Matheon

Control Reduced Interior Point Methods for PDE Constrained Optimization

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Overview and Related Work

Control Reduced Interior Point Methods

- Convergence Results in Function Space
- Discretization error estimates
- Numerical Example

Preprints

- Weiser/Gänzler/Schiela (ZIB '04), Schiela/Weiser (ZIB '05)

Related Work

- M. Hinze (Elimination of the Control)
- M. & S. Ulbrich (Affine Scaling Interior Point Methods)
- M. Ulbrich (Semi-Smooth Newton Methods)
- S. Ulbrich (Primal-Dual IP-Methods for PDE Opt.)
- M. Weiser et al. (Function Space Oriented IP-Methods)

Model Problem in Optimal Control

$$\min \frac{1}{2}\|y-y_d\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L_2(\Omega)}$$

objective functional

$$\text{s.t.} \quad Ly = u \quad on\ \Omega$$

$$y = 0 \quad on\ \partial\Omega$$

PDE constraint

$$-1 \leqslant u \leqslant 1$$

control bounds

- control u, state y
- regular domain in R^d

$$u \in L_2 \Rightarrow y \in H^2 \cap H^1_0$$

L: elliptic differential operator

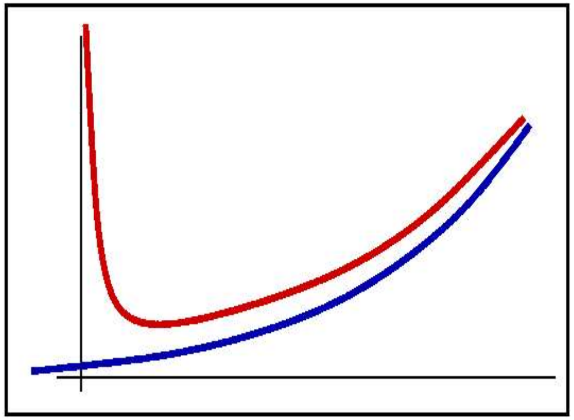
Primal Interior Point Approximation

Augment functional by logarithmic **barrier term**:

$$\min \tfrac{1}{2}\|y-y_d\|^2_{L_2(\Omega)} + \tfrac{\alpha}{2}\|u\|^2_{L_2(\Omega)}$$

$$\boxed{-\mu \int_\Omega \left(\ln(1-u)+\ln(u+1)\right)\, dx}$$

s.t. $L\,y = u$



KKT- conditions:

$$y - y_d + L\lambda = 0$$

$$Ly - u = 0$$

$$\alpha u - \lambda - \frac{\mu}{u+1} + \frac{\mu}{1-u} = 0$$

Elimination of the control:

$$y - y_d + L\lambda = 0$$

$$L\,y - u(\lambda;\mu) = 0$$

Homotopy: $\mu \to 0$

Primal Interior Point Approximation

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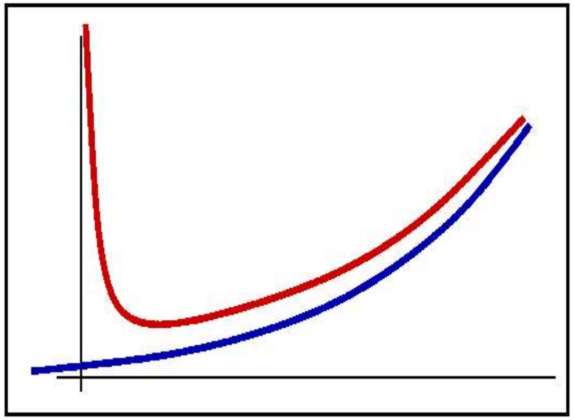

KKT- conditions:

$$y-y_d+L\lambda=0$$

$$Ly-u=0$$

$$\alpha u-\lambda-\frac{\mu}{u+1}+\frac{\mu}{1-u}=0$$

Elimination of the control:

$$y-y_d+L\lambda=0$$

$$Ly-u(\lambda;\mu)=0$$

$$F(v;\mu)=0$$

Homotopy: $\mu\to 0$

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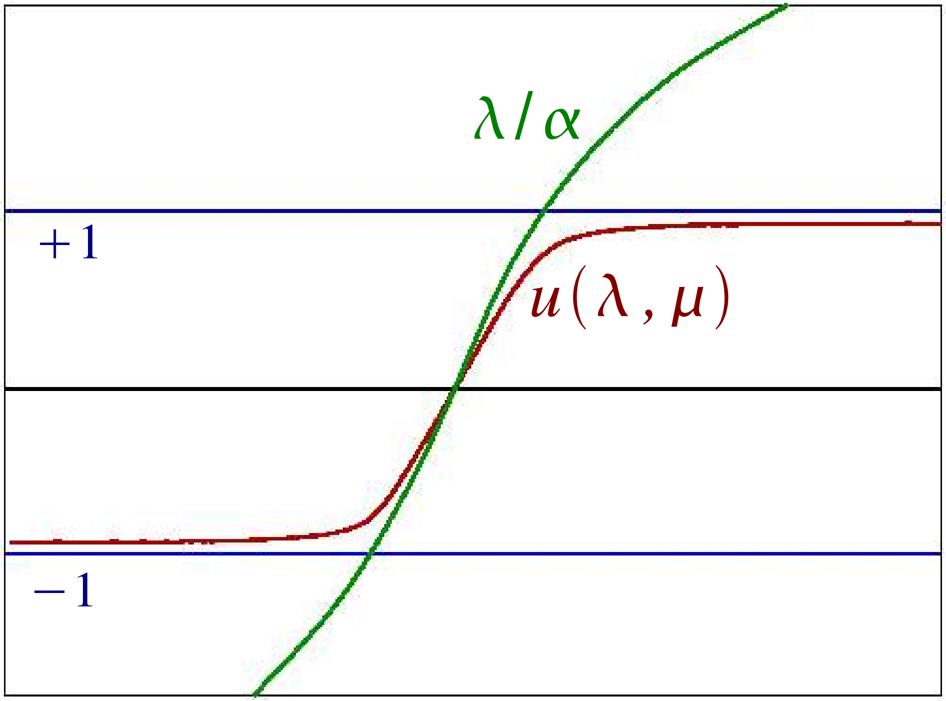
Elimination of the Control

- Use optimality condition to **eliminate** the control **pointwise**
- Idea of *[M. Hinze]* carried over to interior point methods
- Discretization of the **smooth** variables y and λ only

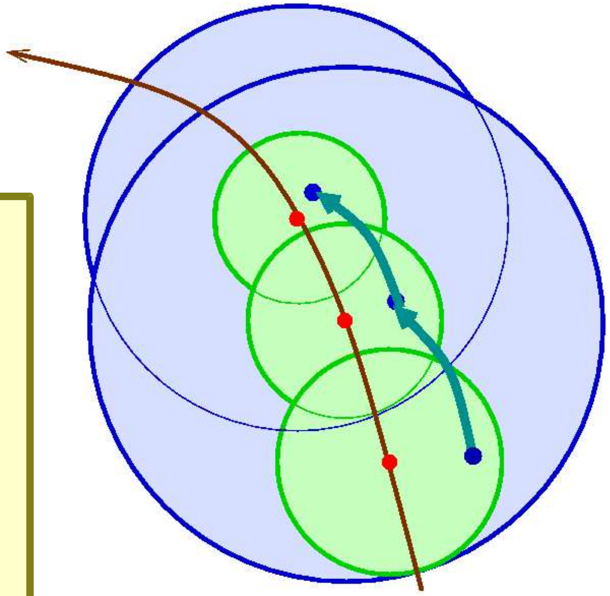


Consequences:

- **fast convergence** of interior point pathfollowing method
- **optimal** discretization error estimates

A Newton Pathfollowing Method

Pathfollowing in function space:



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select $v_0 = (y_0, \lambda_0)$
    for $k = 0, 1, 2 \ldots$
        select $\mu_{k+1}$
        solve $F'(v_k; \mu_{k+1})\Delta v_k = -F(v_k; \mu_{k+1})$
        $v_{k+1} = v_k + \Delta v_k$
```

Linear Convergence

Central Path

$$F(v(\mu);\mu)=0, \qquad \mu>0$$

Slope

$$\eta=\|v_\mu(\mu)\|=O(\mu^{-1/2})$$

Newton Corrector

$$F'(v_k;\mu)\,\Delta v_k=-F(v_k;\mu)$$

Affine inv. Lipschitz constant

$$\omega=O(\mu^{-1/2})$$

$$\sigma\sim\mu/(\eta\cdot\omega)$$

linear convergence:

$$\mu_{k+1}=\sigma\,\mu_k$$

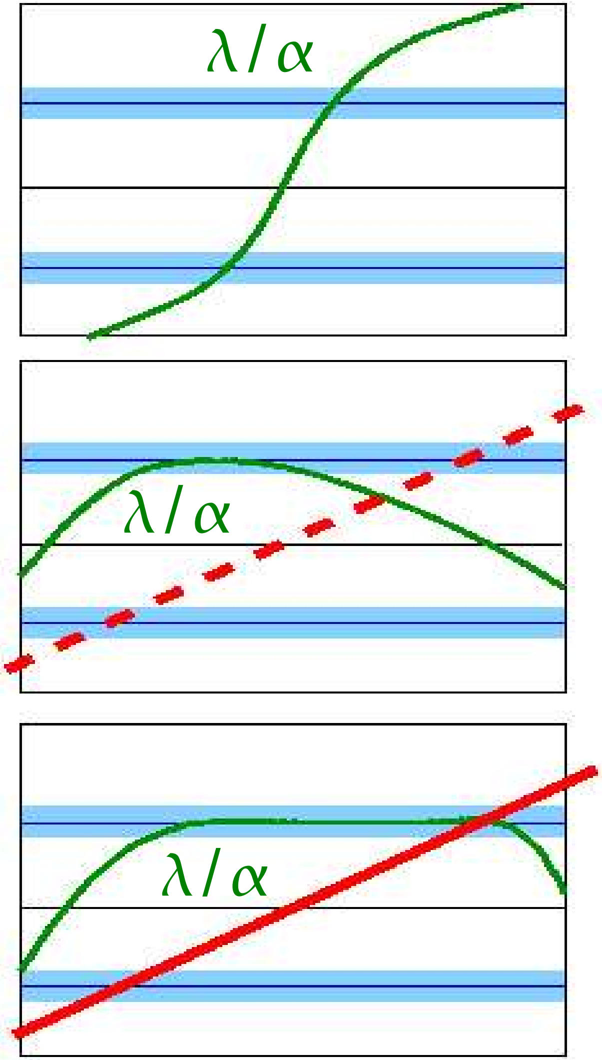
$$\|v_k-v_*\|=O(\sqrt{\mu_k})$$

Complementarity Condition

- Finite dimensional IP-Methods:
  „Strict complementarity implies superlinear convergence“
- Analogue in function space:

$$\left|\left\{x\in\Omega : \left|\lambda(x)\pm\alpha\right|\leqslant e\right\}\right|\leqslant\Gamma e$$

- Similar to *[M. & S. Ulbrich]*:
  „Strong strict complementarity“



Superlinear Convergence

Central Path

$$F(v(\mu);\mu)=0, \qquad \mu>0$$

Slope

$$\eta=O(1-\ln\mu)$$

Newton Corrector

$$F'(v_k;\mu)\Delta v_k=-F(v_k;\mu)$$

Affine inv. Lipschitz constant

$\omega\leqslant c$ uniform in $\mu\to 0$

$$\sigma\sim\mu/(\eta\cdot\omega)$$

convergence of *r-order 2*:

$$\lim_{k\to\infty}\frac{\ln\mu_{k+1}}{\ln\mu_k}=2$$

$$\|v_k-v_*\|=O(\mu_k\ln(\mu_k))$$

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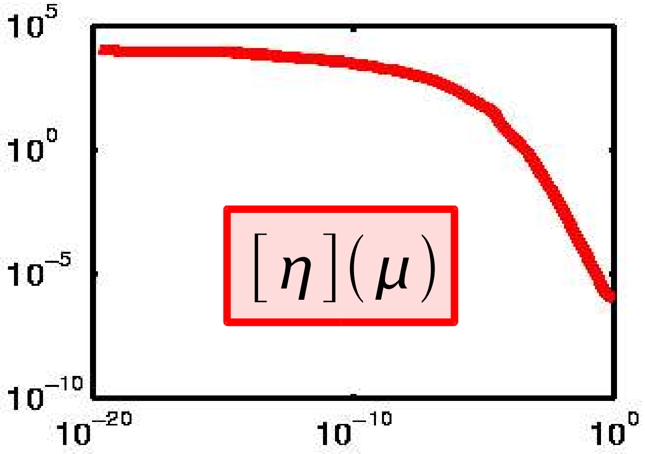
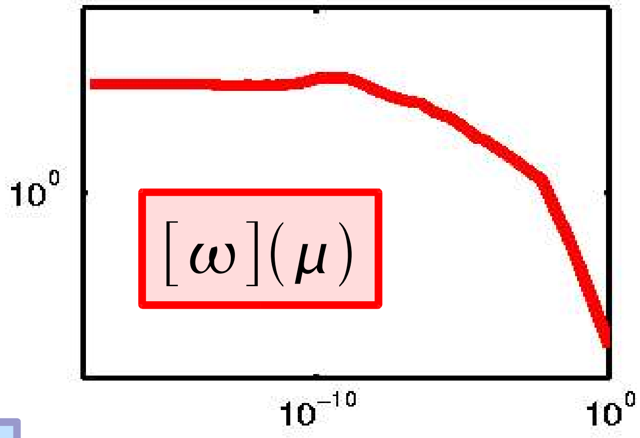
Algorithmic Realisation

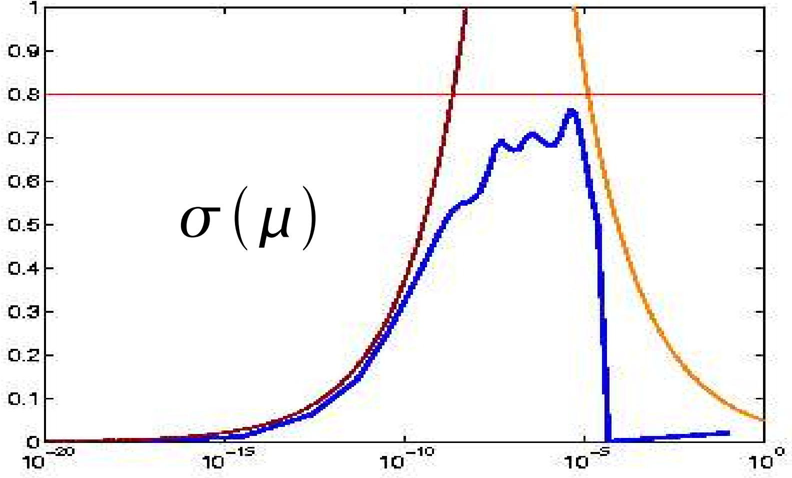
Central Path



Newton Corrector



$$\sigma \sim \mu/(\eta\cdot\omega)$$



Discretization Error Estimates

Finite element discretization of y and λ $(p=1,2)$

Exact assembly of non-grid function u_h *[M. Hinze]*:

$$\|y_h - y(\mu)\|_{H^1} + \|\lambda_h - \lambda(\mu)\|_{H^1} = O(h^p)$$

$$\|y_h - y(\mu)\|_{L_2} + \|\lambda_h - \lambda(\mu)\|_{L_2} + \|u_h - u(\mu)\|_{L_2} = O(h^{p+1})$$

Inexact Assembly

Finite element discretization of y and λ $(p=1,2)$

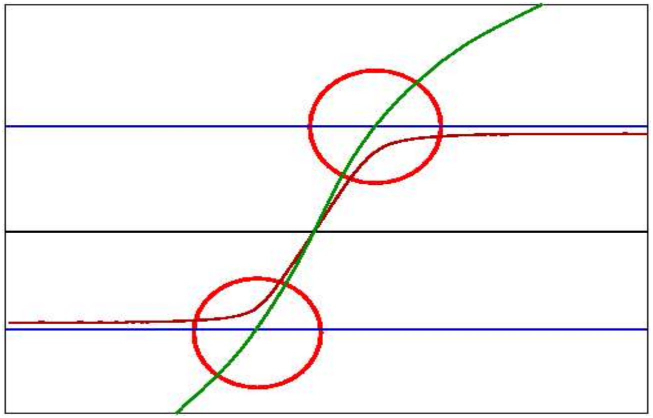
Inexact assembly of non-grid function u_h:

$$\|y_h - y(\mu)\|_{H^1} + \|\lambda_h - \lambda(\mu)\|_{H^1} = O(h^p) \boxed{+ e_\Pi}$$

$$\|y_h - y(\mu)\|_{L_2} + \|\lambda_h - \lambda(\mu)\|_{L_2} + \|u_h - u(\mu)\|_{L_2} = O(h^{p+1}) \boxed{+ e_\Pi}$$

$$e_\Pi = \|u_h - \Pi u_h\|_{L_2} \sim \frac{h_\Pi^2}{\sqrt{\mu}}$$

(u_h develops kinks)



Use adaptive quadrature on refined subgrid h_Π

Numerical Example

$$\min \frac{1}{2}\|y-y_d\|^2_{L_2(\Omega)}+\frac{\alpha}{2}\|u\|^2_{L_2(\Omega)}$$

s.t. $Ly=u \quad on\ \Omega$

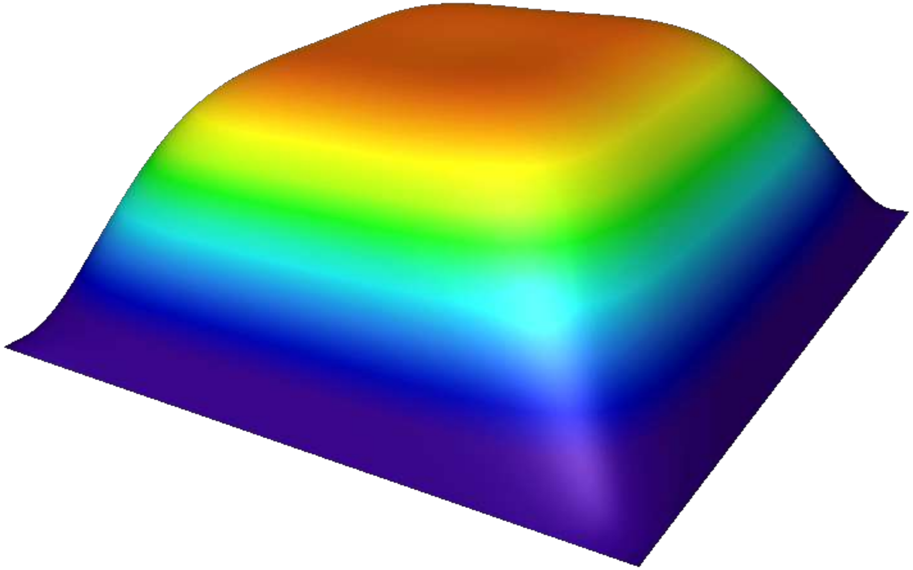
$y=0 \quad on\ \partial\Omega$

$$-40\leqslant u\leqslant 40$$

$$y_d=0.1$$

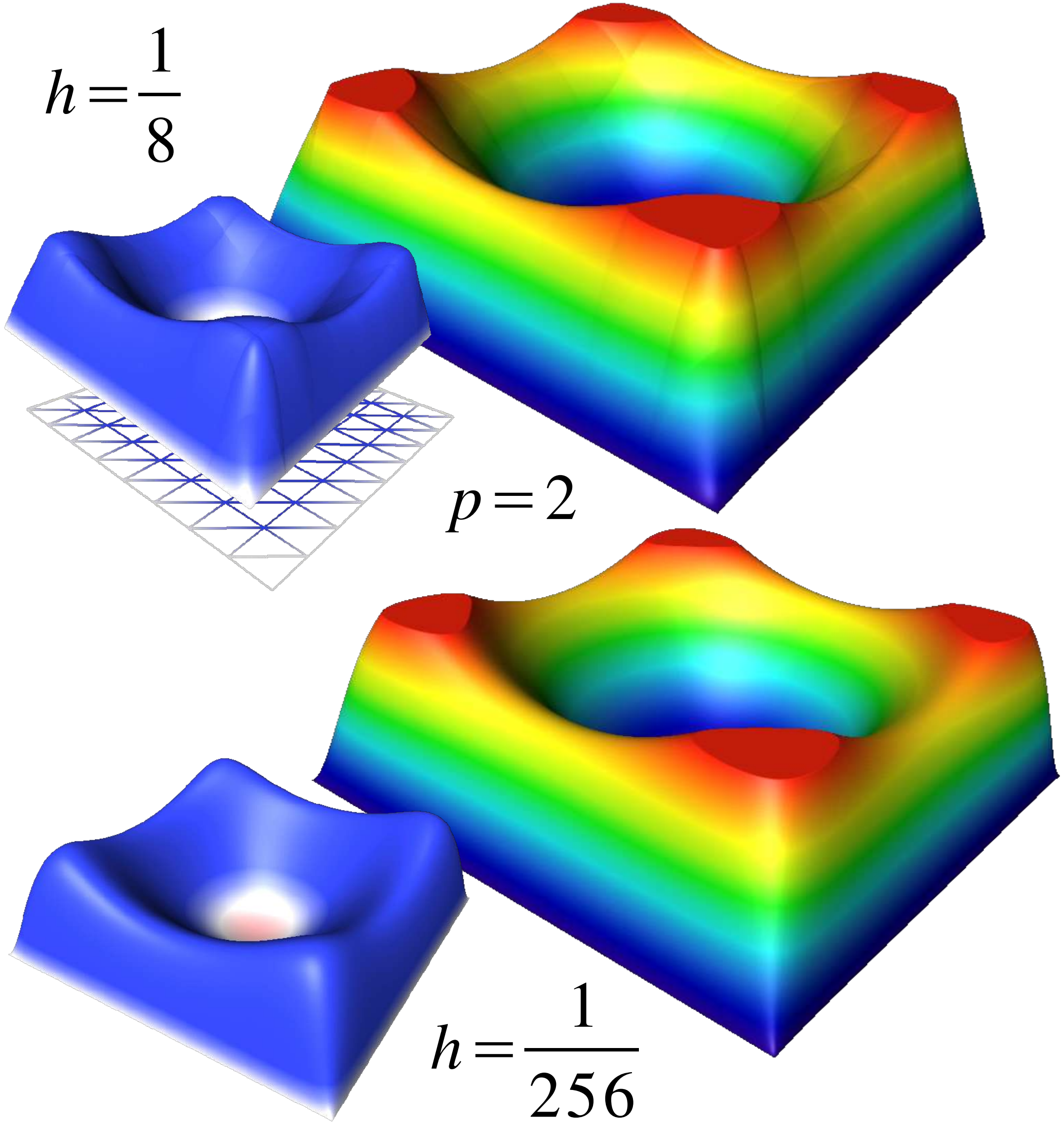
$$\alpha=10^{-6}$$

(JCMSolve/KASKADE)

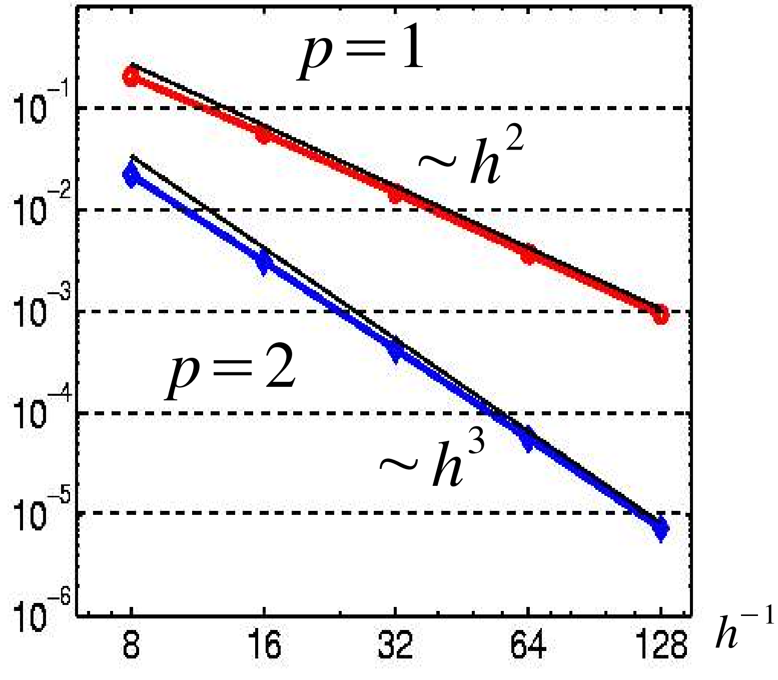


optimal state

Numerical Example

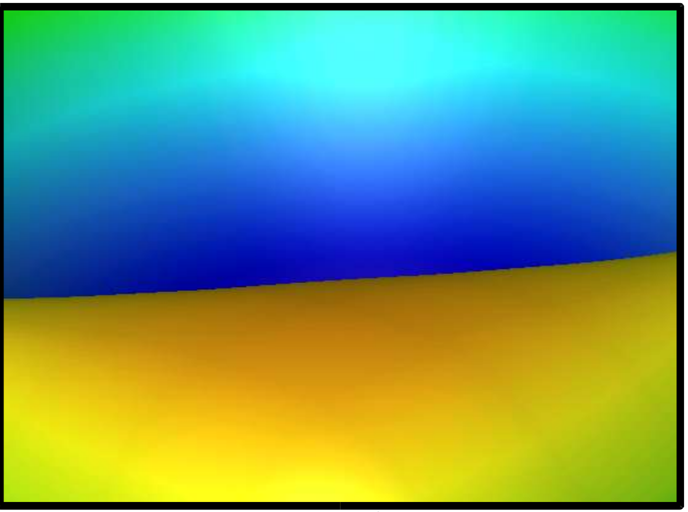


- Optimal error bounds

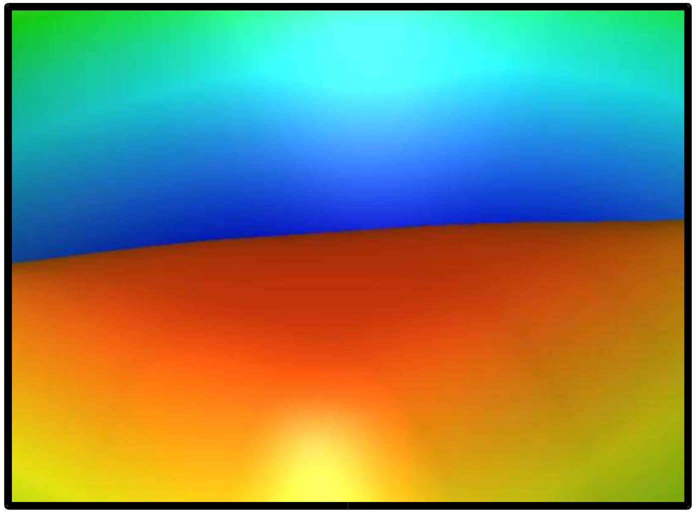


relative L_2-error in control

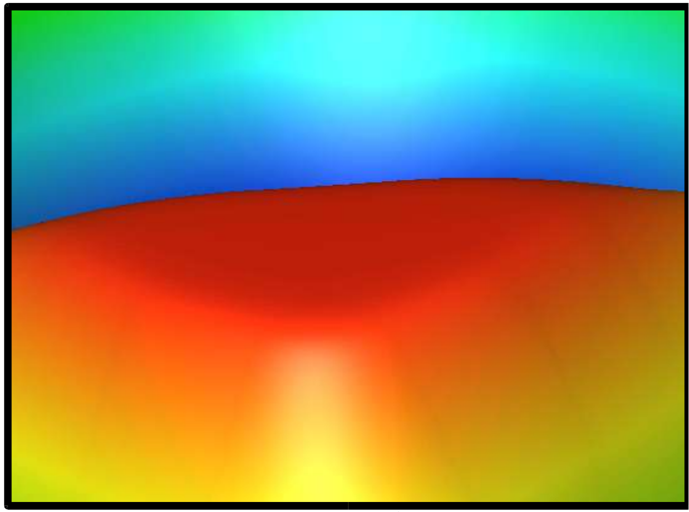
Zoom to Active Set



$\mu = 10^{-4}$

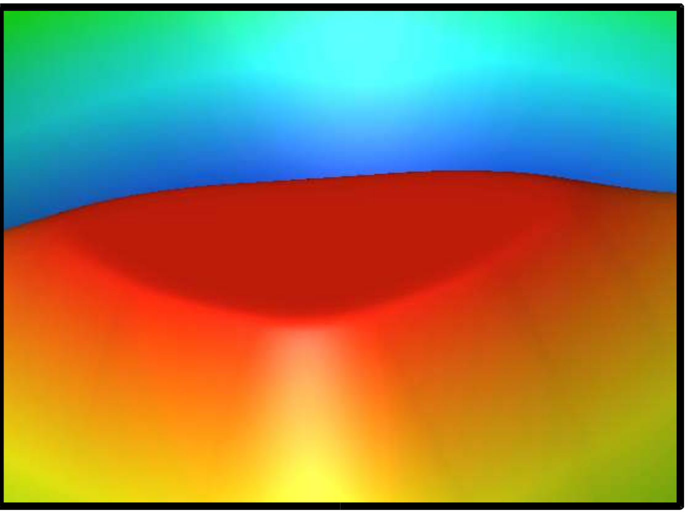


$\mu = 10^{-5}$

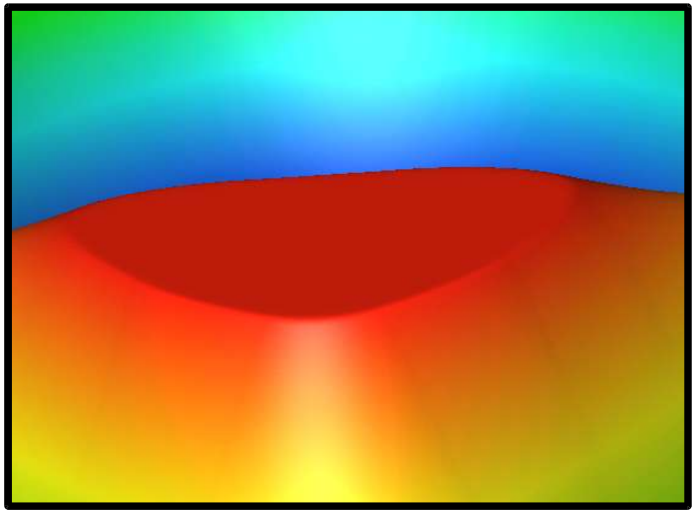


$\mu = 10^{-6}$

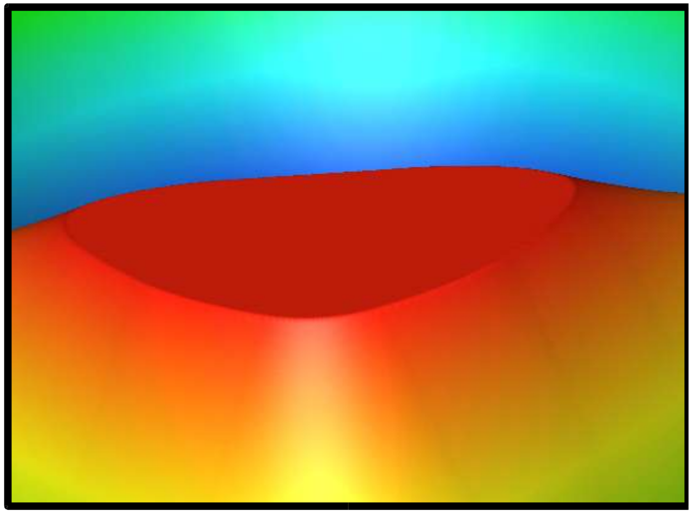
$\mu = 10^{-7}$



$\mu = 10^{-8}$



$\mu = 10^{-10}$



Summary

Control Reduced Methods for optimal control with PDEs

- Superlinear convergence
- Optimal discretization error estimates

Current work

- Integration of all features into a fully adaptive algorithm