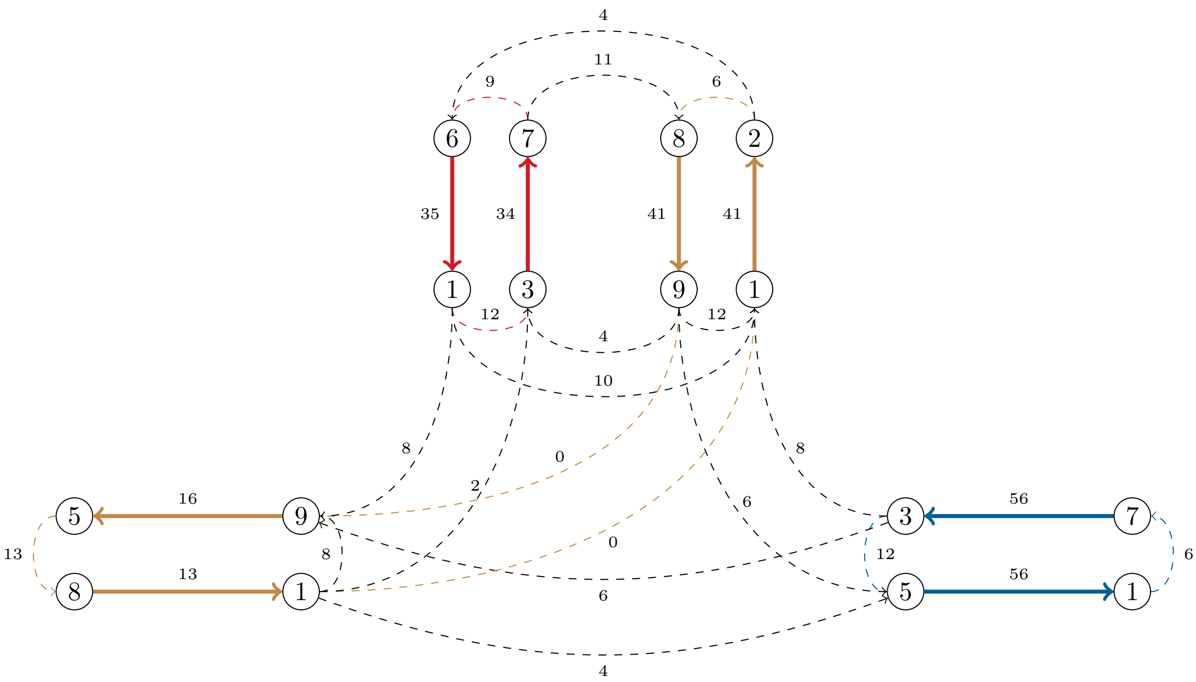
The edge from the bottom left vertex to the top vertex is contained in all 3 cycles of the basis.

Points:

proving (a)	2
proving (b)	2
disproving (c)	2
total	**6**

Exercise 2 **6 points**

Consider the following event-activity network $\mathcal{E}$:



The network $\mathcal{E}$ is equipped with a periodic timetable $\pi : V(\mathcal{E}) \to \{0, 1, \dots, 9\}$ with period time $T = 10$. Each vertex v is labeled with $\pi(v)$. Moreover, each edge carries a length compatible with π. The solid arcs represent driving activities, whereas the dashed arcs mark turnaround activities.

(a) Consider the periodic vehicle schedule S where each of the three lines (red, brown, blue) is run separately. Note that the brown line covers four driving activities. What is the minimum number of vehicles required to operate S?

(b) Is there a better vehicle schedule needing less vehicles than S?

Solution:

(a) A periodic vehicle schedule is a circulation in $\mathcal{E}$ covering each driving activity exactly once. Its number of vehicles is given by $\frac{1}{T}\sum_C \ell(C)$, where the summation ranges over the cycles contained in the circulation, ℓ is their length, and T is the period time.

The schedule S consists of three cycles $C_{\text{red}}, C_{\text{brown}}, C_{\text{blue}}$. Hence the number $n(S)$ of vehicles required for S is given by

$$\begin{aligned}
n(S) &= \frac{\ell(C_{\text{red}}) + \ell(C_{\text{brown}}) + \ell(C_{\text{blue}})}{10} \\
&= \frac{35 + 12 + 34 + 9 + 16 + 13 + 13 + 0 + 41 + 6 + 41 + 0 + 56 + 12 + 56 + 6}{10} \\
&= \frac{90 + 130 + 130}{10} = \frac{350}{10} = 35.
\end{aligned}$$

(b) Periodic vehicle schedules correspond to perfect matchings of the subgraph of $\mathcal{E}$ induced by the turnaround activities. A periodic vehicle schedule with the minimum number of vehicles corresponds to a perfect matching of minimum weight w.r.t. ℓ, or the periodic offsets $p(ij) = (\ell(ij) - \pi(j) + \pi(i))/10$, $ij \in E(\mathcal{E})$. Hence we need to determine with perfect matching corresponds to S and if there is a perfect matching of less weight.

The subgraph induced by the turnaround activities decomposes into four connected components at the top, left, right and center. Since the left and right component consist of a single edge, there is a unique perfect matching. The top component is a complete bipartite graph on $2 \cdot 2$ vertices and has two possible perfect matchings, but both have the same weight w.r.t. ℓ (15). Therefore, savings are only possible in the central component. The schedule S induces the following perfect matching, weights are w.r.t p:

dep $\rightarrow$ $\downarrow$ arr	brown left 9	red top 3	brown top 1	blue right 5
1 brown left	0	0	0	0
1 red top	0	1	1	–
9 brown top	0	1	2	1
3 blue right	0	–	1	1

Thus S produces a perfect matching of the central component with p-weight 2. As can be easily seen from the table, this perfect matching is already of minimum p-weight.

Points:

(a) explanation of method	1
(a) computations	2
(b) proving optimality	3
total	**6**

Exercise 3 **6 points**

Let $\mathcal{E} = (V, E)$ be a weakly connected directed graph. Fix a period time $T \in \mathbb{N}$, $T \geq 2$.

For any periodic event scheduling instance on $\mathcal{E}$ with lower bound vector $\ell \equiv 0$ and upper bound vector $u \equiv T - 1$, the set of feasible periodic timetables is given by the space of all maps $\pi : V \to \{0, 1, \ldots, T-1\}$, which can be identified with $(\mathbb{Z}/T\mathbb{Z})^V$.

Consider the following $\mathbb{Z}$- and $(\mathbb{Z}/T\mathbb{Z})$-linear map φ translating periodic timetables into periodic tensions:

$$\varphi : (\mathbb{Z}/T\mathbb{Z})^V \to (\mathbb{Z}/T\mathbb{Z})^E, \quad (\pi_v)_{v \in V} \mapsto (\pi_w - \pi_v)_{vw \in E}.$$

(a) Taking standard bases on both sides, which special integer matrix represents φ?

(b) Compute $\ker(\varphi) = \{\pi \in (\mathbb{Z}/T\mathbb{Z})^V \mid \varphi(\pi) \equiv_T 0\}$.

(c) How many elements $x \in (\mathbb{Z}/T\mathbb{Z})^E$ are feasible periodic tensions for the above type of periodic event scheduling problems?

Solution:

(a) Let $A = (a_{v,e}) \in \mathbb{Z}^{V \times E}$ be the incidence matrix of $\mathcal{E}$. Then, for $vw \in E$ and $(\pi_v) \in (\mathbb{Z}/T\mathbb{Z})^V$, $\sum_{u \in V} a_{u,vw}\pi_u \equiv_T \pi_w - \pi_v$. In particular, φ is represented by the transpose of the incidence matrix.

(b) We have

$$\ker(\varphi) = \{\pi \in (\mathbb{Z}/T\mathbb{Z})^V \mid \varphi(\pi) \equiv_T 0\} = \{\pi \in (\mathbb{Z}/T\mathbb{Z})^V \mid \forall vw \in E : \pi_v \equiv_T \pi_w\}.$$

Since $\mathcal{E}$ is weakly connected, there is always an oriented path between any two distinct vertices. Since $\pi \in \ker(\varphi)$ if and only if the value of π does not change along every edge,

$$\ker(\varphi) = \{\pi \in (\mathbb{Z}/T\mathbb{Z})^V \mid \pi_v \text{ constant}\} = \{a \cdot (1, 1, \dots, 1) \mid a \in \mathbb{Z}/T\mathbb{Z}\}.$$

(c) An element $x \in \mathbb{Z}^E$ is feasible for a periodic event scheduling problem with lower bounds ℓ and upper bounds u if and only if there is a periodic timetable π such that $x = A^t\pi$ and $\ell \leq x \leq u$, where A is the incidence matrix of $\mathcal{E}$. Since in our situation, we have $\ell = 0$ and $u = T - 1$, we conclude by (a) that an element $x \in (\mathbb{Z}/T\mathbb{Z})^E$ is feasible if and only if x lies in the image of φ. Using (b) and the fundamental theorem on homomorphisms,

$$\#\operatorname{im}(\varphi) = \frac{\#(\mathbb{Z}/T\mathbb{Z})^V}{\#\ker(\varphi)} = \frac{T^{|V|}}{T} = T^{|V|-1}.$$

Points:

(a) naming incidence matrix	1
(a) justification	1
(b) connectivity argument	1
(b) π must be constant	1
(c) number of elements	1
(c) argumentation	1
total	**6**

Exercise 4 **6 points**

The *knapsack minimization problem* is the following: Given n items with costs $c_1, \dots, c_n \in \mathbb{N}$, weights $w_1, \dots, w_n \in \mathbb{N}$, and a weight bound $W \in \mathbb{N}$, find a subset $I \subseteq \{1, \dots, n\}$ of the items such that $\sum_{i \in I} w_i \geq W$ and $\sum_{i \in I} c_i$ is minimal.

Construct a polynomial-time reduction of the knapsack minimization problem to the following line planning problem: Given an undirected graph G, a line pool $\mathcal{L}_0$, line costs $c : \mathcal{L}_0 \to \mathbb{N}$, lower edge-frequency bounds $f^{\min} : E(G) \to \mathbb{N}_0$, and upper line-frequency bounds $F : \mathcal{L}_0 \to \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$

$$\begin{aligned} &\text{minimizing} && \sum_{\ell \in \mathcal{L}} c_\ell && \\ &\text{subject to} && \sum_{\ell \in \mathcal{L} \colon e \in E(\ell)} f_\ell \geq f_e^{\min}, && e \in E(G), \\ & && f_\ell \in \{0, 1, \ldots, F_\ell\}, && \ell \in \mathcal{L}_0. \end{aligned}$$

Hints: As always, $\mathcal{L} = \{\ell \in \mathcal{L}_0 \mid f_\ell \geq 1\}$. Create a line for each item.

Solution:

Given a Knapsack minimization instance as in the description of the exercise, construct the following line planning instance $(\star)$:

- G is a graph containing a single edge e,
- $\mathcal{L}_0$: n lines $\ell_1, \ldots, \ell_n$ on e,
- $c_{\ell_i} := c_i$, $i = 1, \ldots, n$,
- $F_{\ell_i} := w_i$, $i = 1, \ldots, n$,
- $f_e^{\min} := W$.

Claim: The Knapsack minimization instance has a solution I of cost C if and only if $(\star)$ has a solution $(\mathcal{L}, f)$ of cost C.

Proof: $(\Rightarrow)$ Select the line i iff $i \in I$ and set $f_{\ell_i} := F_{\ell_i} = w_i$. Then the minimum frequency requirement on the edge e is met due to $\sum_{i \in I} w_i \geq W$. Moreover $\sum_{i=1}^n c_{\ell_i} = \sum_{i \in I} c_i$.
$(\Leftarrow)$ Select item i iff ℓ_i is in the optimal line plan. Again, the costs are the same. Finally $\sum_{i \in I} w_i = \sum_{i \in I} F_{\ell_i} \geq \sum_{i=1}^n f_{\ell_i} \geq f_e^{\min} = W$. $\square$

Points:

working graph and line pool	1
working costs	1
working edge frequency bounds	1
working line frequency bounds	1
proof of reduction	2
total	**6**