Freie Universität Berlin
FB Mathematik und Informatik
Sommersemester 2017

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Optimierung I

Excercise Sheet 5

Submission: until 17:00 on Monday, Mai 29, 2017

Exercise 5.1 — 10 Points

Let $S \subseteq \mathbb{R}^n$.

a) Show that the linear (conic, affine, convex) hull of S is the smallest linear space (cone, affine space, convex set) which includes S.

b) Determine the dimension of the set $T = \{(x, y) \in \mathbb{R}^2 \mid x+y = 1, x \geq 0, y \geq 0\}$.

c) Show that the affine and the linear hull of S are equal if and only if the origin is included in aff(S).

$$\mathsf{aff}(S) = \mathsf{lin}(S) \Leftrightarrow 0 \in \mathsf{aff}(S)$$

d) Show that for each x of the affine hull of S follows that aff$(S) = x + \text{lin}(S - x)$.

$$\forall x \in \mathsf{aff}(S) \ : \ \mathsf{aff}(S) = x + \mathsf{lin}(S - x)$$

e) Show that the conic and the linear hull of S are equal, if and only if for each $x \in S$ follows that $-x \in \mathsf{cone}(S \setminus \{x\})$.

$$\mathsf{cone}(S) = \mathsf{lin}(S) \Leftrightarrow \forall x \in S \ : \ -x \in \mathsf{cone}(s \setminus \{x\})$$

Exercise 5.2 — 10 Points

Which of the following sets are polyhedra? Proof or give a counterexample.

a)

$$M_1 = \left\{X \in \mathbb{R}^{n\times n} \ : \ a_1^T X a_1 \leq a_2^T X a_2\right\}, \ a_1, a_2 \in \mathbb{R}^n,$$

b)

$$M_2 = \left\{x \in \mathbb{R}^n \ : \ x \geq 0, \sum_{i=1}^n x_i = 1, \sum_{i=1}^n x_i a_i = b_1, \ \sum_{i=1}^n x_i a_i^2 = b_2\right\},$$
$$a_1, \ldots, a_n \in \mathbb{R}, \ b_1, b_2 \in \mathbb{R}.$$

c)

$$M_3 = \{x \in \mathbb{R}^n \ : \ x \geq 0, x^T y \leq 1 \ \forall y \text{ with } \|y\|_2 = 1\}$$

Exercise 5.3 **10 Points**

Consider the following simplified model of producing different types of gasoline and fuel oil from crude oil. Crude oil is transformed by a number of distillation processes into different kinds of oil, which is then blended into different kinds of marketable products.

During distillation of each barrel of crude oil, 14% is transformed into heavy naphtha, 25% into light naphtha, 21% into cracked oil, and 16% into heavy oil, while the remaining 24% of residuum can still be used to produce lube oil. The distillation process costs money: producing 1 barrel of heavy or light naphtha costs \$0.40, of cracked oil \$1.10, of heavy oil \$0.30 and collecting 1 barrel of residuum costs \$0.20. For the blending process certain quality constraints have to be met. High quality gasoline has to consist of at least 70% of heavy naphtha and at least 20% of cracked oil. High quality fuel oil may not contain any residuum, while the low quality may contain at most 15% of the residuum. For marketing reasons the refinery decides to produce at least half as many of the low quality gasoline as of the high quality gas.

Finally, the selling prices per barrel are \$365 for the high quality gasoline, \$260 for the low quality, \$200 for high quality fuel oil, and \$140 for the low quality fuel oil. The lube oil still earns \$45 per barrel. One barrel of crude oil costs the refinery \$90 and for the planning period 1000 barrels will be delivered.

Naturally, the refinery wants to maximize its profit. Set up a linear program that models the problem and solve it using ZIMPL and SCIP.

What happens if you drop the prerequisite of the 1000 barrels of crude oil? Why? How much lube oil is produced? Why?

Homepage of the Lecture: `http://www.zib.de/ss17_Optimierung_I`
Questions?: `klug@zib.de`