Exercise Sheet 1

These exercises relate to the material covered in the lecture of Week 1. Please submit your solutions to these exercises at the beginning of the lecture of Week 2, i.e. 12:00 on 22 October 2015. Environmentally-friendly submissions by e-mail in PDF form are welcomed! The numbers in the margin indicate approximately how many points are available for each part.

Exercise 1.1. Let X be a random variable taking values in the Hilbert space $\mathcal{H} = \mathbb{R}^n$ for $n \geq 1$ with mean $m \in \mathbb{R}^n$ and covariance matrix C defined by

$$C := \mathbb{E}\Big[(X-m)(X-m)^\top\Big] \in \mathbb{R}^{n\times n}$$

(a) Show that C is symmetric and positive semi-definite. **[4]**

(b) Show that if C has a non-trivial kernel, then there exists an open half-space H with $\dim H \geq 1$ such that $X \notin H$ almost surely. Given this result, explain the consequences for C if X is supported everywhere in $\mathbb{R}^n$. Is the converse true? **[6]**

Exercise 1.2. Let $\mu = \mathcal{N}(m, C)$ be a Gaussian measure on $\mathbb{R}^d$ and let $v \in \mathbb{R}^d$. Let T_v be the translation map $x \mapsto x + v$. Show that the push-forward of μ by T_v, namely $\mathcal{N}(m + v, C)$, is equivalent to μ and satisfies

$$\frac{\mathrm{d}(T_v)_*\mu}{\mathrm{d}\mu}(x) = \exp\left(\langle v, x-m\rangle_{C^{-1}} - \frac{1}{2}\|v\|^2_{C^{-1}}\right),$$

sometimes known as the *Cameron–Martin formula*, i.e., for every μ-integrable function f, **[8]**

$$\int_{\mathbb{R}^d} f(x+v)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} f(x)\exp\left(\langle v, x-m\rangle_{C^{-1}} - \frac{1}{2}\|v\|^2_{C^{-1}}\right)\mathrm{d}\mu(x).$$

Exercise 1.3. Let $T\colon \mathcal{H} \to \mathcal{K}$ be a bounded linear map between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, with adjoint $T^*\colon \mathcal{K} \to \mathcal{H}$, and let $\mu = \mathcal{N}(m, C)$ be a Gaussian measure on $\mathcal{H}$. Show that the push-forward measure $T_*\mu$ is a Gaussian measure on $\mathcal{K}$ and that $T_*\mu = \mathcal{N}(Tm, TCT^*)$. **[8]**

Exercise 1.4. For $i = 1, 2$, let $X_i \sim \mathcal{N}(m_i, C_i)$ independent Gaussian random variables taking values in Hilbert spaces $\mathcal{H}_i$, and let $T_i\colon \mathcal{H}_i \to \mathcal{K}$ be a bounded linear map taking values in another Hilbert space $\mathcal{K}$, with adjoint $T_i^*\colon \mathcal{K} \to \mathcal{H}_i$. Show that $T_1X_1 + T_2X_2$ is a Gaussian random variable in $\mathcal{K}$ with

$$T_1X_1 + T_2X_2 \sim \mathcal{N}\big(T_1m_1 + T_2m_2, T_1C_1T_1^* + T_2C_2T_2^*\big).$$

Give an example to show that the independence assumption is necessary. **[8]**

Exercise 1.5. The Woodbury formula / method of Schur complements from basic linear algebra has a natural interpretation in terms of the conditioning of Gaussian random variables. Let $(X, Y) \sim \mathcal{N}(m, C)$ be jointly Gaussian, where, in block form,

$$m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix},$$

and C is self-adjoint and positive definite.

(a) Show that C_{11} and C_{22} are self-adjoint and positive-definite. **[4]**

(b) Show that the Schur complement S defined by $S := C_{11} - C_{12}C_{22}^{-1}C_{12}^*$ is self-adjoint and positive definite, and **[4]**

$$C^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}C_{12}C_{22}^{-1} \\ -C_{22}^{-1}C_{12}^*S^{-1} & C_{22}^{-1} + C_{22}^{-1}C_{12}^*S^{-1}C_{12}C_{22}^{-1} \end{bmatrix}.$$

(c) Hence prove that the conditional distribution of X given that $Y = y$ is Gaussian: **[8]**

$$(X|Y = y) \sim \mathcal{N}\big(m_1 + C_{12}C_{22}^{-1}(y - m_2), S\big).$$