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Lecture 14

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6.2 Single-Depot Aperiodic Vehicle Scheduling

Comparison of periodic and aperiodic scheduling

Consider an event-activity network $\mathcal{P}$ with driving activities $E_d(\mathcal{P})$ and turnaround activities $E_t(\mathcal{P})$. Suppose we are given a periodic timetable π on $\mathcal{P}$ w.r.t. some period time T with corresponding activity durations $x \geq 0$. For an integer $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the event-activity network modeling the aperiodic vehicle scheduling problem for all periodic trips starting between time 0 (included) and time $n \cdot T$ (excluded) as in the running example above. Formally, we set

$$\mathcal{T}_n := E_d(\mathcal{P}) \times \{0, 1, \ldots, n-1\},$$

and for $t_1 = (v_1 w_1, i_1), t_2 = (v_2 w_2, i_2) \in \mathcal{T}_n$, we define

$$t_1 \preceq t_2 \quad :\Leftrightarrow \quad \pi_{v_2} + i_2 T \geq \pi_{v_1} + x_{v_1 w_1} + i_1 T \quad \text{and} \quad w_1 v_2 \in E_t(\mathcal{P}),$$

and let $\mathcal{N}_n := \mathcal{N}(\mathcal{T}_n, \preceq)$. Intuitively, we compare the "real aperiodic" departure time at v_2 of the periodic trip departing within the interval $[i_2 T, i_2 T + T)$ with the "real aperiodic" arrival time at w_1 of the periodic trip departing within $[i_1 T, i_1 T + T)$.

Lemma 1. *Let S_p be an optimal periodic vehicle schedule for $(\mathcal{P}, T, \pi, x)$ and let $S_{a,n}$ be an optimal aperiodic vehicle schedule for $(\mathcal{T}_n, \preceq)$. Then $\nu(S_{a,n}) \leq \nu(S_p)$.*

Proof. Let M_p be a minimum-weight perfect matching of $E_t(\mathcal{P})$ w.r.t. x. For each $w_1 v_2 \in M_p$, let $v_1 w_1$ and $v_2 w_2$ be the preceding and succeeding driving activities, respectively. Define

$$M_{a,n} := \{((a_{v_1 w_1}, i), (d_{v_2 w_2}, i + p_{v_1 w_1} + p_{w_1 v_2})) \mid w_1 v_2 \in M_p, i = 0, \ldots, n - p_{v_1 w_1} - p_{w_1 v_2} - 1\}.$$

Recall that we defined the periodic offset of an activity ij as $p_{ij} := (x_{ij} + \pi_i - \pi_j)/T$.

We claim that $M_{a,n} \subseteq E_t(\mathcal{N}_n)$. We need to check that

$$\pi_{v_2} + (i + p_{v_1 w_1} + p_{w_1 v_2})T \geq \pi_{v_1} + x_{v_1 w_1} + iT.$$

Plugging in the definitions of $p_{v_1 w_1}$ and $p_{w_1 v_2}$,

$$\pi_{v_2} + iT + x_{v_1 w_1} + \pi_{v_1} - \pi_{w_1} + x_{w_1 v_2} + \pi_{w_1} - \pi_{v_2} \geq \pi_{v_1} + x_{v_1 w_1} + iT,$$

or, equivalently,

$$x_{w_1 v_2} \geq 0,$$

which is true by assumption.

It is clear that $M_{a,n}$ is a matching since M_p is. Let $S_{a,n}$ resp. S_p be optimal aperiodic resp.

periodic vehicle schedules. From the matching interpretations of (a)periodic vehicle scheduling, we obtain

$$\nu(S_p) = \sum_{e \in E_d(\mathcal{P})} p_e + \sum_{e \in M_p} p_e.$$

and

$$\nu(S_{a,n}) \leq |E_d(\mathcal{N}_n)| - |M_{a,n}| = n|E_d(\mathcal{P})| - |M_{a,n}|.$$

On the other hand, since M_p is a perfect matching,

$$|M_{a,n}| \geq \sum_{e \in M_p} (n - p_e) - \sum_{e \in E_d(\mathcal{P})} p_e = n|M_p| - \nu(S_p) = n|E_d(\mathcal{P})| - \nu(S_p).$$

Therefore $\nu(S_{a,n}) \leq \nu(S_p)$. $\square$

Exercise. Find an example where $\nu(S_{a,n}) < \nu(S_p)$. In particular, $M_{a,n}$ is not a maximum cardinality matching of $E_t(\mathcal{N}_n)$.

Theorem 2 ($\sim$ Borndörfer, Karbstein, Liebchen, Lindner, 2018)**.** *Let S_p be an optimal periodic vehicle schedule for $(\mathcal{P}, T, \pi, x)$ and let $S_{a,n}$ be an optimal aperiodic vehicle schedule for $(\mathcal{T}_n, \preceq)$. Then $\nu(S_{a,n}) = \nu(S_p)$ for $n \gg 0$.*

6.3 Multi-Depot Aperiodic Vehicle Scheduling

In the multi-depot case, we consider the following input data:

- a finite set $\mathcal{T}$ of trips,
- an acyclic relation $\preceq$ on $\mathcal{T} \times \mathcal{T}$,
- a finite set $\mathcal{D}$ of depots,
- a map $D : \mathcal{T} \to \mathcal{P}(\mathcal{D})$ of feasible depots for each trip.

The depots can stand for actual physical depots, but also for vehicle types, and combinations of both. From this perspective and of course due to geographical reasons, it is reasonable to assume that not every trip can be served by an arbitrary depot.

Definition 3. *Given $(\mathcal{T}, \preceq, \mathcal{D}, D)$ as above, the* multi-depot vehicle scheduling problem *is to find an aperiodic vehicle schedule S for $(\mathcal{T}, \preceq)$ minimizing $\nu(S)$ such that for every chain $t_1 \preceq \cdots \preceq t_r$ in S holds $D(t_1) \cap \cdots \cap D(t_r) \neq \emptyset$.*

The condition $D(t_1) \cap \cdots \cap D(t_r) \neq \emptyset$ implies that there is at least one depot being feasible for the whole chain of trips.

6.3.1 Multi-commodity flow formulation

Again, we want to model the multi-depot vehicle scheduling problem as a network flow problem. We build an event-activity network $\mathcal{N}(\mathcal{T}, \preceq, \mathcal{D}, D)$ as follows:

(1) Create two *depot nodes* p_d and q_d for each depot $d \in \mathcal{D}$.

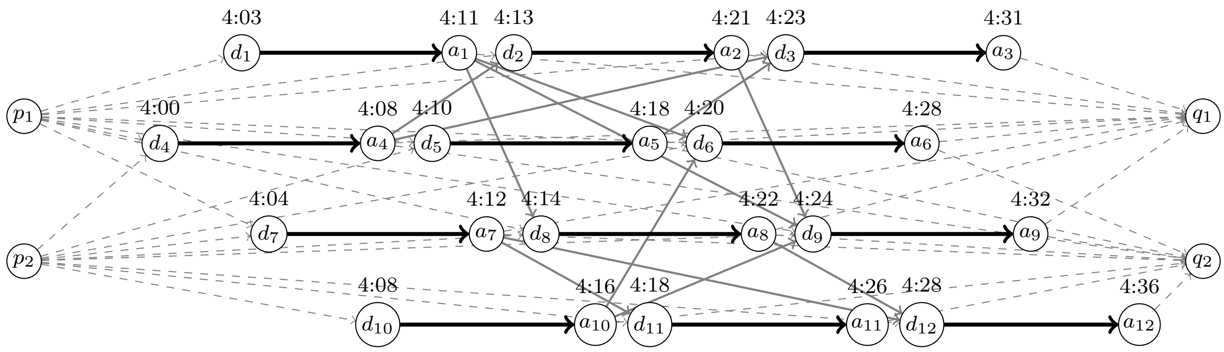
(2) Add *driving activities* (d_t, a_t) for each trip $t \in \mathcal{T}$ (*representing* t).

(3) Add *pull-out activities* (p_d, d_t) for each trip $t \in \mathcal{T}$ and each $d \in \mathcal{D}(t)$.

(4) Add *pull-in activities* (a_t, q_d) for each trip $t \in \mathcal{T}$ and each $d \in \mathcal{D}(t)$.

(5) Add *turnaround activities* (a_{t_1}, d_{t_2}) for each pair $(t_1, t_2) \in \mathcal{T} \times T$ with $t_1 \preceq t_2$.

Example 4. The following is an example network with two depots:



The first idea is to cover all driving activities of $\mathcal{N}(\mathcal{T}, \preceq, \mathcal{D}, D)$ exactly once by a minimum value flow with sources at the nodes p_d and sinks at the sources q_d. However, such a flow does not necessarily respect the condition that every trip is served by a feasible depot: While this is true by construction for the first driving activity after pulling out and the last driving activity before pulling in, this is not guaranteed for intermediate driving activities. Moreover, such a flow might not decompose into edge-disjoint p_d-q_d-paths: If two depots $d \neq d'$ are feasible for some sequence of trips, vehicles starting at p_d might end up at q'_d.

What we need here is one flow for each depot $d \in \mathcal{D}$, and all driving activities along each p_d-q_d-flow represent trips for which d is feasible. So we have to look simultaneously for $|\mathcal{D}|$ flows and select for each flow which activities are feasible.

Let $\mathcal{N}(\mathcal{T}, \preceq, \mathcal{D}, D) = (V, E)$ with driving activities E_d. For each depot $d \in \mathcal{D}$, consider a flow f^d represented by binary variables $f^d_e \in \{0, 1\}$ for each $e \in E$. We can now formulate the multi-depot vehicle scheduling problem as integer program:

$$
\begin{array}{llr}
\text{Minimize} & \displaystyle\sum_{d \in \mathcal{D}} \sum_{e \in \delta^+(p_d)} f_e^d & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta^+(v)} f_e^d - \sum_{e \in \delta^-(v)} f_e^d = 0, & d \in \mathcal{D}, v \in V \setminus \{p_d, q_d\}, \\
& \displaystyle\sum_{d \in \mathcal{D}(t)} f_e^d = 1, & e \in E_d \text{ representing } t, \\
& \displaystyle\sum_{d \notin \mathcal{D}(t)} f_e^d = 0, & e \in E_d \text{ representing } t, \\
& f_e^d \in \{0,1\}, & d \in \mathcal{D}, e \in E.
\end{array}
$$

This is an example of a *multi-commodity flow* problem, i.e., there are several flows which are typically coupled by capacity constraints involving more than one flow.

Theorem 5. *The multi-depot vehicle scheduling problem is solved by the above integer program.*

Proof. Let S be an optimal vehicle schedule and set $f_e^d := 0$ for all $e \in E$ and $d \in \mathcal{D}$. For each chain $t_1 \preceq \dots \preceq t_r$ in S, select a depot $d \in \mathcal{D}$ feasible for the whole chain. Augment the flow f^d by 1 on the path $(p_d, d_{t_1}, a_{t_1}, d_{t_2}, a_{t_2}, \dots, d_{t_r}, a_{t_r}, q_d)$. Proceeding for all chains yields a feasible flow of objective value $\nu(S)$.

Conversely, let $f = (f^d)_{d \in \mathcal{D}}$ be an optimal multi-commodity flow with objective value c. By the structure of the event-activity network, each f^d decomposes into edge-disjoint paths, and all driving activities a.k.a. trips along such a path have d as a common feasible depot. In particular, f gives rise to a vehicle schedule requiring c vehicles. Note that for f^d, the single paths are not necessarily p_d-q_d-paths, but we may change the source to p_d and the sink to q_d without affecting feasibility or optimality. □

Remark 6. There are instances of two-commodity flow problems, i.e. of two coupled network flows, where the solution to the LP relaxation is not integral. In fact, finding an optimal integral two-commodity flow is NP-hard, even if all capacities are 1 (Even, Itai, Shamir, 1975).

Example 7. There is an optimal 2-commodity flow for the previous example decomposing into 3 p_1-q_1-paths and 3 p_2-q_2-paths, hence requiring in total 6 vehicles.