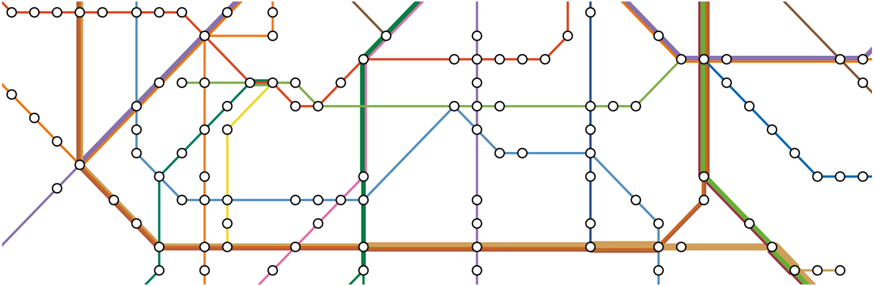
Mathematical Aspects of Public Transportation Networks

Niels Lindner



July 9, 2018

Reminder

Dates

- July 12: no tutorial (excursion)
- July 16: evaluation of Problem Set 10 and Test Exam
- July 19: no tutorial
- July 19/20: block seminar on shortest paths
- August 7 (Tuesday): 1st exam
- October 8 (Monday): 2nd exam

Exams

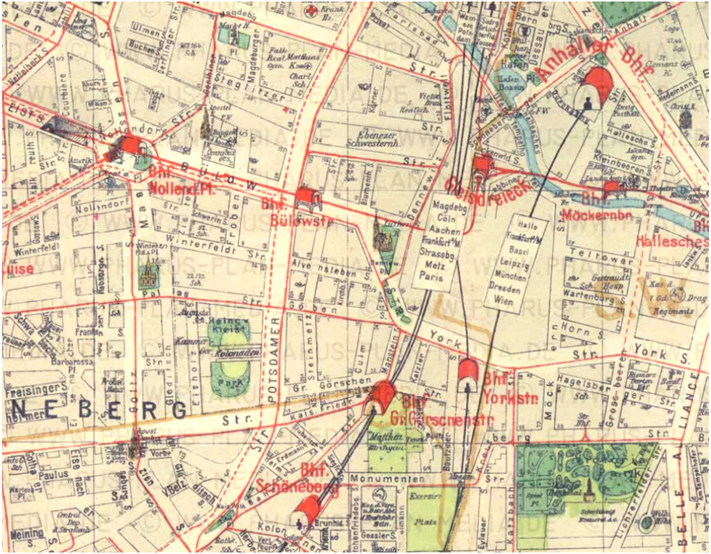
- location: ZIB seminar room
- start time: 10.00am
- duration: 60 minutes
- permitted aids: one A4 sheet with notes

Chapter 6

Metro Map Drawing

§6.1 Metro Maps

Berlin, 1921 (Pharus-Plan)

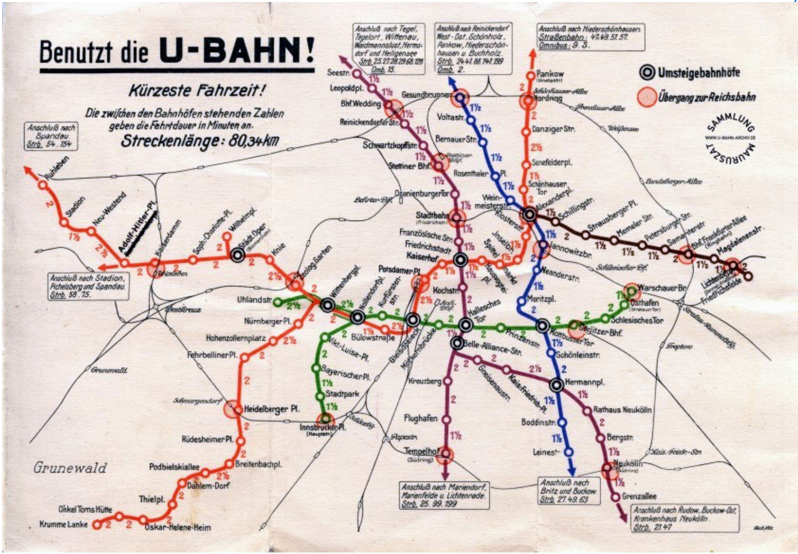


alt-berlin.info

Berlin, 1913 (Hochbahngesellschaft)

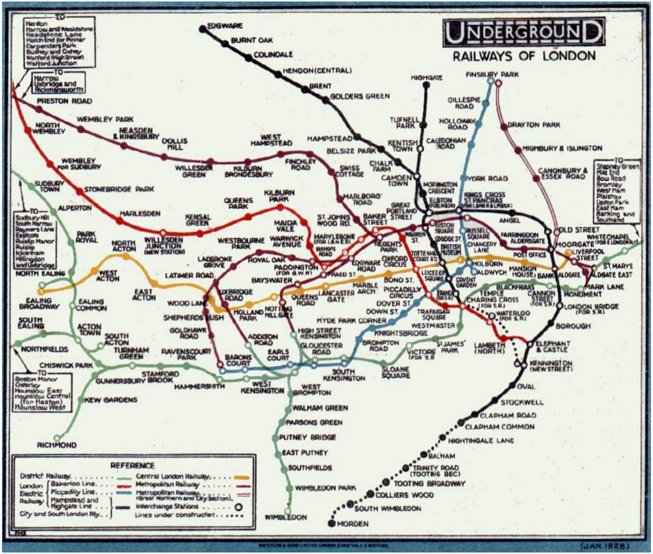
Sammlung Mauruszat, u-bahn-archiv.de

Berlin, 1933 (BVG)



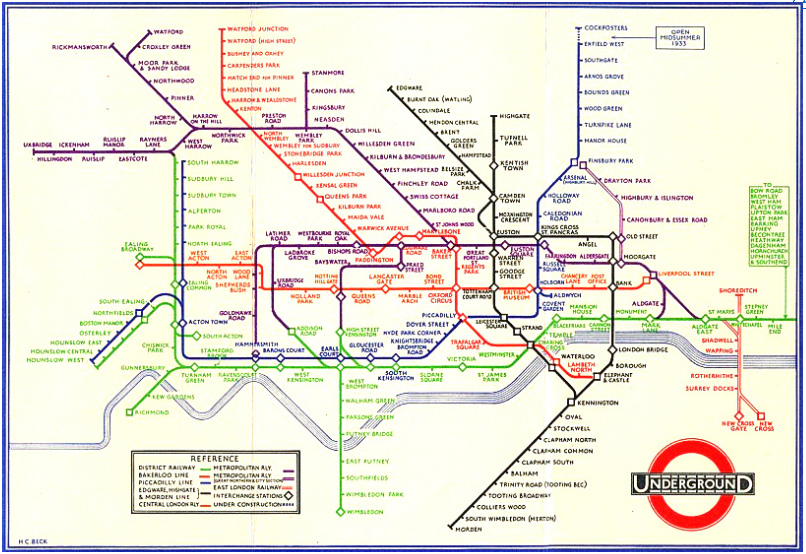
Sammlung Mauruszat, u-bahn-archiv.de

London, 1926 (London Underground)

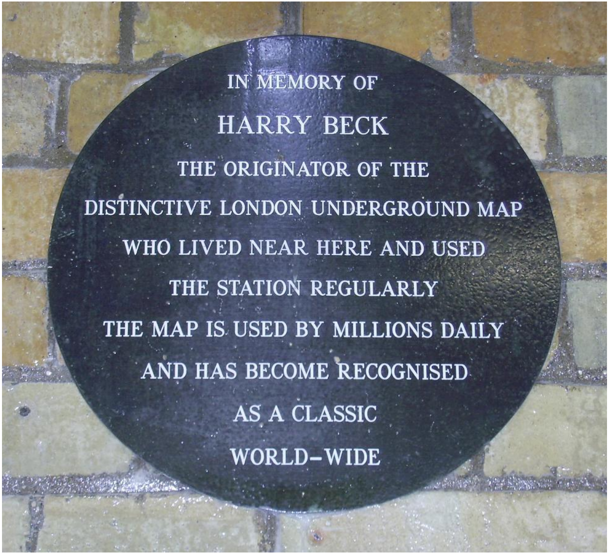


commons.wikimedia.org

London, 1933 (Harry Beck)

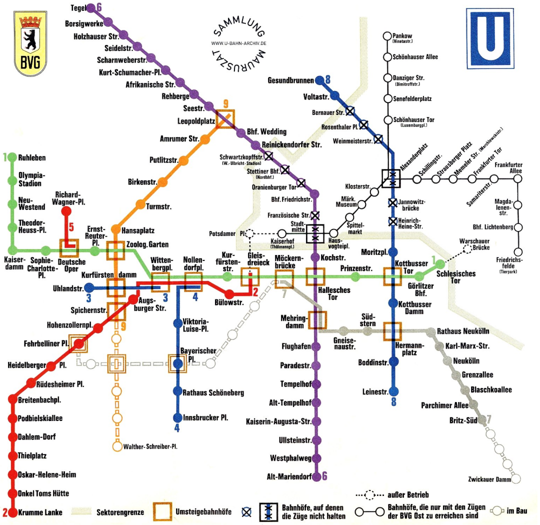


Plaque at Finchley Central station



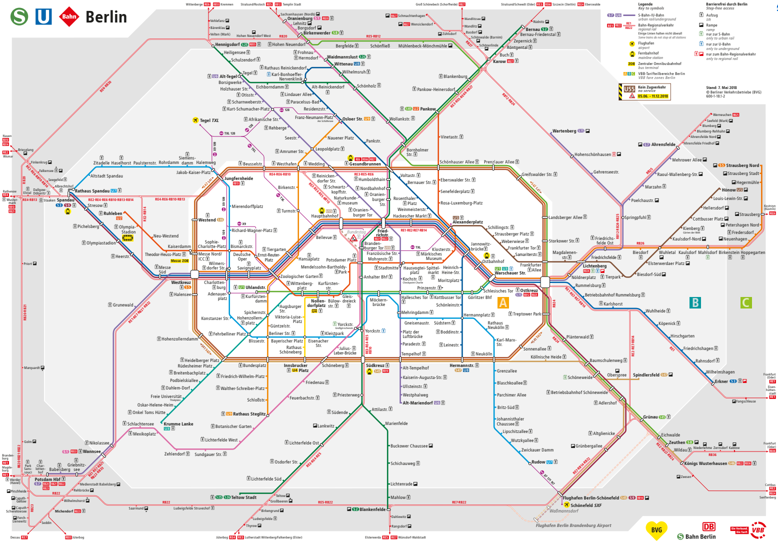
Nick Cooper, commons.wikimedia.org

Berlin, 1968 (BVG-West)



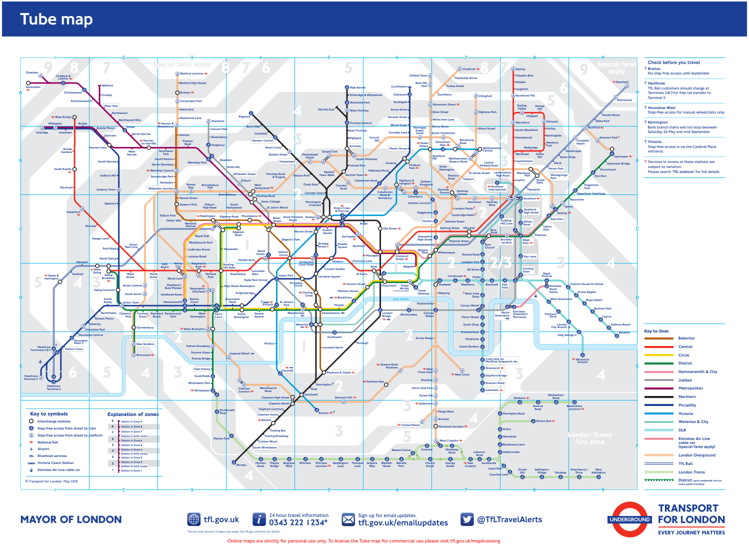
Sammlung Mauruszat, u-bahn-archiv.de

Berlin, 2018 (BVG)



Berliner Verkehrsbetriebe, bvg.de

London, 2018 (Transport for London)



Transport for London, tfl.gov.uk

Saint Petersburg, 2018

Петербургский Метрополитен, metro.spb.ru

Chapter 6

Metro Map Drawing

§6.2 Planar Graphs

Planar Embeddings

Let $G = (V, E)$ be a (simple) graph.

Definition

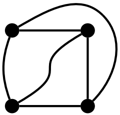
A **planar embedding** of G consists of an injective map $\psi : V \to \mathbb{R}^2$, and a family of continuous and injective functions $f_e : [0,1] \to \mathbb{R}^2$ for each edge $e \in E$ such that

- $f_{vw}(0) = \psi(v)$ and $f_{vw}(1) = \psi(w)$ for all $vw \in E$,
- $f_e((0,1)) \cap \bigcup_{e' \neq e} f_{e'}([0,1]) = \emptyset$ for all $e \in E$.

G is called **planar** if it admits some planar embedding.

Remark

This embeds a graph into the plane using simple Jordan curves.



Faces

Definition

A connected component of $\mathbb{R}^2 \setminus \bigcup_{e \in E} f_e([0,1])$ is called a **face**.

Observations

- There is exactly one unbounded face (Jordan curve theorem).
- The boundary of a bounded face F gives rise to a *face circuit* C_F in G.

Lemma

$\{C_F \mid F$ is a bounded face$\}$ is an undirected cycle basis of G.

Proof.

Let C be a circuit in G. Then $\mathbb{R}^2 \setminus \bigcup_{e \in E(C)} f_e([0,1])$ is the union of a bounded and an unbounded component. Let $F_1, \ldots, F_k$ be the faces contained in the bounded component. Then $C = C_{F_1} + \cdots + C_{F_k}$.
Suppose $\sum_{F \text{ bounded face}} \lambda_F C_F = 0$ for some $\lambda_F \in \mathbb{F}_2$. If e is an edge between a bounded face F and the outer face, then $\lambda_F = 0$. Proceed by induction on the number of bounded faces. □

Euler's formula

Let G be a planar graph with n vertices, m edges, f faces and c connected components.

Corollary (Euler, 1758)

$$f - 1 = m - n + c.$$

Lemma

Suppose that G is connected and $n \geq 3$.

(1) $2m \geq 3f$ (2) $m \leq 3n - 6$ (3) $f \leq 2n - 4$

Proof.

(1) Handshake: $2m = \sum_{v \in V} \deg(v) = \sum_{F \text{ face}} \#\{\text{vertices along } F\} \geq 3f$.
If $n \geq 3$, then the outer face contains at least 3 vertices.

(2) $2m \geq 3f = 3(m - n + 2) = 3m - 3n + 6 \Rightarrow m \leq 3n - 6$.

(3) $3n - 6 \geq m = n + f - 2 \Rightarrow f \leq 2n - 4$. $\square$

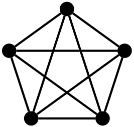
K_5 and $K_{3,3}$

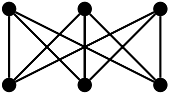
Lemma

The complete graph K_5 is not planar.

Proof.

We have $n = 5$ and $m = \binom{5}{2} = 10$, so $m = 10 > 9 = 3 \cdot 5 - 6 = 3n - 6$. □





Lemma

The complete bipartite graph $K_{3,3}$ is not planar.

Proof.

Every circuit in $K_{3,3}$ has length at least 4. In particular, if $K_{3,3}$ was planar, then $2m \geq 4(f-1) + 3 = 4(m - n + 1) + 3 = 4m - 4n + 7$ and therefore $2m \leq 4n - 7$. But $2m = 2 \cdot 3^2 = 18$ and $4n - 7 = 4 \cdot 6 - 7 = 17$. □

Minors

Definition

Let G and M be undirected graphs. M is a **minor** of G if a series of the following operations transforms G into M:

- delete a vertex
- delete an edge
- contract an edge

Theorem (Wagner, 1937)

An undirected graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

Proof.

($\Rightarrow$) Any minor of a planar graph must be planar.
($\Leftarrow$) Omitted. $\square$

More on Minors

Theorem (Robertson/Seymour, 2004)

A family of graphs is closed under taking minors if and only if it has a finite number of minimal forbidden minors.

Example

Planar graphs: K_5, $K_{3,3}$ Forests: C_3 (circuit on 3 vertices)

Theorem (Robertson/Seymour, 1995)

For any fixed minor M, there is a $\mathcal{O}(n^3)$ algorithm deciding whether a graph on n vertices has M as a minor.

This is nice, but the proof is not constructive. For planarity testing, there are better (and explicit) algorithms:

Theorem (Hopcroft/Tarjan, 1974)

Planarity testing can be done in $\mathcal{O}(n)$ time.

Subdivisions

Definition

Let G and M be undirected graphs. M is a **subdivision** of G if it is obtained from G by consecutively replacing an edge uw with two edges uv, vw, inserting a new vertex v.

The reverse process is called **smoothing**.

Theorem (Kuratowski, 1930)

An undirected graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof.

(⇒) Any subgraph and any smoothing of a planar graph must be planar.
(⇐) Omitted. □

2-bases

Observation

Let G be a planar embedded graph. Then every edge is contained in at most two face circuits.

Definition

A cycle basis $\mathcal{B}$ of a graph is called a **2-basis** if every edge is contained in at most two cycles of $\mathcal{B}$.

Theorem (MacLane, 1937)

A graph is planar if and only if it admits a 2-basis.

Proof (O'Neill, 1973).

Planar graphs have 2-bases. The following would prove the converse:

(1) Admitting a 2-basis is closed under taking subgraphs and smoothings.

(2) K_5 and $K_{3,3}$ do not admit a 2-basis.

Then any graph with a 2-basis does not contain a subdivision of K_5 or $K_{3,3}$, and must hence be planar by Kuratowski's theorem.

MacLane's planarity criterion

Proof of (1).

Let $\{C_1, \ldots, C_\mu\}$ be a 2-basis of a graph $G = (V, E)$.

Deleting an edge $e \in E$: If e is not contained in any cycle, nothing happens. Otherwise this reduces the cyclomatic number by 1. If e is contained in a single cycle (w.l.o.g. C_1), then $\{C_2, \ldots, C_\mu\}$ is a 2-basis of $(V, E \setminus \{e\})$. If e is contained in two cycles (w.l.o.g. C_1, C_2), then $\{C_1 + C_2, \ldots, C_\mu\}$ is a 2-basis of $(V, E \setminus \{e\})$.

Deleting a vertex: Delete first all adjacent edges. Removing an isolated vertex does not affect the cyclomatic number.

Smoothing uv and vw to uw: The columns of uv and vw in the cycle matrix are the same, so smoothing does not change anything about the 2-basis.

MacLane's planarity criterion

Proof of (2).

Let $\{C_1, \dots, C_6\}$ be a 2-basis for K_5. Set $C_7 := \sum_{i=1}^{6} C_i$. Observe that $C_7 \neq 0$ and $\sum_{i=1}^{7} C_{i,e} = 0 \in \mathbb{F}_2$ for all $e \in E$. In particular, every edge is contained in exactly two of the cycles $C_1, \dots, C_7$. Hence $\sum_{i=1}^{7} |E(C_i)| = 2|E(K_5)| = 20$.
On the other hand, $|E(C_i)| \geq 3$ for all i, so $\sum_{i=1}^{7} |E(C_i)| \geq 21$.

Now let $\{C_1, \dots, C_4\}$ be a 2-basis for $K_{3,3}$. Set $C_5 := \sum_{i=1}^{4} C_i$. Again $\sum_{i=1}^{5} C_{i,e} = 0$ for all $e \in E$, so $\sum_{i=1}^{5} |E(C_i)| = 2|E(K_{3,3})| = 18$. Since $|E(C_i)| \geq 4$ for all i, $\sum_{i=1}^{5} |E(C_i)| \geq 20$. $\square$

More on 2-bases

Lemma

The cycle matrix of a 2-basis of a directed graph is totally unimodular. In particular, any 2-basis is integral.

Proof.

Orient all cycles counter-clockwise w.r.t. some planar embedding. Proceed by induction on the size q of a quadratic submatrix A of the cycle matrix. The case $q = 1$ is clear. If $q \geq 2$, there are two cases:

(1) A contains a column with a single non-zero entry. Use Laplace expansion and induction.

(2) All columns of A have at least two non-zero entries. Since we have a 2-basis, there are exactly two non-zeros, a $+1$ and a -1. So all rows of A add up to 0, showing $\det A = 0$.

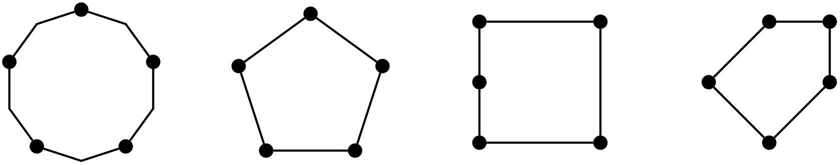
□

Graph drawings

Let G be a planar graph.

Types of planar embeddings

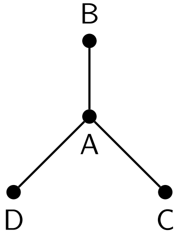
- *polygonal*: All edges are embedded as polygonal arcs.
- *straight line*: All edges are drawn as straight line segments.
- *rectilinear*: All edges are drawn as straight line segments with slopes $k \cdot 90^\circ$, $k \in \mathbb{Z}$.
- *octilinear*: All edges are drawn as straight line segments with slopes $k \cdot 45^\circ$, $k \in \mathbb{Z}$.



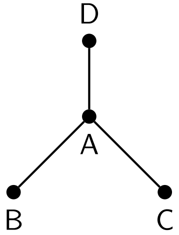
Combinatorial type

Definition

Let $G = (V, E)$ be a planar graph with some planar embedding. For each vertex $v \in V$, the embedding defines an ordering of the neighborhood of v by counter-clockwise sorting of the edges incident to v. This is the **combinatorial type** of the embedding.



A: (B, D, C)



A: (D, B, C)

This defines an equivalence relation on planar embeddings of G.

Straight line drawings

Theorem (Fáry, 1948)

Any simple planar graph has a straight line drawing.

Proof.

Let G be a simple connected planar graph with $n \geq 3$ vertices, embedded into the plane. W.l.o.g. G is maximally planar, i.e., $m = 3n - 6$, and every bounded face is a triangle.

Claim: If uvw is a triangle, then there is a straight line embedding of the same combinatorial type where u, v, w are the vertices along the outer face.

This is easy for $n = 3$. Thus let $n \geq 4$, and let uvw be a triangle. Not all of u, v, w have degree 2 (connectedness). Assume that the other $n - 3$ vertices have degree at least 6. Then

$$\sum_{v \in V} \deg(v) \geq 6(n-3) + 7 = 6n - 11 > 6n - 12 = 2m,$$

contradicting the Handshaking Lemma. □

Straight line drawings

Proof (cont.)

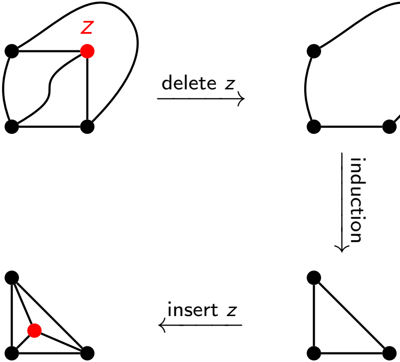
In particular, we find a vertex z different from u, v, w with $\deg(z) \leq 5$. Now remove z from the graph and retriangulate the new face F. The new graph has $n - 1$ vertices and – by induction – admits a straight line embedding of the same combinatorial type where u, v, w are the vertices along the outer face. In particular, the face F has become a simple polygon with at most 5 sides.

By the art gallery theorem (with $\lfloor 5/3 \rfloor = 1$ guards), there is a point inside this polygon that can be connected to all vertices of F by non-crossing straight lines (Short proof: Triangulations of simple polygons admit a 3-coloring.) □

Theorem (De Fraysseix/Pach/Pollack, 1990)

Straight line drawings on a grid can be found in linear time.

Straight line drawings: Example



Rectilinear and octilinear drawings

Let G be a graph.

Theorem (Garg/Tamassia, 2001)

It is NP-hard to decide whether G admits a rectilinear drawing.

Theorem (Tamassia, 1987)

Fix some planar embedding of G. There is a polynomial-time algorithm that decides whether G admits a rectilinear drawing preserving the combinatorial type.

Theorem (Nöllenburg, 2005)

Fix some planar embedding of G. It is NP-hard to decide whether G admits an octilinear drawing preserving the combinatorial type.

Both NP-hardness proofs reduce (variants of) the 3-SAT problem.

Remark

Clearly, if G admits a rectilinear (octilinear) drawing, then $\deg(v) \leq 4$ (≤ 8) for all $v \in V(G)$.

Chapter 6

Metro Map Drawing

§6.3 Octilinear Layout Computation

Requirements

Design Principles (taken from Nöllenburg, 2011)

- Preserve the combinatorial type.
- Octilinear drawing: All edges are drawn as line segments with slopes $k \cdot 45^\circ$ for $k \in \{0, 1, \ldots, 7\}$.
- Lines should avoid sharp bends, and pass straight through interchanges.
- Ensure a minimum distance between stations, and stations and non-incident edges.
- Minimize geometric distortion.
- Use uniform edge lengths.
- Use large angular resolution.
- Place station labels in a readable way.
- ...

Metro Map Layout Problem

Input

- a line network consisting of a graph $G = (V, E)$ of maximum degree 8 and a line cover $\mathcal{L}$
- a planar embedding of G, e.g., by geographical coordinates

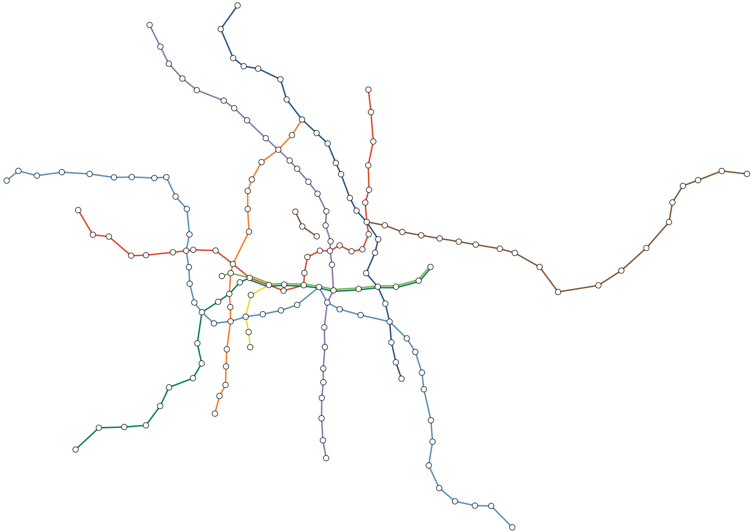
Output

- a map $\psi : V \to \mathbb{R}^2$ inducing an octilinear drawing of G
- satisfying/optimizing design principles

Solution Methods

- metaheuristics (hill climbing, simulated annealing, ant colonies, ...)
- local optimization: least squares
- global optimization: mixed integer programming

U-Bahn Berlin: geographical layout

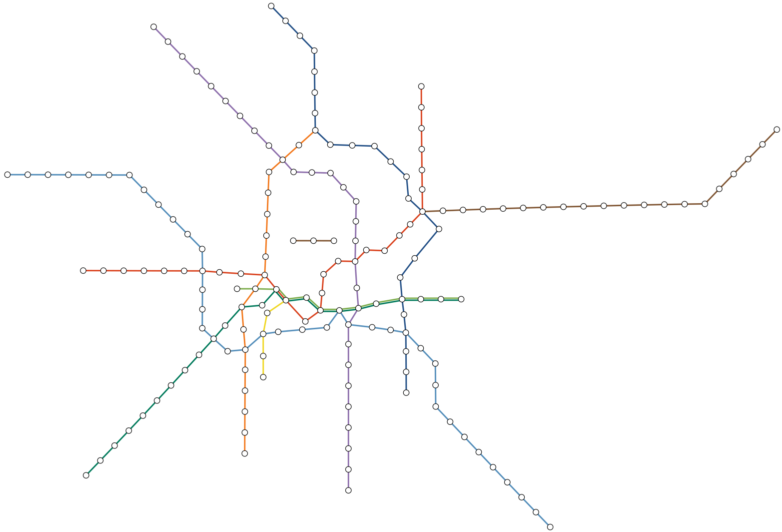


173 vertices, 184 edges, 10 lines

U-Bahn Berlin: curvilinear layout

least squares method (Wang/Chi 2011, Wang/Peng 2016)

U-Bahn Berlin: octilinear layout?



least squares method (Wang/Chi 2011, Wang/Peng 2016), naive python/cvxopt implementation: 36 s

Octilinear Layout MIP

We will use the formulation due to Nöllenburg (2005):

Hard constraints (feasibility)

- octilinearity
- combinatorial type preservation
- minimum edge length
- minimum distance for non-adjacent edges

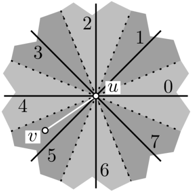
In particular, a feasible solution *guarantees* octilinearity.

Soft constraints (objective)

- bend minimization
- preservation of relative positions for adjacent stations
- minimum total edge length

Octilinear Layout MIP: Basic variables

- *vertex coordinates* $x_v, y_v \in [0, |V|]$ for $v \in V$
- additional vertex coordinates $z_v := x_v + y_v$, $w_v := x_v - y_v$
- *edge directions* $\mathrm{dir}_{vw}, \mathrm{dir}_{wv} \in \{0, 1, \ldots, 7\}$ for $\{v, w\} \in E$
- *original directions* $\mathrm{sec}_{vw} := \left[\left\lfloor \frac{\sphericalangle(v,w)}{45^\circ} + \frac{1}{2} \right\rfloor\right]_8$,
  where $\sphericalangle(v, w) \in (-180^\circ, 180^\circ]$ is the slope of the edge $\{v, w\}$



Nöllenburg (2005)

Octilinear Layout MIP: Directions

- binary variables $\alpha_{i,vw}$ for $i \in \{-1, 0, 1\}$ and $\{v, w\} \in E$ such that $\alpha_{i,vw} = 1 \Leftrightarrow$ edge $\{v, w\}$ is in original direction $+i \cdot 45°$

$$\begin{aligned}
&\alpha_{-1,vw} + \alpha_{0,vw} + \alpha_{1,vw} = 1, && \{v, w\} \in E \\
&\mathrm{dir}_{vw} + M\alpha_{i,vw} \le M + [\mathrm{sec}_{vw} + i]_8, && i \in \{-1, 0, 1\}, \{v, w\} \in E \\
&\mathrm{dir}_{vw} - M\alpha_{i,vw} \ge -M + [\mathrm{sec}_{vw} + i]_8, && i \in \{-1, 0, 1\}, \{v, w\} \in E \\
&\mathrm{dir}_{wv} + M\alpha_{i,vw} \le M + [\mathrm{sec}_{vw} + i + 4]_8, && i \in \{-1, 0, 1\}, \{v, w\} \in E \\
&\mathrm{dir}_{wv} - M\alpha_{i,vw} \ge -M + [\mathrm{sec}_{vw} + i + 4]_8, && i \in \{-1, 0, 1\}, \{v, w\} \in E
\end{aligned}$$

$M \gg 0$ is a large constant

- relating directions with coordinates, e.g., for $\mathrm{sec}_{vw} = 2$ and $\alpha_{0,vw}$:

$$\begin{aligned}
x_v - x_w + M\alpha_{0,vw} &\le M \\
x_v - x_w - M\alpha_{0,vw} &\ge -M \\
y_v - y_w + M\alpha_{0,vw} &\le M - \text{minimum edge length}
\end{aligned}$$

If $\alpha_{0,vw} = 1$, then $x_v = x_w$ and $y_w \ge y_v +$ minimum edge length.

Octilinear Layout MIP: Planar embedding

- preservation of combinatorial type: binary variables $\beta_{i,v}$ for $i \in \{1, \ldots, \deg(v)\}$ and $v \in V$,

$$\sum_{i=1}^{\deg(v)} \beta_{i,v} = 1, \qquad v \in V,$$

$$\mathrm{dir}_{v,w_j} - \mathrm{dir}_{v,w_{[j-1]_{\deg(v)}}} + M\beta_j(v) \geq 1, \qquad v \in V, j = 1, \ldots, \deg(v),$$

  where $w_1, \ldots, w_{\deg(v)}$ are the neighbors of v in counter-clockwise order

- planarity constraints: modeled by binary variables $\gamma_{i,e,e'}$ for $i \in \{0, \ldots, 7\}$ and $e, e' \in E$ non-incident (large number!)

Octilinear Layout MIP: Objective

- total edge length objective: by new variables len_{vw}, $vw \in E$, e.g., for $\text{sec}_{vw} = 2$ and $\alpha_{0,vw}$:

$$y_v - y_w + M\alpha_{0,vw} = M - \text{len}_{vw}$$

  If $\alpha_{0,vw} = 1$, then $x_v = x_w$, and $y_w \geq y_v = \text{len}_{vw}$. Edge lengths are measured w.r.t. $|\cdot|_\infty$: a diagonal line segment $(x,y) \to (x+1, y+1)$ has length 1.
- bend objective: for each three consecutive stations (u, v, w) on a line of the network, add the constraints

$$\text{bend}_{uvw} = \begin{cases} |\text{dir}_{uv} - \text{dir}_{vw}| & \text{if } |\text{dir}_{uv} - \text{dir}_{vw}| \leq 4 \\ 8 - |\text{dir}_{uv} - \text{dir}_{vw}| & \text{if } |\text{dir}_{uv} - \text{dir}_{vw}| \geq 5 \end{cases}$$

- relative position objective:

$$-M \cdot \text{rel}_{vw} \leq \text{dir}_{vw} - \text{sec}_{vw} \leq M \cdot \text{rel}_{vw}, \quad vw \in E$$

- objective function: $\lambda_1 \sum_{vw} \text{len}_{vw} + \lambda_2 \sum_{uvw} \text{bend}_{uvw} + \lambda_3 \sum_{vw} \text{rel}_{vw}$

Octilinear Layout MIP: Practical Aspects

Size

Suppose that G has n vertices and m edges. Let $m' := \sum_{\ell \in \mathcal{L}} |E(\ell)|$. Then the MIP formulation uses uses $O(n + m' + m^2)$ variables and constraints.

Example

The Berlin U-Bahn example needs

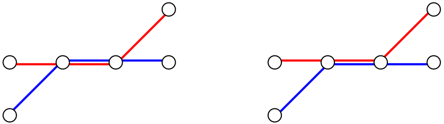
- 3 046 variables (1 623 binary, 546 integer, 877 continuous) and 7 149 constraints *without* planarity constraints.
- 137 158 variables (135 735 binary, 546 integer, 877 continuous) and 560 361 constraints *with* planarity constraints

Pre-processing

- Do not use all planarity constraints (heuristics, lazy constraints).
- Contract vertices of degree 2.

Crossing minimization

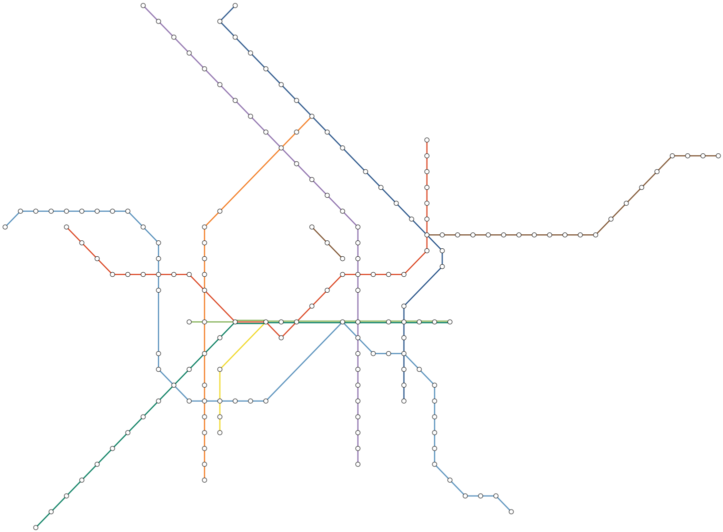
Consider a line network $(G, \mathcal{L})$. The **metro line crossing minimization problem** is to find an ordering of the lines at each station such that the total number of crossings of lines is minimized.



Results

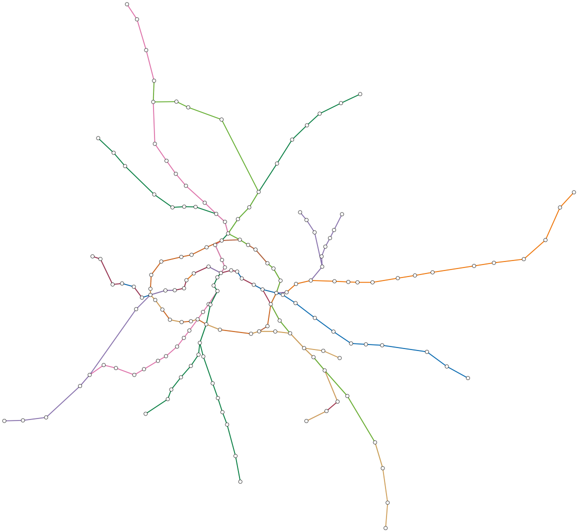
- The problem is in general NP-hard (Fink/Pupyrev, 2013).
- There is an integer programming formulation (Asquith et. al., 2008).
- The key step is to determine the ordering of the lines at their terminals.
- For a fixed ordering at the terminals, there is a polynomial-time algorithm.

U-Bahn Berlin: octilinear layout!

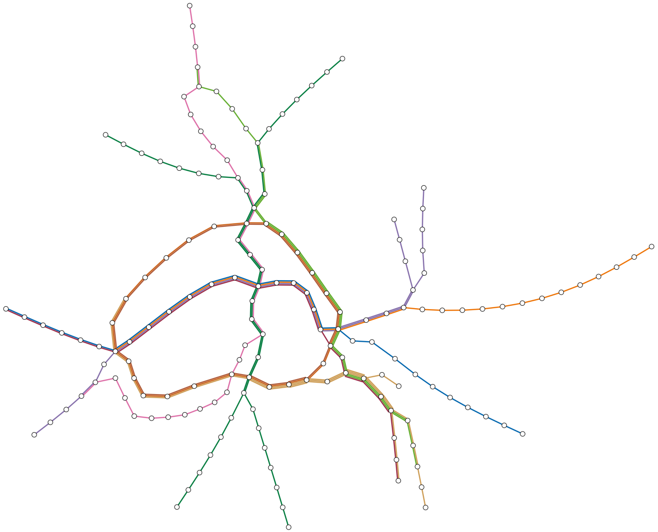


mixed integer programming method (Nöllenburg 2005), solution found by CPLEX 12.7.1 after 66 s, optimality: 15 min 55 s

S-Bahn Berlin: geographical layout

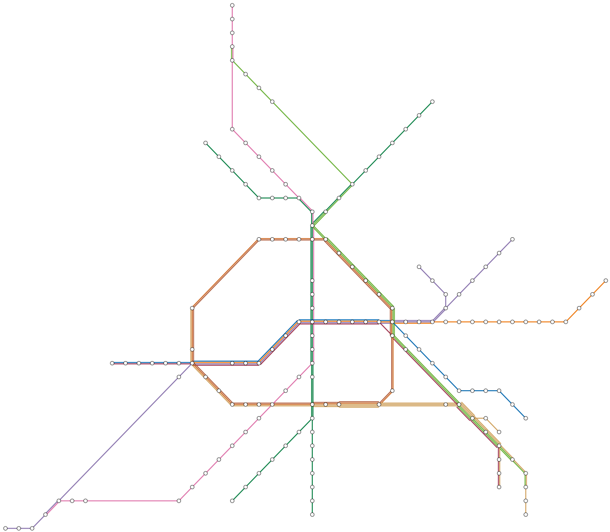


S-Bahn Berlin: curvilinear layout



least squares method, naive python/cvxopt implementation: 31 s

S-Bahn Berlin: octilinear layout



mixed integer programming method, solution found by CPLEX 12.7.1 after 25 s, optimality gap: $\leq 3\%$