Thermalization of Rate-Independent Processes by Entropic Regularization

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Joint work with **M. Koslowski** (Purdue), **F. Theil** (Warwick) and **M. Ortiz** (Caltech).

Problem Setting

- Gradient descent on a connected Riemannian manifold $(\mathcal{Q}, g)$ in an energetic potential $E\colon [0, T] \times \mathcal{Q} \to \mathbb{R}$ with respect to a dissipation potential $\Psi\colon [0, T] \times \mathrm{T}\mathcal{Q} \to [0, +\infty)$:

$$\partial\Psi(t, z(t), \dot{z}(t)) \ni -\mathrm{D}E(t, z(t)). \tag{RI}$$

- Each $\Psi(t, x, \cdot)$ is 1-homogenous: the dissipation is a Finsler structure on $\mathcal{Q}$, continuous and non-degenerate w.r.t. g. This makes the evolution rate-independent (a.k.a. quasi-static): the solution operator commutes with monotone reparametrizations of time.
- (RI) models stick-slip dynamics, dry friction, evolution of some material properties (*e.g.* the Barkhausen effect in magnetization).
- We analyse a positive-temperature perturbation of (RI). As an application, this model explains the creep effects shown by such systems at positive temperature.

Incremental Problem

- The discrete time incremental formulation of (RI) is, given times $\{t_i = ih \mid i = 0, \dots, T/h\}$ and the state z_i at time t_i, to find the state z_{i+1} at time t_{i+1} that minimizes

$$W(z_i, z_{i+1}) := E(t_{t+1}, z_{i+1}) - E(t_i, z_i) + h\Psi\big(\mathrm{Log}_{z_i}(z_{i+1})/h\big).$$

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- To model the effect of a heat bath with power $\theta > 0$ (*i.e.* injects energy θh over $[t_i, t_{i+1}]$), we posit that the random next state Z^h_{i+1} has probability distribution $\rho(\cdot|z_i)\,\mathrm{dVol}_g$ on $\mathcal{Q}$ that minimizes

$$\int_{\mathcal{Q}} \big[W(z_i, \cdot)\rho(\cdot|z_i) + \theta h\,\rho(\cdot|z_i)\log\rho(\cdot|z_i)\big]\,\mathrm{dVol}_g$$

$$\textit{i.e.} \quad \rho(z_{i+1}|z_i) \propto \exp\left(-\frac{W(z_i, z_{i+1})}{\theta h}\right)$$

  and consider the Markov chain Z^h with such transition probabilities.
- For 2-homogeneous Ψ, this procedure corresponds to adding Itō noise. What is the continuous-time limit for 1-homogeneous Ψ?

Incremental Distribution

Quick back-of-envelope calculations in $\mathrm{T}_{z_i}\mathcal{Q}$ yield

$$\mathbb{E}\Big[\mathrm{Log}_{z_i}(Z_{i+1})\Big|Z_i = z_i\Big] \approx -\theta h\, \mathrm{D}\widetilde{\Psi}^\star\big(t_i, z_i, \mathrm{D}E(t_i, z_i)\big),$$

$$\mathbb{V}\Big[\mathrm{Log}_{z_i}(Z_{i+1})\Big|Z_i = z_i\Big] \approx (\theta h)^2\, \mathrm{D}^2\widetilde{\Psi}^\star\big(t_i, z_i, \mathrm{D}E(t_i, z_i)\big).$$

Conjecture

The variance is essentially negligible, and so the limit process as $h \to 0$ is a deterministic flow along the vector field on the RHS of the expression for the mean:

$$\dot{y}(t) = -\theta\, \mathrm{D}\widetilde{\Psi}^\star\big(t, y(t), \mathrm{D}E(t, y(t))\big),$$

$$\textit{i.e.} \quad \mathrm{D}\widetilde{\Psi}\big(t, y(t), -\theta^{-1}\dot{y}(t)\big) = \mathrm{D}E(t, y(t)), \qquad \text{(NL)}$$

$$\text{(If } \Psi \text{ is even)} \quad \mathrm{D}\widetilde{\Psi}\big(t, y(t), \theta^{-1}\dot{y}(t)\big) = -\mathrm{D}E(t, y(t)),$$

i.e. the non-linear $\widetilde{\Psi}$-gradient descent in E.

Effective Dissipation — Cramer Transform

Definitions

The effective dissipation potential $\widetilde{\Psi}$ on the previous slide is the Cramer transform of Ψ, defined for each $(t, x) \in [0, T] \times \mathcal{Q}$ by

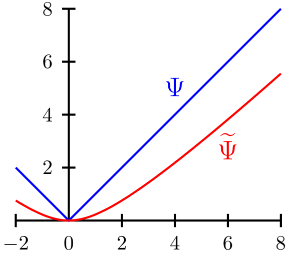
$$\widetilde{\Psi}^\star(t, x, \ell) := \log \int_{\mathrm{T}_x \mathcal{Q}} \exp\big(-(\langle \ell, v \rangle + \Psi(t, x, v)) \big) \, \mathrm{d}v, \qquad \ell \in \mathrm{T}_x^* \mathcal{Q},$$

$$\widetilde{\Psi}(t, x, v) := \sup \left\{ \langle \ell, v \rangle - \widetilde{\Psi}^\star(t, x, \ell) \,\middle|\, \ell \in \mathrm{T}_x^* \mathcal{Q} \right\}, \qquad v \in \mathrm{T}_x \mathcal{Q}.$$

Example

$\Psi(v) := \sigma \|v\|_2$ on $\mathbb{R}^n$, $\sigma > 0$,

$$\widetilde{\Psi}^\star(\ell) = -\frac{n+1}{2} \log \left(\sigma^2 - \|\ell\|_2^2 \right)$$



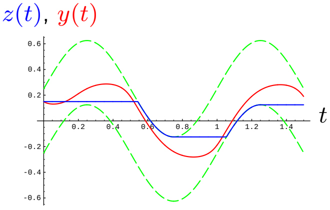
Theorem

Under technical conditions, the piecewise-constant interpolants of the discrete time Markov chain Z^h converge in probability as $h \to 0$ to y, the solution of (NL), *i.e.*

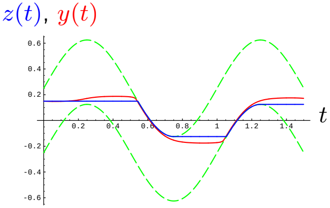
$$\mathrm{D}\widetilde{\Psi}(t, y(t), -\theta^{-1}\dot{y}(t)) = \mathrm{D}E(t, y(t))$$

with the same initial condition. That is, for all $\delta > 0$,

$$\lim_{h \to 0} \mathbb{P}\left[\sup_{t \in [0,T]} d_{(\mathcal{Q},g)}\big(Z^h(t), y(t)\big) \geq \delta \right] = 0.$$



(a) $\theta = 1$



(b) $\theta = \frac{1}{10}$

Figure: Comparison of the original rate-independent process z (blue) that solves (RI) and the thermalized process y (red) that solves (NL).

Technical Conditions

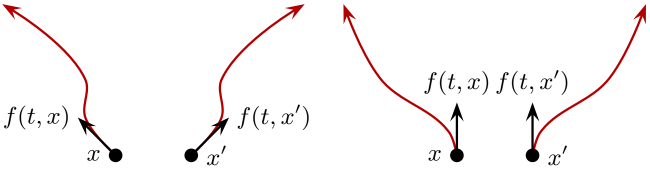
- Main technical condition (for the moment!): the vector field

$$f(t, x) := -\mathrm{D}\widetilde{\Psi}^{\star}\big(t, x, \mathrm{D}E(t, x)\big)$$

should admit a spacetime neighbourhood of the solution y in which, for any two initial conditions (t, x) and (t, x') and small enough $h > 0$,

$$d_{(\mathcal{Q},g)}\big(\mathrm{Exp}_x(hf(t, x)), \mathrm{Exp}_{x'}(hf(t, x'))\big) \leq d_{(\mathcal{Q},g)}(x, x').$$

- This is can be seen as a combination of two criteria:
  - the vector field f should not be outward-pointing;
  - the curvature of $(\mathcal{Q}, g)$ should not be strongly positive.



Andrade's creep law (1910)

For soft metals under constant subcritical stress, strain grows initially $\sim t^{1/3}$ and later $\sim t$.

- Work on $\mathcal{Q} = (0, +\infty)$ with energy gradient $\mathrm{D}E(t, x) \equiv \ell$ and the Finsler dissipation $\Psi(t, x, v) = \sigma x |v|$, *i.e.* linear strain hardening.
- Solutions to the effective evolution (NL)

$$y(0) = 1, \quad \mathrm{D}\widetilde{\Psi}(t, y(t), -\theta^{-1}\dot{y}(t)) = \ell$$

$$\text{i.e.} \quad \dot{y}(t) = \frac{2\theta\ell}{(\sigma y(t))^2 - \ell^2}$$

do indeed grow $\sim t^{1/3}$, in accordance with Andrade's creep law:

$$y(t) = \left(C + \frac{6\theta\ell t}{\sigma^2} \right)^{1/3}.$$