Freie Universität Berlin
FB Mathematik und Informatik
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Optimization 2

Exercise Sheet 8

Submission: Wednesday, 20.12.2017, 12:00

Exercises:
Rigorous mathematical proofs/arguments are expected if not stated otherwise. You are allowed to work in groups of two.

Homepage of the Lecture: `http://www.zib.de/ws17_Optimierung_II`
Questions?: `beckenbach@zib.de`

Exercise 8.1 **4+6 Points**

Let $v_1, \dots, v_n, w_1, \dots, w_n, W \in \mathbb{N}_{\geq 1}$ be given positive integers. Consider the $0,1$ Knapsack Problem

$$\begin{aligned} \max \sum_{i=1}^{n} v_i x_i \\ \sum_{i=1}^{n} w_i x_i \leq W \\ x_i \in \{0,1\} \ \forall i = 1, \dots, n. \end{aligned}$$

For every $i = 1, \dots, n$ and $w \in \{1, \dots, W\}$ we denote by $m[i,w]$ the maximum value of items of weight less than w using the first i items, i.e.,

$$m[i,w] := \max\{\sum_{j \in S} v_j : S \subseteq \{1, \dots, i\}, \sum_{j \in S} w_j \leq w\},$$

in particular, $m[n,W]$ equals the optimal value of the knapsack problem. We set $m[0,w] := 0$ for all $w \in \{1, \dots, W\}$.

(a) Show that $m[i,w]$ satisfies the following recursion:

$$m[i,w] := \begin{cases} m[i-1,w] & \text{if } w_i > w \\ \max\{m[i-1,w], m[i-1,w-w_i] + v_i\} & \text{if } w_i \leq w \end{cases}.$$

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(b) Look at the following algorithm for the $0, 1$ Knapsack Problem.

```
for w = 1 to W do
 | m[0, w] = 0
end
for i = 1 to n do
 | for w = 1 to W do
 | | if w_i > w then
 | | | m[i, w] = m[i − 1, w]
 | | else
 | | | m[i, w] = max{m[i − 1, w], m[i − 1, w − w_i] + v_i}
 | | end
 | end
end
```

Calculate its time and space complexity.

Does this algorithm run in polynomial time?

Exercise 8.2 — 5 Points

Given an undirected graph $G = (V, E)$ and a positive integer $k \leq n$. The degree constrained spanning tree problem asks whether a spanning tree exists in which no vertex has degree greater than k.

Show that this decision problem is $\mathcal{NP}$-complete.

Exercise 8.3 — 1+4 Points

Let $(E, \mathcal{I}, c)$ be a constrained optimization problem of the form $\min\{c(I) : I \in \mathcal{I}\}$, where $\mathcal{I}$ is a family of subsets of E, and $c : E \to \mathbb{Z}$ an integral function on E. We denote this optimization problem by Π_O and its associated decision problem $\min\{c(I) : I \in \mathcal{I}\} \leq B$ by Π_D.

(a) Show that $\Pi_O \in \mathcal{P}$ implies $\Pi_D \in \mathcal{P}$.

(b) Assume that for every instance of Π_O of size n an upper bound $U(n)$ is given with $U \in \mathcal{O}(2^n)$, and $\Pi_D \in \mathcal{P}$. Derive a polynomial time algorithm for Π_O using a polynomial time algorithm for Π_D