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Exercise Sheet 2

These exercises relate to the material covered in the lecture of Week 2, and possibly previous weeks' lectures and exercises. Please submit your solutions to these exercises at the beginning of the lecture of Week 3, i.e. 12:00 on 29 October 2015. Environmentally-friendly submissions by e-mail in PDF form are welcomed! The numbers in the margin indicate approximately how many points are available for each part.

Exercise 2.1. Let $\mathcal{V}$ be a vector space over $\mathbb{K}$, equipped with a semi-definite inner product $\langle \cdot, \cdot \rangle$. Given $v_1, \dots, v_n \in \mathcal{V}$, the *Gram matrix* defined by these vectors is

$$G(v_1, \dots, v_n) = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix}.$$

(a) Show that, if $\mathcal{V} = \mathbb{K}^n$ with the usual inner product, then $G(v_1, \dots, v_n) = V^* V$, where the i-th column of the matrix V is v_i and V^* is the conjugate transpose of V. **[4]**
(b) Show that $G(v_1, \dots, v_n)$ is conjugate-symmetric (i.e. Hermitian). **[4]**
(c) Show that $\det G(v_1, \dots, v_n) \geq 0$. Show that if the $v_1, \dots, v_n$ are linearly dependent, then $\det G(v_1, \dots, v_n) = 0$, and that this statement becomes an 'if and only if' statement provided that the inner product on $\mathcal{V}$ is positive definite. **[4]**
(d) Prove the Cauchy–Schwarz inequality using the case $n = 2$. **[4]**

Exercise 2.2. This exercise concerns closest-point approximation in Banach spaces. The aim is to show that in Banach spaces (even finite-dimensional ones), closest-point approximation is not as simple as series truncation. Let $R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map that rotates the Euclidean plane about the origin by a fixed angle $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.
(a) Let $\| \cdot \|_1$ be the usual 1-norm on $\mathbb{R}^2$. Show that the function $\| \cdot \|_\theta$ defined by

$$\|(x, y)\|_\theta := \|R_\theta (x, y)\|_1$$

is a Banach norm on $\mathbb{R}^2$. **[2]**
(b) Find $x^* \in \mathbb{R}$ to minimise $\|(x^*, 0) - (1, 1)\|_\theta$. Hint: draw, with respect to the norm $\| \cdot \|_\theta$, open balls of various radii centred on $(1, 1)$. Compare your result for x^* with the value of $x^\dagger \in \mathbb{R}$ that minimises $\|(x^\dagger, 0) - (1, 1)\|_2$. For some $0 \neq \theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, draw a picture in $\mathbb{R}^2$ showing the corresponding x^*, and $x^\dagger$. What do you notice? **[8]**

Exercise 2.3. Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a domain, let $\mathcal{H} \subseteq L^2(\mathcal{X}, \mathrm{d}x; \mathbb{R})$ be a closed subspace of the Lebesgue space of square-integrable real-valued functions on $\mathcal{X}$, with its usual inner product and norm. Let $(e_k)_{k \in \mathbb{N}}$ be a complete orthogonal basis for $\mathcal{H}$. This exercise concerns series of the form

$$u = \sum_{k \in \mathbb{N}} u_k e_k \tag{2.1}$$

in $\mathcal{H}$, where the coefficients u_k are either real numbers or real-valued random variables.
(a) Show that, if $\|e_k\|_{\mathcal{H}}$ is bounded uniformly in k, and if the coefficients are square-summable, i.e. if $(u_k)_{k \in \mathbb{N}} \in \ell^2$, then the series (2.1) is Cauchy, and hence convergent, with respect to $\| \cdot \|_{\mathcal{H}}$. **[4]**
(b) Show that, if each $e_k \colon \mathcal{X} \to \mathbb{R}$ is a continuous function, such that $\|e_k\|_\infty$ is bounded uniformly in k, and if the coefficients are summable, i.e. if $(u_k)_{k \in \mathbb{N}} \in \ell^1$, then (2.1) defines a continuous function $u \colon \mathcal{X} \to \mathbb{R}$. Hint: consider the uniform norm instead of the norm in $\mathcal{H}$. **[4]**

(c) Let (Θ, μ) be a probability space. Suppose that the coefficients $u_k \in L^2(\Theta, \mu; \mathbb{R})$ are real-valued random variables. Specify conditions on $(u_k)_k$ and $(e_k)_k$ so that we consider u, as defined by (2.1), to be an element of $L^2(\Theta, \mu; \mathbb{R}) \otimes \mathcal{H}$ or as an element of $L^2(\Theta, \mu; \mathcal{H})$. **[4]**

(d) Suppose that $u_k \sim \mathcal{U}(-k^{-p}, k^{-p})$ are independently uniformly distributed in the interval $[-k^{-p}, k^{-p}] \subset \mathbb{R}$. Suppose also that the $\|e_k\|_\infty$ are bounded uniformly in k and that each e_k is a continuous function. Find conditions on $p \geq 0$ for u to be a well-defined $\mathcal{H}$-valued random variable, and for u to be almost surely continuous. **[4]**

(e) Repeat part (d) in the case that $u_k \sim \mathcal{N}(0, k^{-p})$ are independently normally distributed with mean zero and variance k^{-p}. You may use the fact that, if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$\mathbb{E}[|X|^q] = \sigma^q \frac{2^{q/2}\Gamma(\frac{q+1}{2})}{\sqrt{\pi}} \quad \text{for all } q > -1,$$

where Γ denotes the Gamma function. (Note: revisit this exercise once the lectures have covered the Karhunen–Loève theorem.) **[4]**

(f) In the scientific computing environment of your choice, plot some sample realisations of u (2.1) for the case when the $u_k \sim \mathcal{U}(-k^{-p}, k^{-p})$ are independently uniformly distributed and for the case when $u_k \sim \mathcal{N}(0, k^{-p})$ are independently normally distributed, given the Hilbert space

$$\mathcal{H} := \overline{\{e_k \mid k \in \mathbb{N}\}} \subset L^2([0,1], \mathrm{d}x; \mathbb{R})$$
$$e_k(x) := \sin(k\pi x).$$

Show numerical evidence for your results concerning the exponent p in parts (d–e). (Note: include your program code in your submitted solutions.) **[4]**

(g) Optional further exercise: try the analogue of parts (d–f) to generate random fields on the cube $[0,1]^d$. You should discover some interesting dependency upon the dimension, which can be explained in several ways.