Hypersurfaces with defect

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Remark (Lefschetz principle + base change)

This generalizes to varieties over arbitrary characteristic 0 fields.

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$$h^i(\mathbb{A}^n) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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- $h^i(X) = 0$ for $i > \dim X$ if X is affine.

Let K be a field of characteristic 0, $n \geq 3$.
Let $X = \{f = 0\} \subseteq \mathbb{P}^n_K$ be a hypersurface of degree d over K.

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- $$h^{n-1}(X) = \sum_{k=1}^{n-1} \left(\dim_K \frac{K[x_0, \dots, x_n]}{\left\langle \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n} \right\rangle} \right)_{kd-n-1} + n \bmod 2.$$

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- Explicit computations (reduction of pole order)

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Definition

X has **defect** if $h^i(X) \neq h^i(\mathbb{P}^n)$ for some $n \leq i \leq n+\dim\Sigma$.

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- threefold hypersurfaces with at most rational singularities containing a non-$\mathbb{Q}$-Cartier divisor, because

$$\operatorname{rk} \mathrm{CH}^1(X) \leq h^4(X).$$

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Theorem (–)

Suppose $X \subseteq \mathbb{P}^n$, $n \geq 3$, has at most isolated singularities. Then

$$X \text{ has defect } \Rightarrow \tau \geq \frac{d-n+1}{n^2+n+1},$$

where τ denotes the global Tjurina number of f.

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- Choosing a hyperplane avoiding Σ, identify

$$H^n_\Sigma(X) \cong H^n(\mathbb{A}^n \setminus (X \cap \mathbb{A}^n))/V,$$

  where V is the inverse image of $H^{n-1}(X \cap \mathbb{A}^n)$ under the Poincaré residue map.

Proof ingredients II

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$$\begin{array}{ccc} K[x_0,\dots,x_n]_{kd-n-1} & \xrightarrow{\text{restriction}} & K[x_1,\dots,x_n] \\ {\scriptstyle g \mapsto g\Omega/F^k}\downarrow{\scriptstyle \text{surj.}} & & {\scriptstyle \text{surj.}}\downarrow{\scriptstyle h \mapsto h\omega/f^k} \\ \operatorname{Gr}_P^k H^n(\mathbb{P}^n \setminus X) & \xrightarrow{\text{restriction}} & \operatorname{Gr}_P^k H^n_\Sigma(X) \end{array}$$

using the pole-order filtration P by forms of pole order k.

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- Using the space V:

$$h \in J^3 \quad \Rightarrow \quad h\omega/f^k = 0 \quad \text{in } \mathrm{Gr}^1_P\, H^n_\Sigma(X).$$

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$$\begin{array}{ccc} K[x_0,\ldots,x_n]_{kd-n-1} & \xrightarrow{\text{restriction}} & K[x_1,\ldots,x_n]/(\langle f\rangle + J^3) \\ g\mapsto g\Omega/F^k \big\downarrow \text{surj.} & & \text{surj.} \big\downarrow h\mapsto h\omega/f^k \\ \mathrm{Gr}_P^k\, H^n(\mathbb{P}^n\setminus X) & \xrightarrow{\text{restriction}} & \mathrm{Gr}_P^k\, H^n_\Sigma(X) \end{array}$$

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- If X has defect, then the bottom map is not surjective for some $k \geq 1$. So the top map is not surjective for some $k \geq 1$.
- Commutative algebra:

$$\dim_K K[x_1,\ldots,x_n]/(\langle f\rangle + J^3) \geq d-n+1$$

$$\Rightarrow \tau = \dim_K K[x_1,\ldots,x_n]/(\langle f\rangle + J) \geq \frac{d-n+1}{n^2+n+1}.$$

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Theorem (Deligne, Katz/Messing, Kedlaya)

The two theories give the same Betti numbers for smooth projective varieties over finite fields.

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Theorem (Proper-smooth base change)

If X has a smooth proper lift $\mathcal{X}$ with generic fiber $\mathcal{X}_\eta$, then

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Theorem (Baldassarri/Chiarellotto)

Let $\mathcal{X}$ be a smooth proper scheme over a dvr and let $\mathcal{D}$ be a relative snc divisor. Then $H^i_{rig}(\mathcal{X}_s \setminus \mathcal{D}_s) \cong H^i_{dR}(\mathcal{X}_\eta \setminus \mathcal{D}_\eta)$ for all i.

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Remark

- If X has at most isolated singularities, then X has defect in étale cohomology $\Leftrightarrow h^n(X) \neq h^n(\mathbb{P}^n)$.

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- If X has at most isolated singularities, then X has defect in étale cohomology $\Leftrightarrow h^n(X) \neq h^n(\mathbb{P}^n)$.
- If X has at most isolated weighted homogeneous singularities, X has defect in rigid cohomology $\Leftrightarrow h^n(X) \neq h^n(\mathbb{P}^n)$.

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If X lifts to a hypersurface in $\mathbb{P}^n$ with defect, then

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Can one always lift defect?

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Example

Let $S \subseteq \mathbb{P}^4$ be a non-liftable surface defined over $\mathbb{F}_q$ (Vakil). Let X be a hypersurface containing S. Then S cannot be a $\mathbb{Q}$-Cartier divisor on X. $\Rightarrow X$ has defect.

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Then: X *has defect* $\Rightarrow \displaystyle\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2\left\lceil \frac{k_x}{2} \right\rceil \geq d.$

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Corollary (–)

$$\lim_{d \to \infty} \frac{\#\{f \in \mathbb{F}_q[x_0, \ldots, x_n]_d \mid \{f = 0\} \textit{ no defect}\}}{\#\mathbb{F}_q[x_0, \ldots, x_n]_d} \geq \frac{1}{\zeta_{\mathbb{P}^n}(n+3)}.$$