Freie Universität Berlin
FB Mathematik und Informatik
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Optimierung I

Excercise Sheet 6

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Exercise 6.1 — 10 Points

Given a polyhedron $P(A,b)$ defined by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Show the following statements:

a) $\dim(P(A,b)) = 5$.

b) $-x_i \leq 0$ defines a facet of $P(A,b)$ for all $i = 1,\dots,5$.

c) $x_5 \leq 1$ defines a facet of $P(A,b)$.

d) Which of the inequalities are redundant?

Exercise 6.2 — 10 Points

a) Proof:

Let $\mathcal{P} = \mathcal{P}^{=}(A,b) \subseteq \mathbb{R}^n$ be a polyhedron, $x \in \mathcal{P}$. The following statments are equivalent:

i) x is a vertex of $\mathcal{P}$

ii) $\mathsf{rank}\, A_{.\,\mathsf{supp}(x)} = |\,\mathsf{supp}(x)|$

iii) $\{A_{.j}\}_{j\in\mathsf{supp}(x)}$ is linear independent

b) A matrix $A \in \mathbb{R}^{m\times n}$ is called totally unimodular if each square submatrix of A has determinant equal to $0,\ +1,\ or -1$. In particular, each entry of a totally unimodular matrix is $0,\ +1,\ or -1$.

Let $A \in \mathbb{R}^{m\times n}$ totally unimodular and $b \in \mathbb{Z}^m$. Proof the following statement:

If A is totally unimodular, then the polyhedron $P^{=}(A, b)$ has only integer vertices.

Hint: Use a) and Cramer's Rule.

Exercise 6.3 **10 Points**

a) Show that each non-trivial face of a polyhedron is the intersection of facets of the polyhedron.

b) Let P be a polyhedron of dimension d, and let F be a non-trivial face of P whose dimension k is less than d. Prove that there exist faces $F_{k+1}, F_{k+2}, \ldots, F_{d-1}$ of P such that

(i) $F \subseteq F_{k+1} \subseteq F_{k+2} \subseteq \ldots \subseteq F_{d-1} \subseteq P$,

(ii) $\dim(F_{k+i}) = k + i,$ für $i = 1, \ldots, d - k - 1$.

Hint: induction over d-k

Homepage of the Lecture: `http://www.zib.de/ss17_Optimierung_I`
Questions?: `klug@zib.de`