Analysis of the Effect of a Heat Bath on a Rate-Independent System

Marisol Koslowski[1], Michael Ortiz[2], **Tim Sullivan**[2] and Florian Theil[3]

[1] School of Mechanical Engineering, Purdue University, U.S.A.

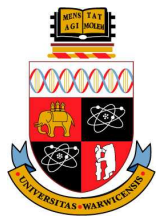
[2] Graduate Aerospace Laboratories, California Institute of Technology, U.S.A.

[3] Mathematics Institute, University of Warwick, U.K.

Rate-Independent Systems: Modeling, Analysis, and Computations

Banff International Research Station

29 August–3 September 2010



Outline

1. Introduction
2. Gradient Descents
3. Thermalized Gradient Descent — Generalities
4. Thermalized Gradient Descent — 1-Homogeneous Dissipation
5. Application: Andrade Creep
6. Conclusions and Outlook

Introduction

- Many deterministic physical processes are modeled by evolution laws that incorporate not just energetics but also dissipation, *i.e.* loss of energy due to frictional effects. Mathematically, these processes can be modelled as gradient descents — in the case of 1-homogenous dissipation, they are rate-independent processes.
- Also, many physical processes are subject to some randomness, perhaps due to the presence of a heat bath, which supplies additional (uncorrelated) energy to the process.

Question

How does a gradient descent (in particular, a rate-independent system) behave when placed in contact with a heat bath?

To investigate this, we introduce a notion of thermalized gradient descent.

Introduction

- The thermalization procedure is quite general, and in the case of 2-homogenous dissipation it corresponds to additive Itō noise.
- In the case of 1-homogeneous dissipation (a rate-independent system), something unexpected happens: the heat bath destroys the rate-independence in a controlled way and yields a deterministic nonlinear gradient descent; the thermalized dissipation potential is a "smoothing out" of the original one.
- As a toy model, consider a rough block sitting on a sandpaper table and subject to forces (springs, external loads, *& c.*) weaker than the frictional resistance of the sandpaper/block interface. Intuition suggests that
  - at "zero temperature", the block shouldn't move at all;
  - at "positive temperature" (shaking the table), the block might move — will it move deterministically? randomly?

Some Notation

- P will denote a finite partition of an interval of time $[0, T]$:

$$P = \{0 = t_0 < t_1 < t_2 < \cdots < t_N = T\}.$$

- Δ will denote the backward difference operator acting on sequences, finite or infinite, taking values in $\mathbb{R}$, $\mathbb{R}^n$, or any vector space $\mathcal{Z}$:

$$\Delta z_i := z_i - z_{i-1}.$$

- $\overline{z}^{(P)} : [0, T] \to \mathcal{Z}$ will denote the piecewise-constant càdlàg interpolation of $z_0, z_1, \ldots, z_N \in \mathcal{Z}$ on the partition P:

$$\overline{z}^{(P)}(t) := z_i \text{ for all } t \in [t_i, t_{i+1}).$$

- $[\![P]\!]$ will denote the mesh size of P:

$$[\![P]\!] := \max_{i=1,\ldots,N} |\Delta t_i|.$$

Gradient Descents — The Basics

- Many evolutionary models for physical processes are posed in terms of a quantity/field of interest, $z\colon [0,T] \to \mathcal{Z}$, $\mathcal{Z}$ being some (suitably nice) linear space (*e.g.* Hilbert, Banach, $\mathrm{BV}(\Omega;\mathbb{R}^3)$, ...).
- The evolution of z is determined by an initial condition, an energetic potential $E\colon [0,T] \times \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ and a dissipation potential $\Psi\colon \mathcal{Z} \to [0,+\infty]$.

Example

In $\mathcal{Z} = \mathbb{R}^n$ with dissipation $\Psi = \frac{1}{2}\|\cdot\|_2^2$, we have the classical gradient descent

$$\dot{z}(t) = -\nabla E(t, z(t)).$$

Along a trajectory, the energy satisfies the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t, z(t)) = -\|\dot{z}(t)\|_2^2 + (\partial_t E)(t, z(t)).$$

Gradient Descents — Energetic Solutions

Definition

$z\colon [0,T] \to \mathcal{Z}$ is said to be an energetic solution of the gradient descent problem in E and Ψ if z is absolutely continuous, satisfies the prescribed intitial condition, and, Lebesgue-a.e. in $[0,T]$, the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t, z(t)) = -\big(\Psi(\dot{z}(t)) + \Psi^\star(\mathrm{D}E(t, z(t)))\big) + (\partial_t E)(t, z(t)),$$

where $\Psi^\star\colon \mathcal{Z}^\star \to \mathbb{R} \cup \{+\infty\}$ is the convex conjugate of Ψ:

$$\Psi^\star(\ell) := \sup\{\langle \ell, x\rangle - \Psi(x) \mid x \in \mathcal{Z}\}.$$

Much of this carries over to state spaces with no linear structure: see Ambrosio, Gigli & Savaré (2008), *Gradient Flows in Metric Spaces and in the Space of Probability Measures*.

Gradient Descents — Incremental Formulation

- Typically, existence and uniqueness questions for gradient descents, as well as computation, are addressed using the Moreau–Yosida (implicit Euler) incremental formulation: given $z_i \approx z(t_i)$, find $z_{i+1} \approx z(t_{i+1})$ to minimize

$$\mathscr{W}_{i+1} : \tilde{z} \mapsto \underbrace{E(t_{i+1}, \tilde{z}) - E(t_i, z_i)}_{\text{energy difference}} + \underbrace{\Delta t_{i+1} \Psi\left(\frac{\tilde{z} - z_i}{\Delta t_{i+1}}\right)}_{\text{dissipation}}. \qquad (\mathrm{MY})$$

- The idea now is to generate a thermalized gradient descent Z by seeking densities $\rho(t, z)$ that minimize a functional in which (MY) competes with a entropy term.
- For simplicity, we now restrict our attention to $\mathcal{Z} = \mathbb{R}^n$, and "entropy" will mean relative entropy with respect to n-dimensional Lebesgue measure.

Thermalized Gradient Descent

Consider the following incremental problem for the PDF $\rho(t, \cdot)$ of Z_t with respect to Lebesgue measure on $\mathbb{R}^n$ at times $0 = t_0 < t_1 < \ldots < t_N = T$:

- Consider the "prior" density $\rho_i \approx \rho(t_i, \cdot)$ on $\mathbb{R}^n$.
- Find a new joint density $\rho_{i,i+1}(\cdot, \cdot)$ on $(\mathbb{R}^n)^2$, with first marginal ρ_i, that minimizes

$$\tilde{\rho} \mapsto \iint \left[\mathscr{W}_{i+1} \tilde{\rho} + \varepsilon \tilde{\rho} \log \tilde{\rho} \right],$$

  where

$$\mathscr{W}_{i+1}(z_i, z_{i+1}) = E(t_{i+1}, z_{i+1}) - E(t_i, z_{i+1}) + \Delta t_{i+1} \Psi \left(\frac{\Delta z_{i+1}}{\Delta t_{i+1}} \right)$$

  is the Moreau–Yosida "cost" of changing from state z_i to state z_{i+1} in the absence of the heat bath.
- Integrate/marginalize over the first slot of $\rho_{i,i+1}(\cdot, \cdot)$ to get a "posterior" density ρ_{i+1} on $\mathbb{R}^n$ for time t_{i+1}.

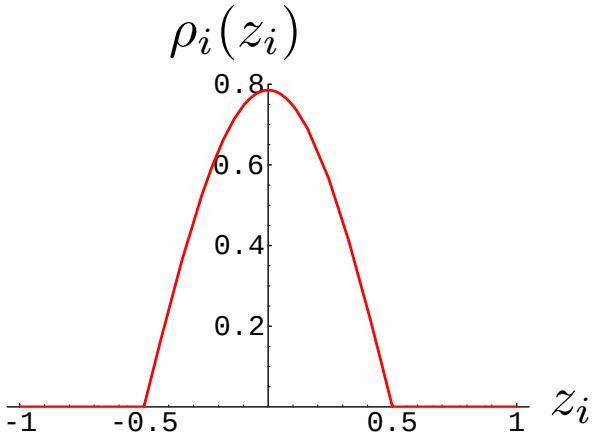
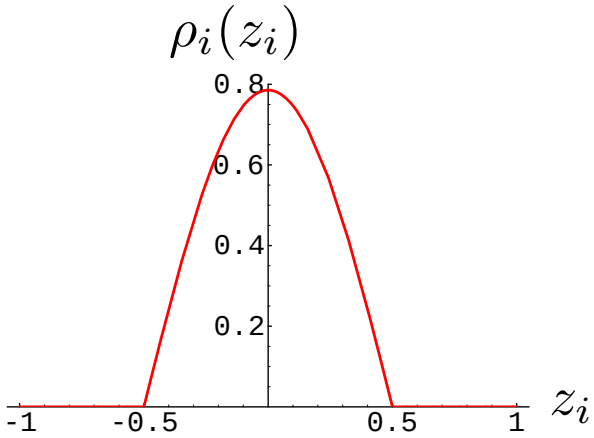


Figure: One step of the incremental thermalized gradient descent problem.



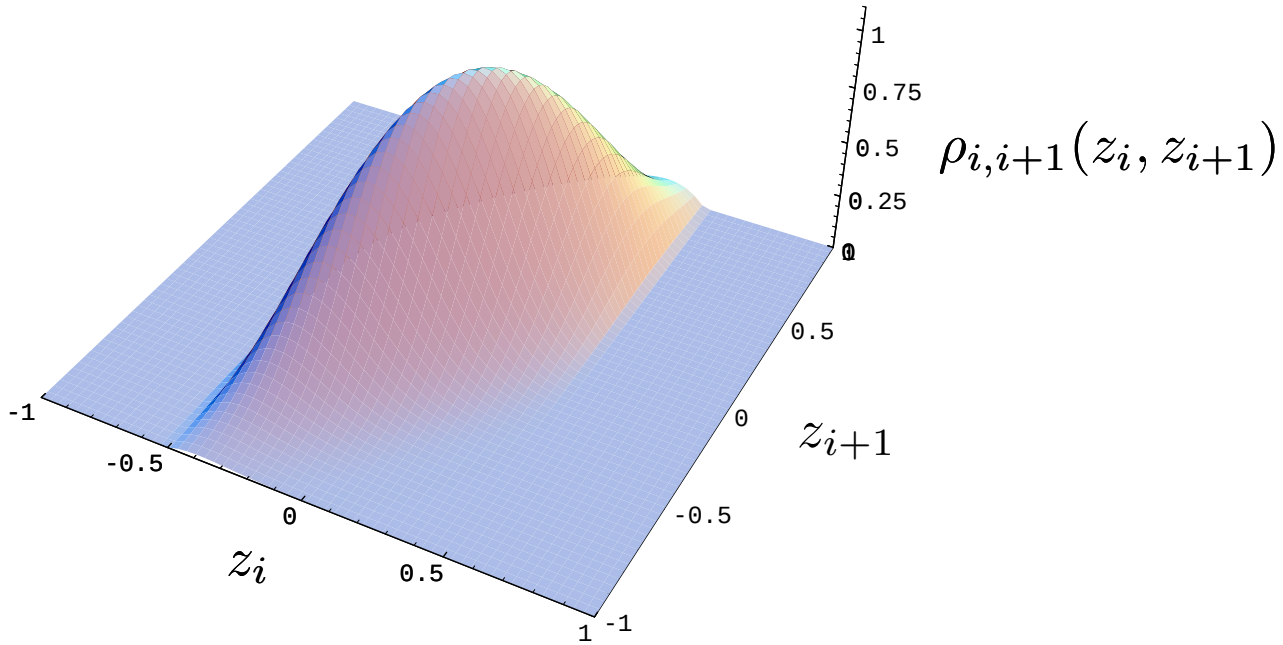
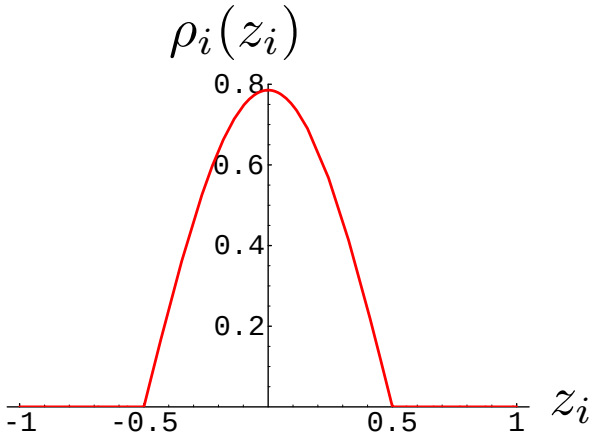
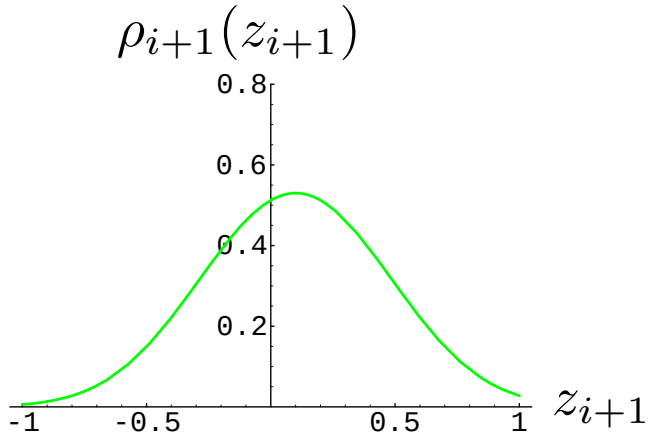


Figure: One step of the incremental thermalized gradient descent problem.





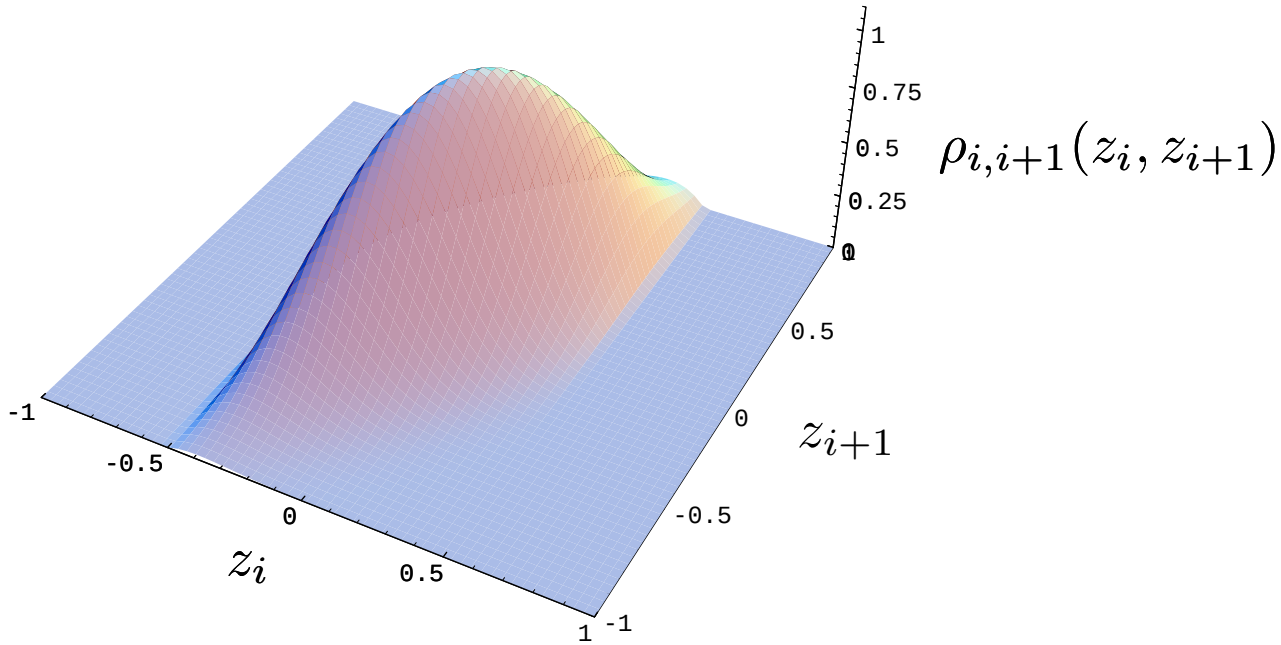


Figure: One step of the incremental thermalized gradient descent problem.

Thermalized Gradient Descent

Lemma

Subject to mild regularity and growth conditions, the single-step increments follow a Gibbs–Boltzmann-type distribution with respect to the Moreau–Yosida incremental cost function:

$$\rho_{i+1}(z_{i+1}|z_i) = \frac{1}{C(z_i)} \exp\big(-\mathscr{W}_{i+1}(z_i, z_{i+1})/\varepsilon\big).$$

Definition

On a partition P of $[0,T]$, we will call the Markov chain $Z^{(P)}$ so generated the (discrete-time) thermalized gradient descent in E and Ψ.

Question. What limit, if any, does the càdlàg process $\overline{Z}^{(P)}$ have as $[\![P]\!] \to 0$?

Thermalized Gradient Descent

Example

For "nice" potentials E and the 2-homogeneous dissipation potential

$$\Psi := \frac{1}{2}\| \cdot \|_2^2,$$

the incremental scheme makes sense. As $[\![P]\!] \to 0$, the piecewise constant càdlàg interpolation $\overline{Z}^{(P)} : \Omega \times [0,T] \to \mathbb{R}^n$ of $Z^{(P)}$ converges in law on $\mathcal{C}^0([0,T];\mathbb{R}^n)$ to the solution Y of the Itō stochastic gradient descent

$$\dot{Y}(t) = -\nabla E(t, Y(t)) + \sqrt{\varepsilon}\,\dot{W}(t);$$

i.e., the thermalized gradient descent scheme is a plausible model for "linear kinetics + noise".

Thermalizing a Rate-Independent Process

Consider a "nice" rate-independent system in $\mathbb{R}^n$:

- an energetic potential $E\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ — convex, time derivative in $W^{1,\infty}$, space derivative in $\mathcal{C}^1$;
- a dissipation potential $\Psi\colon \mathbb{R}^n \to [0,+\infty)$ — homogeneous of degree one and strictly convex (*i.e.* non-degenerate). Ψ is the convex conjugate of the characteristic function of a convex compact set $\mathscr{E} \subsetneq (\mathbb{R}^n)^*$ that has $0 \in \mathring{\mathscr{E}}$, the *elastic region*:

$$\Psi(v) = \sup\{\langle \ell, v \rangle \mid \ell \in \mathscr{E}\}.$$

Study the process $Z^{(P)}$ by studying its increments:

$$\Delta Z_i^{(P)} := Z_i^{(P)} - Z_{i-1}^{(P)}.$$

What are the moments of the increments, $\mathbb{E}\left[\Delta Z_i^{(P)}\right]$, $\mathrm{Var}\left[\Delta Z_i^{(P)}\right]$ & *c.*?

Effective (Dual) Dissipation Potential

Definition

Define an effective dual dissipation potential

$$\widetilde{\Psi}^{\star} \colon (\mathbb{R}^n)^* \to \mathbb{R} \cup \{+\infty\}$$

by

$$\widetilde{\Psi}^{\star}(\ell) := \log \int_{\mathbb{R}^n} \exp\left(-\left(\langle \ell, x \rangle + \Psi(x)\right)\right) \, \mathrm{d}x.$$

(*Cf.* the cumulant-generating function of the measure $\mathrm{d}\psi = e^{-\Psi(x)} \, \mathrm{d}x$)
Define the effective dissipation potential by convex conjugation:

$$\widetilde{\Psi}(x) = \widetilde{\Psi}^{\star\star}(x) := \sup\left\{ \langle \ell, x \rangle - \widetilde{\Psi}^{\star}(\ell) \,\middle|\, \ell \in (\mathbb{R}^n)^* \right\}.$$

Note that $\widetilde{\Psi}$ is determined purely by the dissipation functional Ψ (or, equivalently, the elastic region $\mathscr{E} \subsetneq (\mathbb{R}^n)^*$ associated to Ψ).

$$\widetilde{\Psi}^\star(\ell) := \log \int_{\mathbb{R}^n} \exp\left(-\left(\langle \ell, x\rangle + \Psi(x)\right)\right) \, \mathrm{d}x.$$

$$\widetilde{\Psi}(x) := \sup_{\ell} \left[\langle \ell, x\rangle - \widetilde{\Psi}^\star(\ell)\right].$$

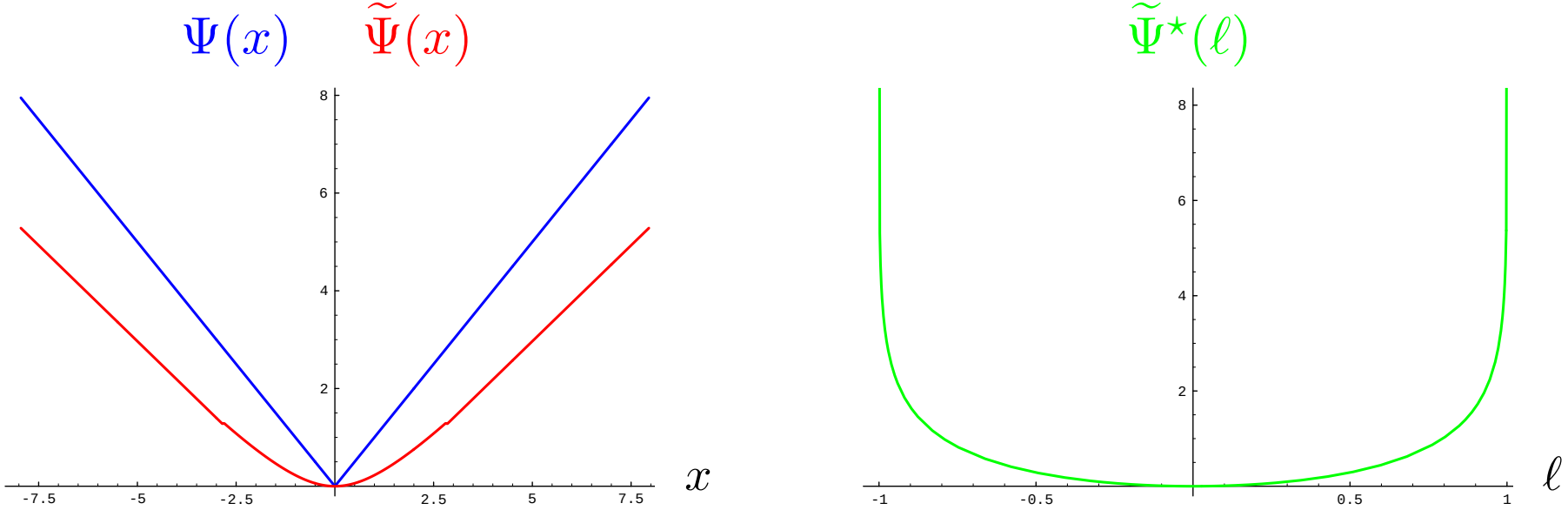


Figure: The effective (dual) dissipation potential in dimension one for $\Psi(x) = |x|$. $\widetilde{\Psi}^\star$ blows up like the logarithm of the distance to the yield surface $\partial\mathscr{E}$. $\widetilde{\Psi}$ is smooth and is asymptotic to Ψ at infinity.

Explicit Examples

Example

If Ψ is the weighted ℓ^1 norm $\Psi(x) := \sigma_1|x^1| + \ldots + \sigma_n|x^n|$, with weights $\sigma_i > 0$, then (up to an additive constant)

$$\widetilde{\Psi}^\star(\ell) = -\sum_{i=1}^{n} \log\left(\sigma_i^2 - |\ell_i|^2\right).$$

Example

If Ψ is a multiple of the Euclidean norm, $\Psi(x) := \sigma\|x\|_2$, $\sigma > 0$, then

$$\begin{aligned}\widetilde{\Psi}^\star(\ell) &= \log \int_{\mathbb{S}^{n-1}} \frac{(n-1)!}{\left(\langle \ell, \omega\rangle + \sigma\right)^{-n}} \,\mathrm{d}\mathcal{H}^{n-1}(\omega) \\ &= -\frac{n+1}{2} \log\left(\sigma^2 - \|\ell\|_2^2\right).\end{aligned}$$

Why $\widetilde{\Psi}$?

- The reason that $\widetilde{\Psi}$ is so important is that the change of variables $Z_{i+1} \rightsquigarrow \Delta Z_{i+1}/\varepsilon_{i+1}$ yields (modulo higher-order error terms):

$$\mathbb{E}\left[\Delta Z_{i+1} \middle| Z_i = z_i\right] \approx -\varepsilon_{i+1} \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big);$$

$$\mathrm{Var}\left[\Delta Z_{i+1} \middle| Z_i = z_i\right] \approx -\varepsilon_{i+1}^2 \left\|\mathrm{D}^2\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big)\right\|_2 \ll \varepsilon_{i+1}.$$

- Hence, it looks like the continuous-time limit with $\varepsilon_{i+1} = \theta\Delta t_{i+1}$ should satisfy the deterministic ordinary differential equation

$$\dot{y}(t) = -\theta \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t, y(t))\big)$$

Why $\widetilde{\Psi}$?

- The reason that $\widetilde{\Psi}$ is so important is that the change of variables $Z_{i+1} \rightsquigarrow \Delta Z_{i+1}/\varepsilon_{i+1}$ yields (modulo higher-order error terms):

$$\mathbb{E}\big[\Delta Z_{i+1} \big| Z_i = z_i\big] \approx -\varepsilon_{i+1} \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big);$$

$$\mathrm{Var}\big[\Delta Z_{i+1} \big| Z_i = z_i\big] \approx -\varepsilon_{i+1}^2 \left\|\mathrm{D}^2\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big)\right\|_2 \ll \varepsilon_{i+1}.$$

- Hence, it looks like the continuous-time limit with $\varepsilon_{i+1} = \theta \Delta t_{i+1}$ should satisfy the deterministic ordinary differential equation

$$\dot{y}(t) = -\theta \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t, y(t))\big)$$

$$\textit{i.e.,}\ \mathrm{D}\widetilde{\Psi}\left(-\frac{\dot{y}(t)}{\theta}\right) = \mathrm{D}E(t, y(t)),$$

Why $\widetilde{\Psi}$?

- The reason that $\widetilde{\Psi}$ is so important is that the change of variables $Z_{i+1} \rightsquigarrow \Delta Z_{i+1}/\varepsilon_{i+1}$ yields (modulo higher-order error terms):

$$\mathbb{E}\big[\Delta Z_{i+1} \big| Z_i = z_i\big] \approx -\varepsilon_{i+1} \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big);$$

$$\mathrm{Var}\big[\Delta Z_{i+1} \big| Z_i = z_i\big] \approx -\varepsilon_{i+1}^2 \left\|\mathrm{D}^2\widetilde{\Psi}^\star\big(\mathrm{D}E(t_{i+i}, z_i)\big)\right\|_2 \ll \varepsilon_{i+1}.$$

- Hence, it looks like the continuous-time limit with $\varepsilon_{i+1} = \theta \Delta t_{i+1}$ should satisfy the deterministic ordinary differential equation

$$\dot{y}(t) = -\theta \mathrm{D}\widetilde{\Psi}^\star\big(\mathrm{D}E(t, y(t))\big)$$

$$\textit{i.e.,}\ \mathrm{D}\widetilde{\Psi}\left(-\frac{\dot{y}(t)}{\theta}\right) = \mathrm{D}E(t, y(t)),$$

$$\textit{i.e.,}\ \mathrm{D}\widetilde{\Psi}\left(\frac{\dot{y}(t)}{\theta}\right) = -\mathrm{D}E(t, y(t)).$$

Convergence Result

Theorem (Convergence to Nonlinear Gradient Descent)

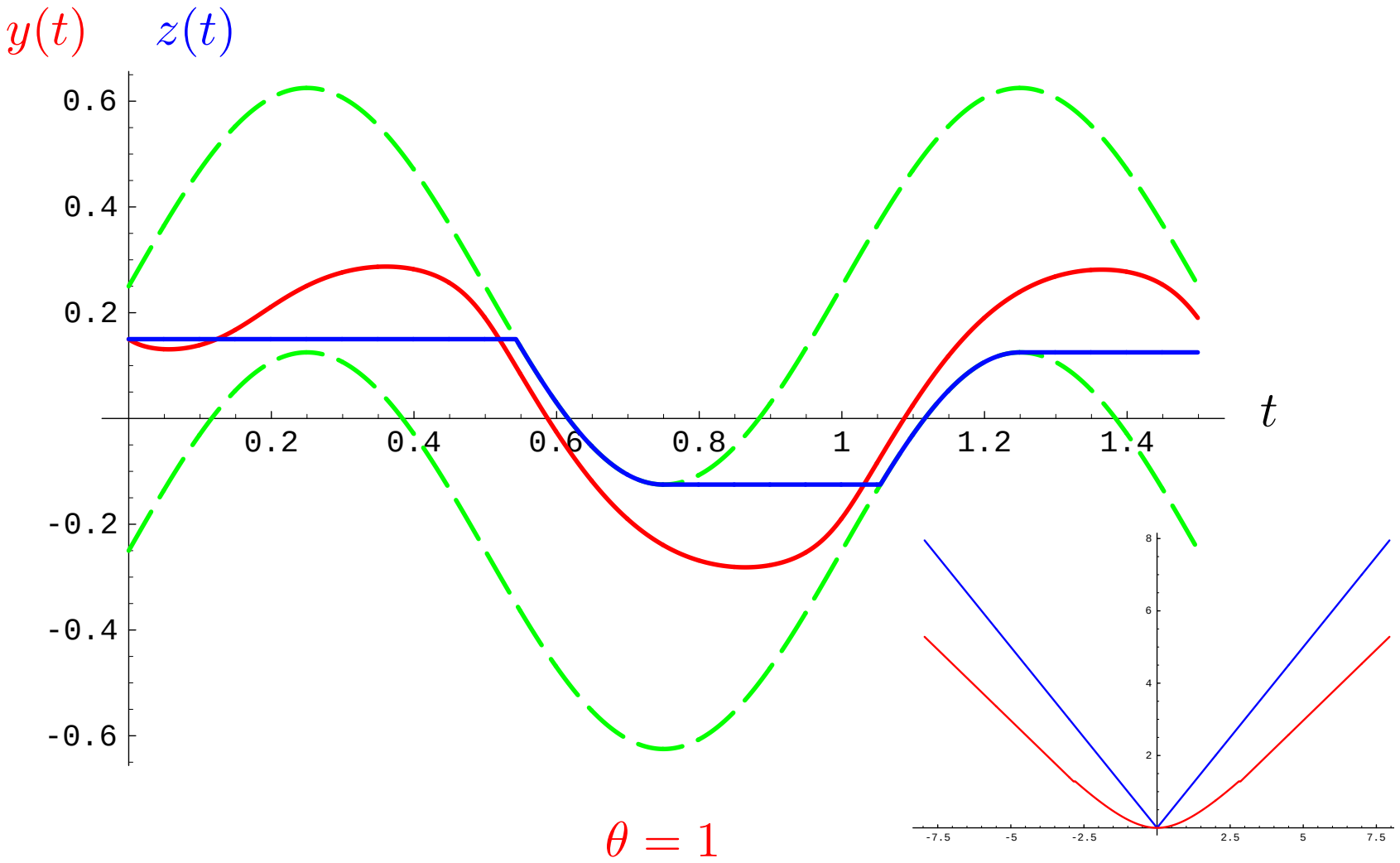
If $(t,x) \mapsto \widetilde{\Psi}^\star(\mathrm{D}E(t,x))$ *is convex in* x *for each* $t \in [0,T]$*, then the piecewise constant càdlàg interpolation* $\overline{Z}^{(P)} : \Omega \times [0,T] \to \mathbb{R}^n$ *converges in probability as* $[\![P]\!] \to 0$ *to the solution of*

$$\mathrm{D}\widetilde{\Psi}\left(-\frac{\dot{y}(t)}{\theta}\right) = \mathrm{D}E(t,y(t)).$$

More precisely, for any $\lambda > 0$*, as* $[\![P]\!] \to 0$*,*

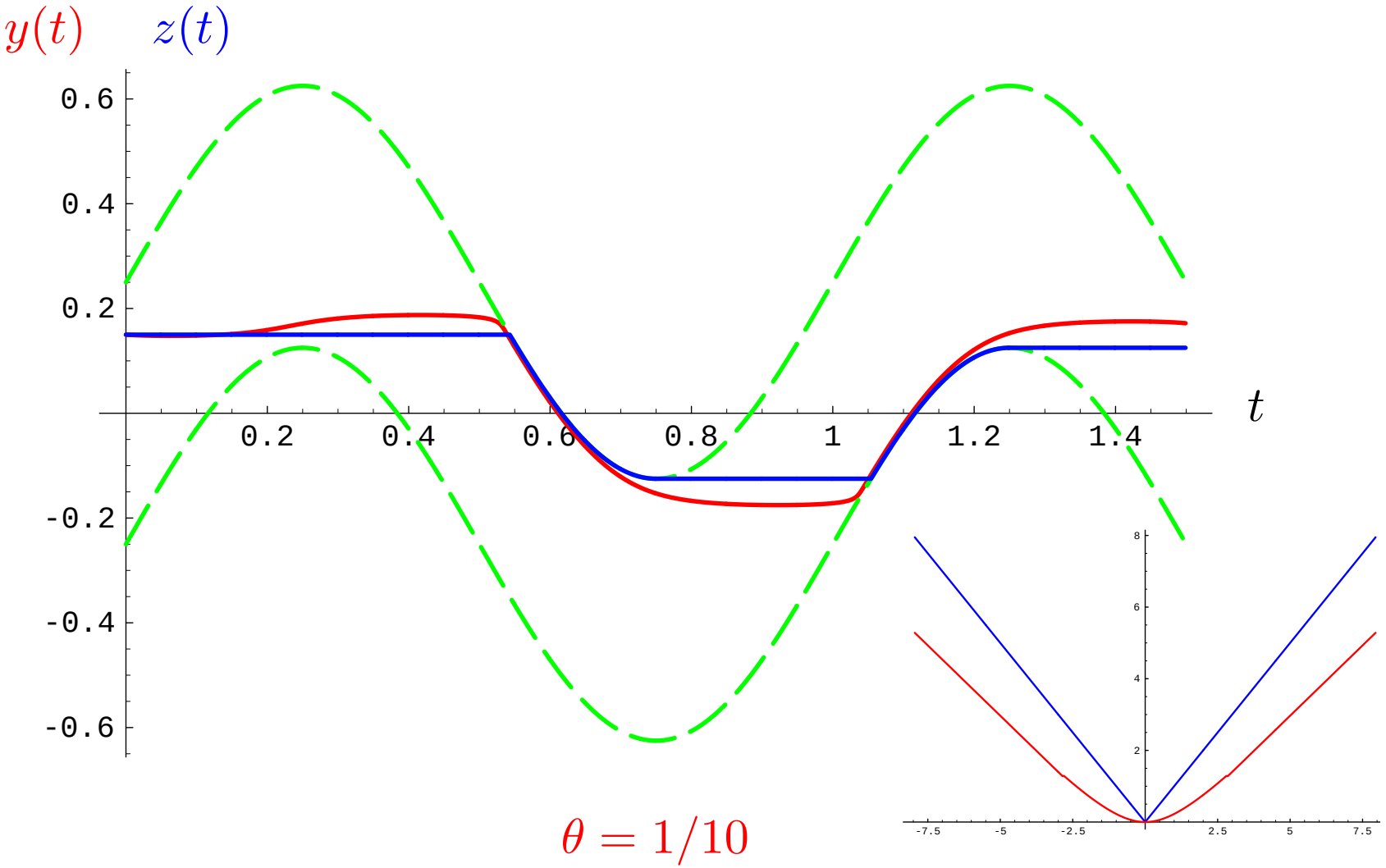
$$\mathbb{P}\left[\sup_{t\in[0,T]} \left\|\overline{Z}^{(P)}(t) - y(t)\right\|_2 \geq \lambda\right] \in O\big([\![P]\!]^{1/2}\big).$$

(If $\mathrm{D}^2 E \equiv 0$*, then the order of convergence is* $O([\![P]\!])$*.)*



Indicated in green is the frontier of the stable region,

$$\mathcal{S}(t) := \{x \in \mathbb{R}^n \mid -\mathrm{D}E(t,x) \in \mathscr{E}\}.$$



Indicated in green is the frontier of the stable region,

$$\mathcal{S}(t) := \{x \in \mathbb{R}^n \mid -\mathrm{D}E(t,x) \in \mathscr{E}\}.$$

Andrade's Creep Law

- In 1910, Andrade reported that as a function of time, t, the creep deformation, ξ, of soft metals at constant temperature and applied stress can be described by a power law $\xi(t) \sim t^{1/3}$. Similar behavior has been observed in many classes of materials, including non-crystalline materials.
- Morally, macroscopic creep should be observed as a change in the mean of the microscopic slip field.
- We will study this using the Koslowski–Cuitiño–Ortiz phase field model for a material sample along a single slip plane, thought of as the unit torus, $\mathbb{T}^2$.
- The phase field $u \in H^{1/2}(\mathbb{T}^2; \mathbb{R})$ measures the total amount of slip at each point: $u(x) \in \mathbb{R}$ is the slip (in multiples of the Burgers vector b) at $x \in \mathbb{T}^2$.

KCO Phase Field Model — Details

In the KCO model the energetic potential is

$$E(s,u) := \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{\mu b^2}{4}\frac{1}{1/K+d/2}|\hat{u}(k)|^2 - (s\cdot b)\int_{\mathbb{T}^2} u(x)\,\mathrm{d}x,$$

where $\hat{u}$ is the Fourier transform of u, $k\in\mathbb{Z}^2$ is the wavenumber,

$$K := \frac{1}{1-\nu}\sqrt{k_1^2+k_2^2} - \frac{\nu}{1-\nu}\frac{k_2^2}{\sqrt{k_1^2+k_2^2}},$$

b is the Burgers vector, s is the applied shear stress, ν is Poisson's ratio and d is the interplanar distance. For $\mu>0$ (the shear modulus), the dissipation potential is

$$\Psi(u) = \mu\sum_{p\in\mathcal{O}}\int_{\mathbb{B}_r(p)}|u(x)|\,\mathrm{d}x.$$

Reduction of the KCO Model

- In KCO, the dissipation Ψ is not positive-definite on $H^{1/2}(\mathbb{T}^2;\mathbb{R})$: it is concentrated over discs of radius $0 < r \ll 1$ centred on a (random) point set of obstacles, $\mathcal{O}$. Hence, only the values of the phase field over $\mathcal{O}$ experience dissipation.
- Schur complementation (minimizing out those degrees of freedom that do not experience dissipation) yields a reduced finite-dimensional model posed in $\mathbb{R}^{\mathcal{O}}$. The reduced energetic potential is a quadratic form in $\xi := u|_{\mathcal{O}}$:

$$E_{\mathrm{red}}(s,\xi) = \frac{1}{2}\xi \cdot G\xi - (s\cdot b)\tau\cdot\xi + \frac{h}{2}(s\cdot b)^2$$

and the reduced dissipation Ψ_{red} is a non-degenerate weighted ℓ^1 norm.

Mean Field Model

- Write ξ as the sum of its mean and the fluctuation about the mean:

$$\bar{\xi} := \frac{1}{\#\mathcal{O}} \sum_{p \in \mathcal{O}} \xi_p \in \mathbb{R}, \quad \hat{\xi} := \xi - \bar{\xi}\mathbf{1}.$$

- Creep is a "bulk phenomenon" in the sense that it is a change of the mean field $\bar{\xi}$ of the overall phase field. Numerical experiments bear out the hypothesis that the fluctuations $\hat{\xi}$ of the phase field about its mean are small (in the sense that they dissipate very little energy).
- Therefore, we narrow the focus of our study to the one-dimensional mean field $\bar{\xi}$. The resulting model is of the form

$$E_{\mathrm{MF}}\big(s, \bar{\xi}\big) = -(s \cdot b)\bar{\tau}\bar{\xi}; \quad \Psi_{\mathrm{MF}}\big(\bar{\xi}\big) = \sigma\big|\bar{\xi}\big|,$$

where $\sigma = \mu \cdot \pi r^2 \cdot \#\mathcal{O}$.

Convergence Result

- In this case, the effective dual dissipation potential for the mean field is
$$\widetilde{\Psi}^{\star}_{\mathrm{MF}}\big(\bar{\ell}\big) = -\log\left(\sigma^2 - (s\cdot b)^2\bar{\tau}^2\right).$$
- Under the assumption of linear strain hardening (*i.e.* $\sigma = \sigma_0\bar{\xi}$) the resulting effective ordinary differential equation is
$$\frac{\mathrm{d}\bar{\xi}(t)}{\mathrm{d}t} = \frac{2\theta(s\cdot b)\bar{\tau}}{\big(\sigma_0\bar{\xi}(t)\big)^2 - (s\cdot b)^2\bar{\tau}^2}.$$
- For fixed θ and $0 < (s\cdot b)\bar{\tau} \ll \sigma$ (*i.e.* constant temperature and constant small applied stress), solutions grow in accord with Andrade's creep law:
$$\bar{\xi}(t) \sim t^{1/3}.$$

Conclusions

Given a reference measure and hence a notion of entropy, gradient descents can be thermalized and the resulting Markov chain models analyzed.

- In the case of 2-homogeneous dissipation, the TGD scheme describes the effect of adding Itō noise.
- In the case of 1-homogeneous dissipation, the TGD scheme yields a deterministic gradient flow in an effective dissipation potential $\widetilde{\Psi}$ that is a nonlinear transformation ("smoothing-out") of the original dissipation potential Ψ.
- As an application, this analysis can be used to derive Andrade's creep law for soft metals under the assumption of linear strain hardening.

Outlook

Can this analysis be extended to

- second-order equations of motion (inertial effects, non-Markovian processes)? Difficulty: increments are very badly behaved, variances can blow up in short time.
- infinite-dimensional and/or non-linear state spaces? Difficulty: with respect to what reference measure will entropy be calculated?

"Commutativity Question". The Andrade creep law should follow directly from applying the TGD method to the Koslowski–Cuitiño–Ortiz model in $H^{1/2}(\mathbb{T}^2)$ and *then* examining the mean field, not passing to a mean field model first and then applying TGD.

Bibliography & Acknowledgements

- A. Garroni & S. Müller. "Γ-limit of a phase-field model of dislocations." *SIAM J. Math. Anal.* 36 (**6**) 1943–1964. 2005.
- A. Garroni & S. Müller. "A variational model for dislocations in the line tension limit." *Arch. Ration. Mech. Anal.* 181 (**3**) 535–578. 2006.
- M. Koslowski, A. M. Cuitiño, M. Ortiz. "A phase-field theory of dislocation dynamics, strain hardening and hysteresis in ductile single crystals." *J. Mech. Phys. Solids* 50 (**12**) 2597–2635. 2002.
- T. J. Sullivan, M. Koslowski, F. Theil, M. Ortiz. "On the behavior of dissipative systems in contact with a heat bath: Application to Andrade creep." *J. Mech. Phys. Solids* 57 (**7**) 1059–1077. 2009.
- T. J. Sullivan. *Analysis of Gradient Descents in Random Energies and Heat Baths.* Ph.D. thesis, University of Warwick, U.K. 2009.

THE UNIVERSITY OF WARWICK

Portions of this work were supported by the Engineering and Physical Sciences Research Council and by the University of Warwick Research Development Fund.

EPSRC Pioneering research and skills