Plastic Evolutions

Scaling Limits of Deterministic Evolutions in Random Potentials

Timothy J. Sullivan[1] and Florian Theil[1]

[1]Mathematics Institute, University of Warwick, U.K.

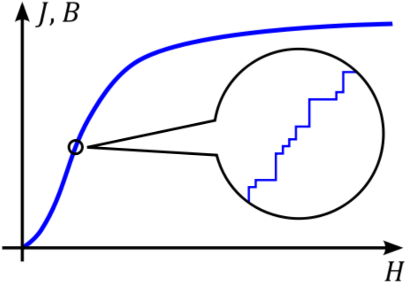
Postgraduate Open Day
Warwick Mathematics Institute
4 February 2009

Dissipation, Plasticity and Rate-Independence

- Heuristically speaking, a *dissipative system* is an evolutionary system in which energy can be irreversibly lost as time passes.
- In materials science and related disciplines, dissipation can manifest itself as *plasticity* — as opposed to *elasticity*.
- "Reasonable" (first-order) approximations to plastic evolutions tend to exhibit *rate-independence*. The evolution equations that describe such systems tend to be succinct, but hard to solve explicitly because the time derivative is contained inside a strong nonlinearity.
- Heuristically, though, such non-linear equations should arise as scaling limits of more well-behaved evolutions.

Examples of dissipative evolutions:

- a block sliding/being pushed across a rough surface;
- crack growth in brittle materials: $\gamma \in \mathrm{BV}([0,T];\mathbb{R}^n)$;
- evolution of a magnetic domain under an applied field (the Barkhausen effect);



The notion of *rate-independence* makes precise the notion that a non-autonomous dynamical system has no preferred timescale, or that it evolves only as fast as its time-dependent inputs.

Definition (Rate-independence)

Consider a state space $\mathcal{Q}$, and suppose that for each initial condition $x_0 \in \mathcal{Q}$ and "external load" $\ell \colon \mathcal{T} \to \mathcal{Q}^*$ ($\mathcal{T} \subseteq \mathbb{R}$ some interval of time), there is a (possibly non-unique) "solution" $q = q(x_0, \ell) \colon \mathcal{T} \to \mathcal{Q}$. This solution operator is said to be rate-independent if, whenever $\alpha \colon \mathcal{T}' \to \mathcal{T}$ is a strictly increasing diffeomorphism,

$$q(x_0, \ell \circ \alpha) = q(x_0, \ell) \circ \alpha \colon \mathcal{T}' \to \mathcal{Q}.$$

I.e. the solution operator commutes with strictly monotone reparametrizations of time.

- Some rate-independent processes can be seen as generalizations of ordinary differential equations like

$$\dot{z}(t) = -\nabla E(t, z(t)).$$

Indeed, such an ODE is a very bad model for plasticity, rate-independence, hysteresis loops & c. (it is "too smooth").

- Consider a block with position $z(t)$ at time t, resting on a fixed rough surface with "roughness" $\mu > 0$ and subject to a time-dependent load ℓ.
  - If $|-\nabla E(t, z(t))| < \mu$, then $\dot{z}(t) = 0$.
  - If $|-\nabla E(t, z(t))| > \mu$, then $\dot{z}(t) \neq 0$ — exactly how fast the block moves will depend on how inertial effects are treated, but at least we should have that

$$\operatorname{sgn}(\dot{z}(t)) = \operatorname{sgn}(-\nabla E(t, z(t))).$$

A (nastily nonlinear) way of summarising these rules, in the absence of inertia, is the *differential inclusion*

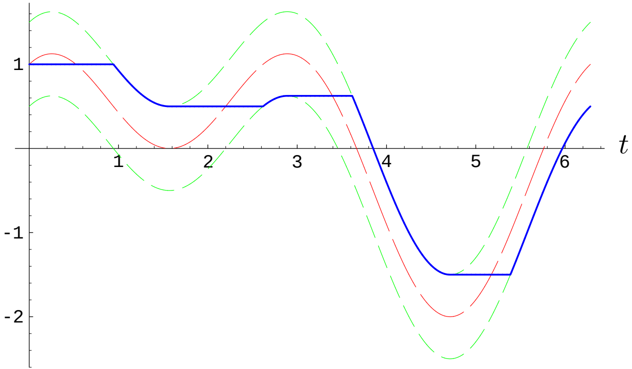
$$\partial\Psi(\dot{z}(t)) \ni -\nabla E(t, z(t)),$$

where the convex function $\Psi(v) := \mu|v|$ is called the *dissipation potential* and $\partial\Psi$ denotes the set-valued *subderivative*:

$$\partial\Psi(v) := \begin{cases} \{-\mu\}, & v < 0; \\ [-\mu, +\mu], & v = 0; \\ \{+\mu\}, & v > 0. \end{cases}$$

What does such an evolution look like?

$z(t)$, $\ell(t)$, $(\ell(t) \pm \mu)/k$



A rate-independent evolution z in the energetic potential $E(t,x) := \frac{1}{2}kx^2 - \ell(t)x$ with dissipation potential $\Psi(v) := \mu|v|$. Note the stable region $\mathcal{S}(t)$, where dissipation is stronger than the potential gradient.

Rate-independent processes have a number of formulations:

- the subdifferential ("sweeping process") formulation used so far;
- a dual subdifferential formulation posed in the dual space $\mathcal{Q}^*$;
- local and global formulations in terms of *stable states* and *energy inequalities*, e.g. the global formulation that
  - (stability) for almost all $t \in [0, T]$ and all $y \in \mathcal{Q}$,

$$E(t, z(t)) \leq E(t, y) + \Psi(y - z(t));$$

  - (energy inquality) for all $[s, t] \subseteq [0, T]$,

$$E(t, z(t)) + \int_s^t \Psi(\mathrm{d}z(t)) \leq E(s, z(s)) + \int_s^t (\partial_\tau E)(\tau, z(\tau)) \, \mathrm{d}\tau.$$

Scaling Limits

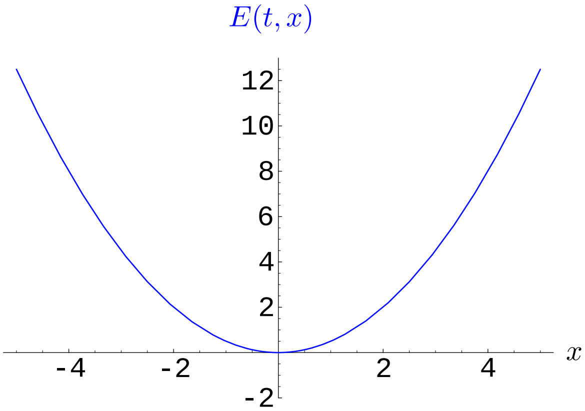
Intuitively, the rate-independent plastic evolution

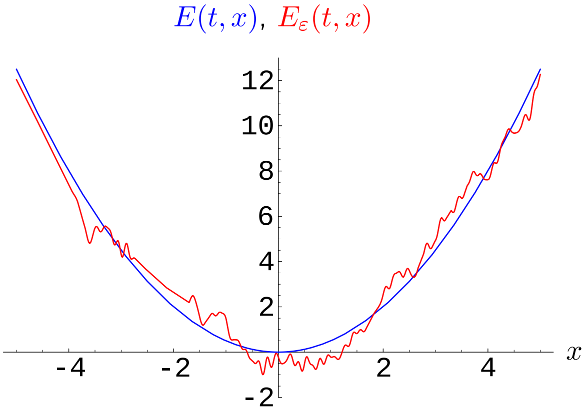
$$\partial\Psi(\dot{z}(t)) \ni -\nabla E(t, z(t))$$

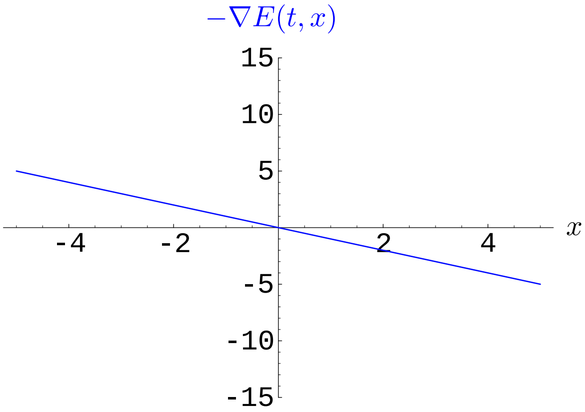
ought to arise as a suitable scaling limit (a "zooming-out") of a reversible evolution in a "wiggly" version of the energetic potential E,

$$\dot{z}_\varepsilon(t) = -\nabla E_\varepsilon(t, z_\varepsilon(t)).$$

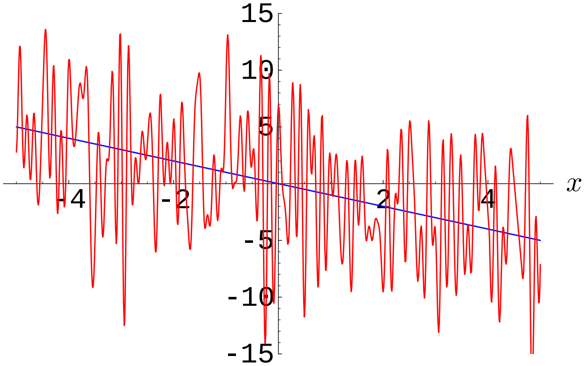
> *"The physical insight [. . .] is that the macroscopic dynamics may depend essentially on microstructural events like getting stuck in local minima. The goal is to derive an averaged equation for the macroscopic variable, z, that includes the effect of the microstructure."*
> — *Menon (2002)*


$E(t,x)$
12
10
8
6
4
2
-2
-4
-2
2
4
x


$E(t, x)$, $E_\varepsilon(t, x)$
12
10
8
6
4
2
-2
-4
-2
2
4
x


$-\nabla E(t,x)$
15
10
5
-5
-10
-15
-4
-2
2
4
x

$-\nabla E(t,x)$, $-\nabla E_\varepsilon(t,x)$



- For example, consider

$$E_\varepsilon(t, x) := E(t, x) + \varepsilon G(x/\varepsilon)$$

so, by the chain rule,

$$-\nabla E_\varepsilon(t, x) = -\nabla E(t, x) - \nabla G(x/\varepsilon),$$

where G is a fixed (but perhaps randomly chosen) scalar potential with bounded gradient.

- Note that this scaling preserves the gradient of G while making it more wiggly, so ∇E_ε does not converge as $\varepsilon \to 0$ (unless G is very boring). Therefore, all our limiting arguments will necessarily have to be about *weak limits*, i.e. convergence of trajectories $z_\varepsilon \to z$.

Set-up for Scaling Results

For definiteness, consider

- $V : \mathbb{R} \to [0, +\infty)$ convex with V' Lipschitz;
- $\ell \colon [0, T] \to \mathbb{R}^*$ Lipschitz;

$$E(t, x) := V(x) - \ell(t)x;$$

- $G \colon \mathbb{R} \to \mathbb{R}$ having surjective, continuous derivative $G' \colon \mathbb{R} \to [\mu_-, \mu_+]$;

$$E_\varepsilon(t, x) := V(x) - \ell(t)x + \varepsilon G(x/\varepsilon).$$

Force the process $z_\varepsilon \colon [0, T] \to \mathbb{R}$ to equilibriate quickly by taking

$$\dot{z}_\varepsilon(t) = -\frac{1}{\varepsilon} E'_\varepsilon(t, z_\varepsilon(t)).$$

Theorem (Abeyaratne–Chu–James (1996); Menon (2002))

With the notation of the previous slide, let G' be periodic. Then $z_\varepsilon \colon [0, T] \to \mathbb{R}$ solving

$$\dot{z}_\varepsilon(t) = -\frac{1}{\varepsilon} E'_\varepsilon(t, z_\varepsilon(t))$$

converges pointwise as $\varepsilon \to 0$ to $z \colon [0, T] \to \mathbb{R}$ solving

$$\partial\Psi(\dot{z}(t)) \ni -E'(t, z(t))$$

$$\Psi(v) := \begin{cases} \min G' \cdot v, & v \leq 0; \\ \max G' \cdot v, & v \geq 0. \end{cases}$$

Moreover, up to a subsequence, $z_\varepsilon \to z$ uniformly and $\dot{z}_\varepsilon \overset{*}{\rightharpoonup} \dot{z}$ in $L^\infty([0, T]; \mathbb{R})$.

Definition (Property (✠))

Fix $\mu_- \leq \mu_+$. A function $g\colon \mathbb{R} \to \mathbb{R}$ is said to have property (✠) if

- g is continuous;
- the image of g is $[\mu_-, \mu_+]$;
- define $D_0^+ \geq 0$ to be the least $x > 0$ such that $g(x) = \mu_-$; inductively define D_{n+1}^+ to be the least positive number such that g takes both values μ_- and μ_+ in the interval

$$\left(\sum_{i=0}^{n} D_i^+ , \sum_{i=0}^{n+1} D_i^+ \right] ;$$

  and define $D_n^- \leq 0$ similarly. Then require that
  - $D_n^\pm$ exists and is finite for all n;
  - $\sum_{n=0}^{\infty} D_n^\pm = \pm\infty$;
  - $\lim_{n\to\infty} \left(D_{n+1}^\pm / \sum_{i=0}^{n} D_i^\pm \right) = 0$.

Example

If $g \colon \mathbb{R} \to [\mu_-, \mu_+]$ is continuous, periodic and surjective, then g has property (✠).

Example

Let $g \colon \Omega \times \mathbb{R} \to [\mu_-, \mu_+]$ be a doubly reflected Brownian motion (Wiener process). Then, for almost all $\omega \in \Omega$, $g(\omega, \cdot)$ has property (✠).

Example

If $g \colon \Omega \times \mathbb{R} \to [\mu_-, \mu_+]$ is any sample continuous and surjective process for which the $D_n^{\pm}$ are IID with finite variance, then g almost surely has property (✠).

Theorem (S.–Theil)

With the notation of the previous slides, suppose that G has surjective derivative $G' \colon \mathbb{R} \to [\mu_-, \mu_+]$. Then G' has property (✠) if and only if any $z_\varepsilon \colon [0, T] \to \mathbb{R}$ solving

$$\dot{z}_\varepsilon(t) = -\frac{1}{\varepsilon} E'_\varepsilon(t, z_\varepsilon(t))$$

converges pointwise as $\varepsilon \to 0$ to $z \colon [0, T] \to \mathbb{R}$ solving

$$\partial\Psi(\dot{z}(t)) \ni -E'(t, z(t))$$

$$\Psi(v) := \begin{cases} \mu_- v, & v \leq 0; \\ \mu_+ v, & v \geq 0. \end{cases}$$

Moreover, if z is continuous, then the convergence is uniform.

The key step in the proof of our theorem is to show that property (✠) is a necessary and sufficient condition for the zeroes of the vector field $-E'_\varepsilon(t, \cdot)$ to "fill up" the stable region $\mathcal{S}(t)$ for z in the sense of Kuratowski's limit inferior for sequences of subsets of metric spaces:

$$(\maltese) \iff \mathcal{S}(t) = \operatorname*{Li}_{\varepsilon \to 0} Z_\varepsilon(t) := \left\{ x \in \mathbb{R} \,\middle|\, \limsup_{\varepsilon \to 0} \operatorname{dist}(x, Z_\varepsilon(t)) = 0 \right\},$$

where

$$Z_\varepsilon(t) := \{ y \in \mathbb{R} \mid E'_\varepsilon(t, y) = 0 \}.$$

Summary

- Rate-independent processes are good approximations for dissipative systems in the absence of inertial effects.
- They feature strong nonlinearities and exhibit irreversibility.
- Morally, macroscopic dissipation should be a consequence of fine microstructure; the microscale evolution should be (more) linear and reversible.
- A large class of possible microstructures (those with property (✠) for given upper and lower bounds) all give rise to the same rate-independent macroscopic behaviour.

Where Next?

- What about non-convex energies E?
- What about higher-dimensional state spaces? We have some preliminary results in $\mathbb{R}^n$, where G is realized as a sum of small "dents" centred on the points of a Poisson point process.
- In applications to materials science, the state space is usually infinite-dimensional: for example, given a body $\Omega \subseteq \mathbb{R}^3$, we consider the space of deformations of that body,

$$\begin{aligned}\mathcal{Q} &:= \mathrm{SBV}(\Omega; \mathbb{R}^3) \\ &:= \left\{ u \colon \Omega \to \mathbb{R}^3 \,\middle|\, \nabla u = f \,\mathrm{d}\mathcal{H}^n + g \,\mathrm{d}\mathcal{H}^{n-1} \right\},\end{aligned}$$

and E is an integral functional, usually horribly non-convex.