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Problem Set 8

due: December 9, 2019

We derive a new set of valid inequalities (*comb inequalities*) for the TSP.

Exercise 1 **8 points**

Let $T_1, \dots, T_s \subseteq V(K_n)$ be s pairwise disjoint sets, where $s \geq 3$ is odd. Additionally, let $H \subseteq V(K_n)$ fulfill $T_i \cap H \neq \emptyset$ and $T_i \setminus H \neq \emptyset$ for $i = 1, \dots, s$. Consider an incidence vector $x \in \{0,1\}^{E(K_n)}$ of a Hamiltonian circuit in K_n.

(a) Show that for any $i \in \{1, \dots, s\}$ holds

$$\sum_{e \in \delta(H) \cap E(K_n[T_i])} x_e + \sum_{e \in \delta(T_i)} x_e \geq 3.$$

(b) Deduce from (a) that

$$\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{s} \sum_{e \in \delta(T_i)} x_e \geq 3s + 1. \tag{1}$$

(c) Prove – recalling the cut formulation of the subtour elimination constraints – that

$$\sum_{e \in E(K_n[H])} x_e + \sum_{i=1}^{s} \sum_{e \in E(K_n[T_i])} x_e \leq |H| + \sum_{i=1}^{s} |T_i| - \frac{3s+1}{2}. \tag{2}$$

Here, for $X \subseteq V(K_n)$, $E(K_n[X])$ denotes the set of edges in K_n with both endpoints in X.

Exercise 2 **8 points**

For a complete graph K_n consider the *subtour polytope* described by

$$\begin{aligned}
\sum_{e \in \delta(v)} x_e &= 2, & v &\in V(K_n), \\
\sum_{e \in E(K_n[X])} x_e &\leq |X| - 1, & X &\subsetneq V(K_n),\ X \neq \emptyset, \\
0 \leq x_e &\leq 1, & e &\in E(K_n).
\end{aligned} \tag{3}$$

The incidence vectors of Hamiltonian circuits in K_n are the integral points in (3). However, as noticed in the example from the lecture shown in Figure 1, the solutions obtained after optimizing over the subtour polytope (note that there are no integrality constraints!) can be fractional. Recall that the optimal fractional solution has an objective value of 9, and there is an optimal TSP tour with an objective value of 10.

(a) Prove that on K_6, any $(T_1, \dots, T_s, H)$ as in Exercise 1 satisfies $|H| = 3$, $s = 3$, and $|T_1| = |T_2| = |T_3| = 2$.

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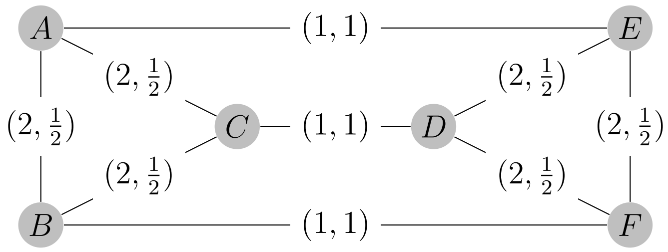

Figure 1: TSP instance (K_6, c) and optimal subtour polytope solution x^*. The labels (c_e, x^*_e) at every edge $e \in K_6$ show the edge costs c_e and the solution value x^*_e of the variable x_e that was gained after optimizing over the subtour polytope. The edges that are not shown in the picture have $c_e = 3$ and $x^*_e = 0$.

(b) Find a comb inequality of the form (1) or (2) that is violated by the fractional solution x^* shown in Figure 1.

(c) Conclude with the help of (b) that an optimal integral TSP tour for the instance of Figure 1 has cost 10.

Exercise 3 **4 points**

For integer $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, consider the integer program

(4) $$\text{Minimize } c^t x \text{ subject to } Ax \leq b, x \geq 0, x \in \mathbb{Z}^n.$$

Let $\alpha \in \mathbb{Q}^n$ and $\beta \in \mathbb{Q}$.

(a) Suppose that $\alpha^t x \leq \beta$ is a valid inequality for all feasible solutions to (4). Prove that $\lfloor\alpha\rfloor^t x \leq \lfloor\beta\rfloor$ is a valid inequality for (4), where $\lfloor\cdot\rfloor$ means rounding down component-wise.

(b) Show that if $\alpha^t x \leq \beta$ is valid for (4), α is integral, β is non-integral and $\alpha^t x^* = \beta$ holds for some feasible solution x^* to the LP relaxation of (4), then $\lfloor\alpha\rfloor^t x \leq \lfloor\beta\rfloor$ is a cutting plane separating x^* from the convex hull of the feasible solutions to (4).

(c) Use a violated comb inequality for the fractional TSP solution x^* in Figure 1 (cf. Exercise 2b) to find α and β satisfying the prerequisites for the statement in (b).