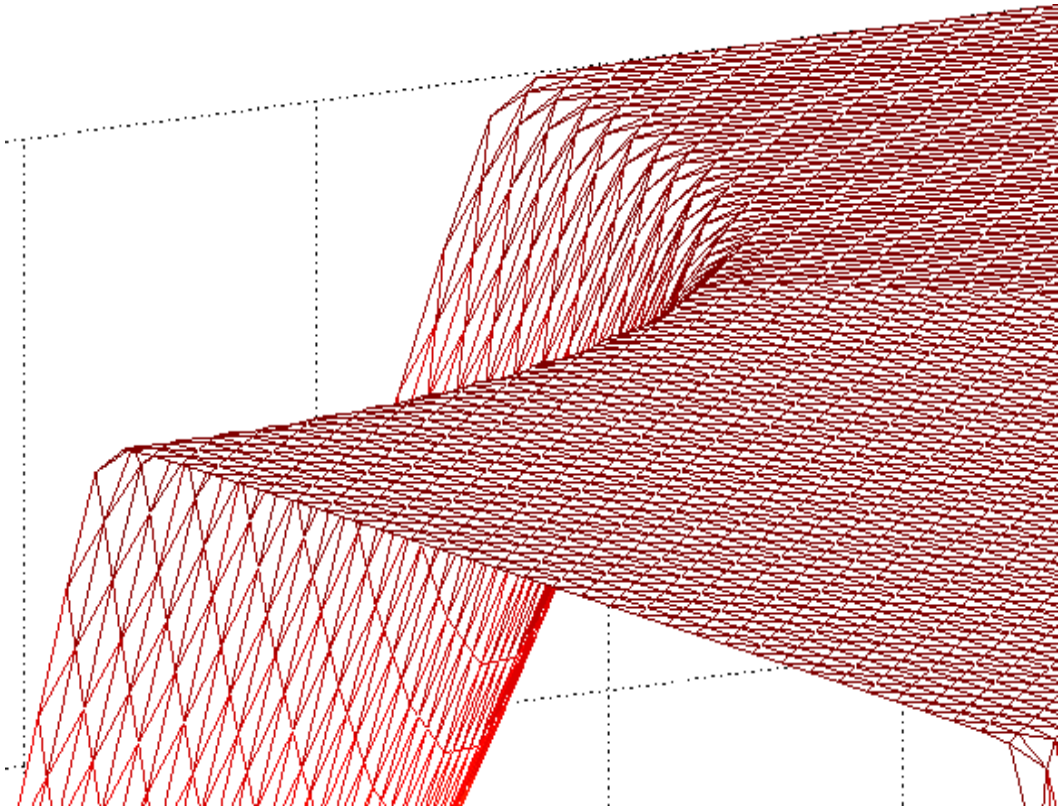

Interior Point Methods for the Numerical Solution of Optimal Control Problems

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Anton Schiela
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Optimal Control Problem

$$\min_{u\in L_2,\, y\in H_0^1} \frac{1}{2}\|y-y_d\|_{L_2}^2+\frac{\alpha}{2}\|u\|_{L_2}^2$$

subject to $\quad Ly+u=0$

$$-1\le u\le 1$$

$$Ly=\nabla\cdot(A\nabla y)+ay$$

$A:\Omega\to\mathbb{R}^{2\times 2}$ uniformly spd

$a:\Omega\to\mathbb{R}$ nonnegative

KKT conditions

$$\exists\lambda\in H_0^1,\underline{\eta},\overline{\eta}\in L^2:$$

$$y-y_d+L\lambda=0$$

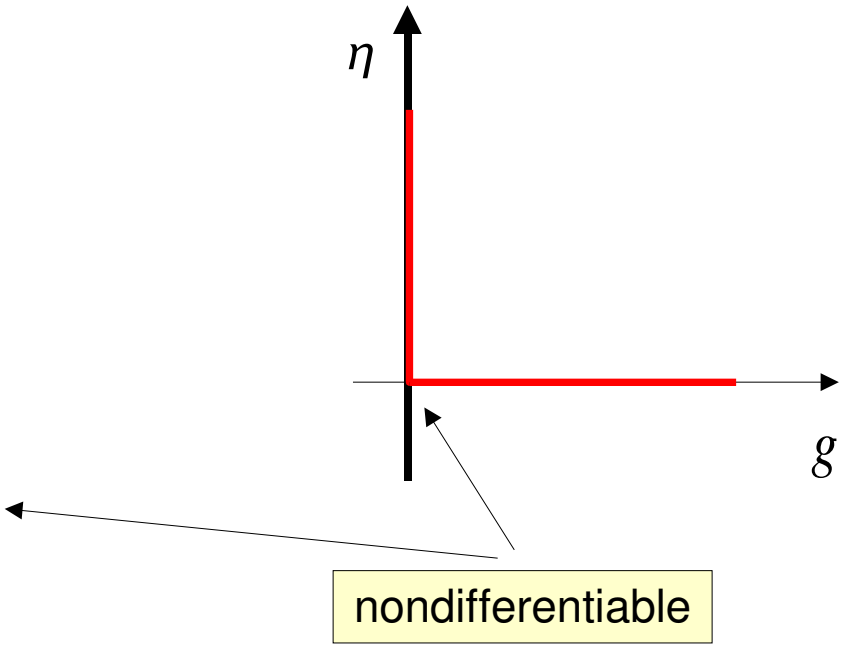
$$\alpha u+\lambda-\underline{\eta}+\overline{\eta}=0$$

$$Ly+u=0$$

$$\underline{\eta}(1+u)=0$$

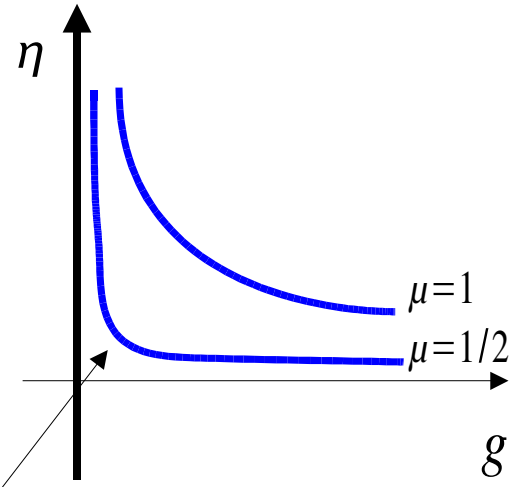
$$\overline{\eta}(1-u)=0$$

$$\underline{\eta},\overline{\eta},1+u,1-u\ge 0$$



Primal-Dual Interior Point Method

$$
\begin{aligned}
y - y_d + L\lambda &= 0 \\
\alpha u + \lambda - \underline{\eta} + \overline{\eta} &= 0 \\
Ly + u &= 0 \\
\underline{\eta}(1+u) &= \mu \\
\overline{\eta}(1-u) &= \mu \\
\underline{\eta}, \overline{\eta}, 1+u, 1-u &\geq 0
\end{aligned}
$$



differentiable for $\mu > 0$

Homotopy in $\mu \to 0$ defines the central path $v(\mu)$

$$F(v(\mu);\mu) = 0$$

Theoretical results

Existence of Central Path:

For all $\mu > 0$ there is $v(\mu)$ with $F(v(\mu); \mu) = 0$.

Convergence of Central Path:

$\mu \to v(\mu)$ is continuously differentiable with $\|v(\mu) - v(0)\|_\infty \leq \text{const}\sqrt{\mu}$

select $v_{0,}\mu_0>0,\sigma<1$
for $k=0,\ldots$

$\mu_{k+1}=\sigma\mu_k$

solve $F'(v_k;\mu_{k+1})\Delta v_k=-F(v_k;\mu_{k+1})$

$v_{k+1}=v_k+\Delta v_k$

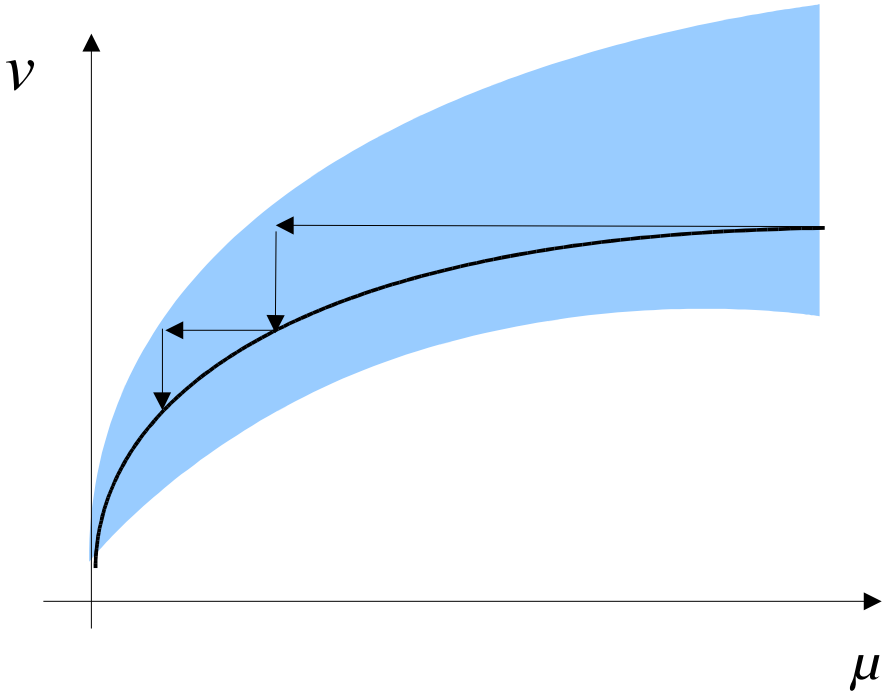
linear convergence:

$\|v_0-v(\mu_0)\|$ sufficiently small

σ sufficiently large

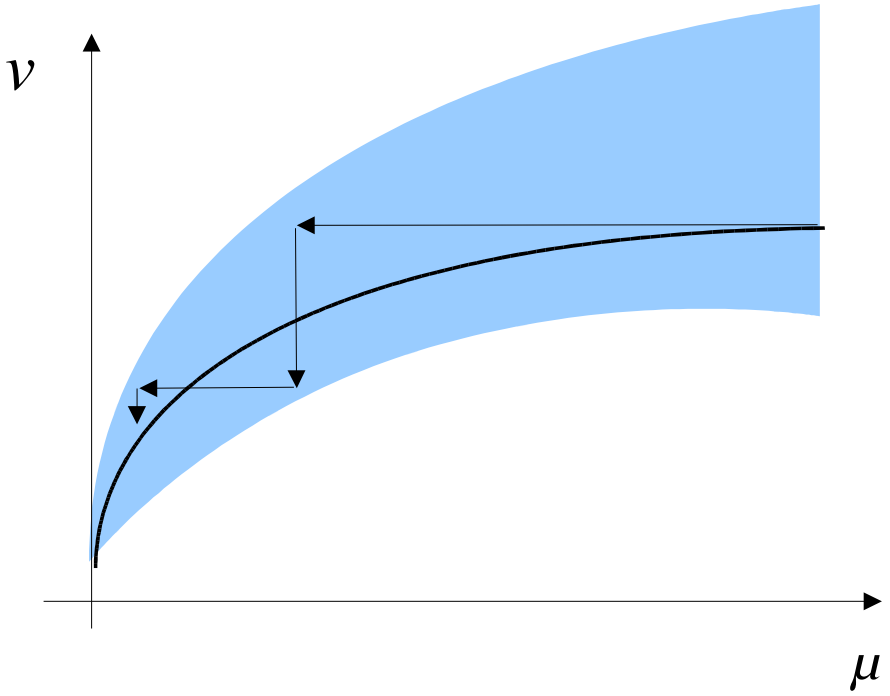
$\Rightarrow \quad \|v_k-v(0)\|_\infty\leq \text{const}\,\sigma^{k/2}$

The Central Path: Continuation



exact continuation

corrector $F'(v;\mu)\Delta v = -F(v;\mu)$



inexact continuation

corrector $F'(v;\mu)\delta v = -F(v;\mu)+r$

select $v_0, \mu_0 > 0, \sigma < 1, \delta > 0$
for $k = 0, \ldots$

$$\mu_{k+1} = \sigma \mu_k$$

solve $F'(v_k; \mu_{k+1}) \Delta v_k = -F(v_k; \mu_{k+1})$ for δv_k

with tolerance $\|\delta v_k - \Delta v_k\| \le \delta \|\Delta v_k\|$

$$v_{k+1} = v_k + \delta v_k$$

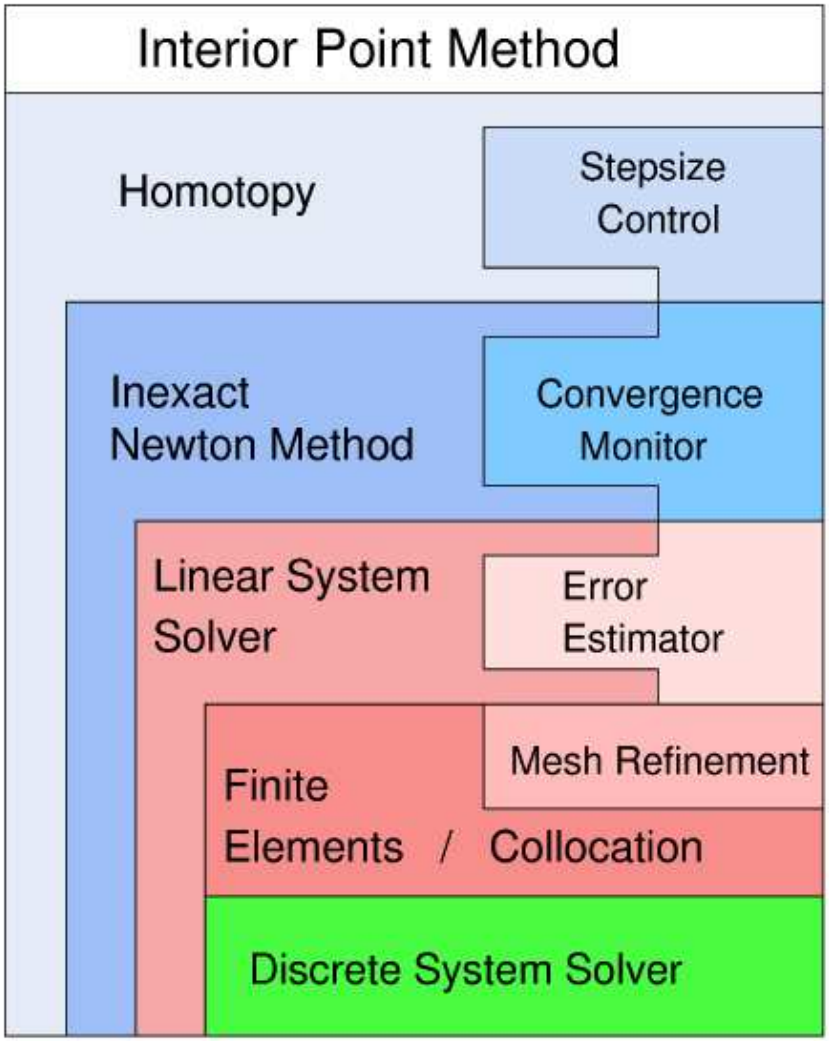
numerical ingredients
- adaptive stepsize control
- convergence monitor
- stepsize reduction
- accuracy matching
- error estimator
- mesh refinement
- nonlinearity estimators

linear convergence:

$\|v_0 - v(\mu_0)\|$ sufficiently small

σ sufficiently large $\Rightarrow$ $\|v_k - v(0)\|_\infty \le \text{const}\, \sigma^{k/2}$

δ sufficiently small



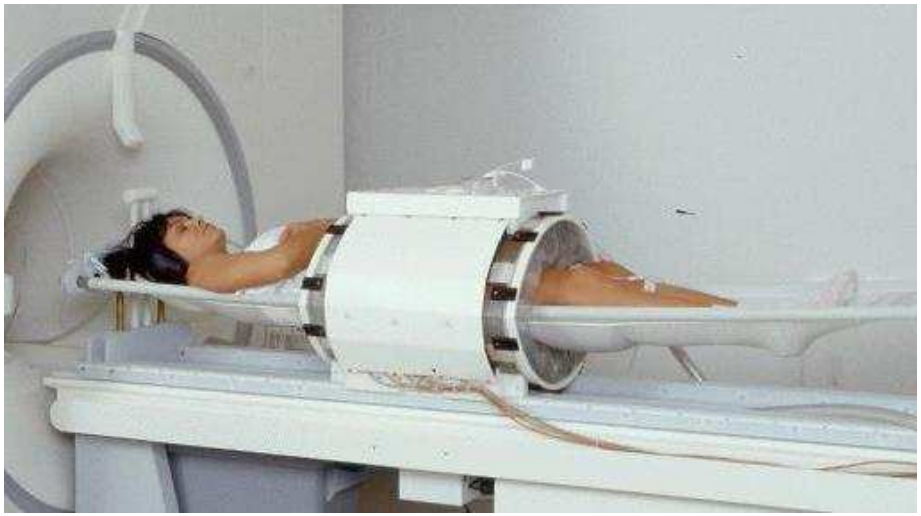
inequality constrained optimal control → sequence of nonlinear problems

nonlinear equation → sequence of linear problems

infinite dimensional linear equation → sequence of finite dimensional equations

Medicine

- cancer therapy
- tumor heating by microwaves
- temperature constraints for healthy tissue



Mathematical Modelling

- time harmonic Maxwell equations
- nonlinear Bio-Heat-Transfer-Equation
- control and state constraints

Optimization

- optimization of antenna parameters
- identification of perfusion

$$\min \int_{\text{tumor}} (T - T_t)^2 \, d\xi$$

subject to

$$-\operatorname{div}(\kappa \nabla T) + cW(T)(T - T_a) = \mathrm{SAR}(u)$$

$$T \le T_{\lim}(\xi)$$

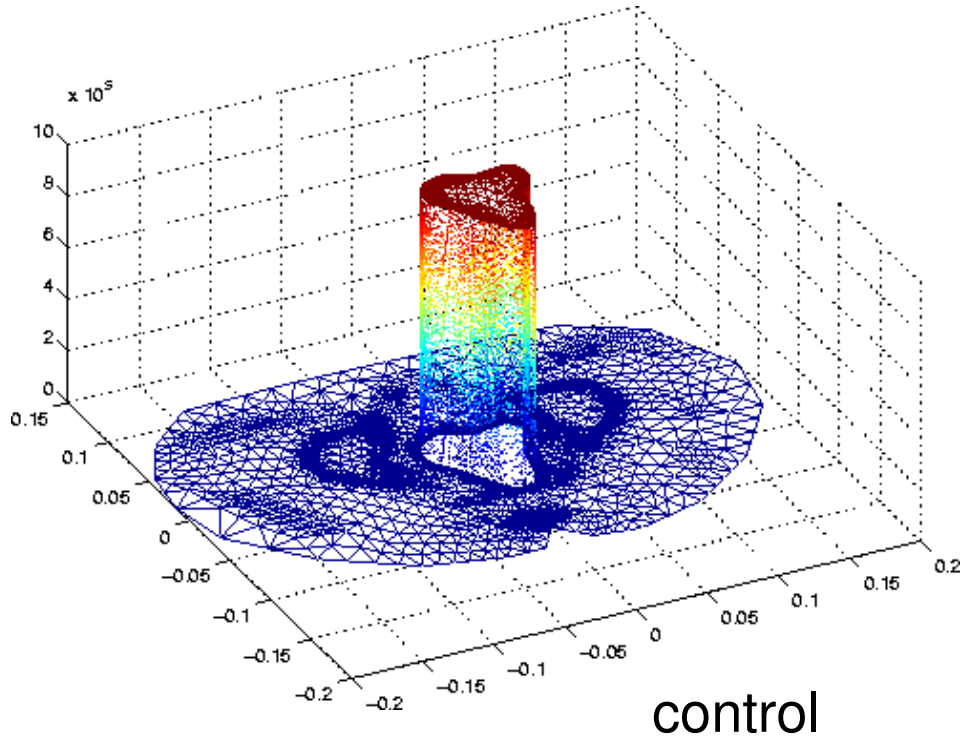
$$a \le u \le b$$

$$\min \int_\Omega (T - T_t)^2 + \alpha u^2 \, d\xi$$

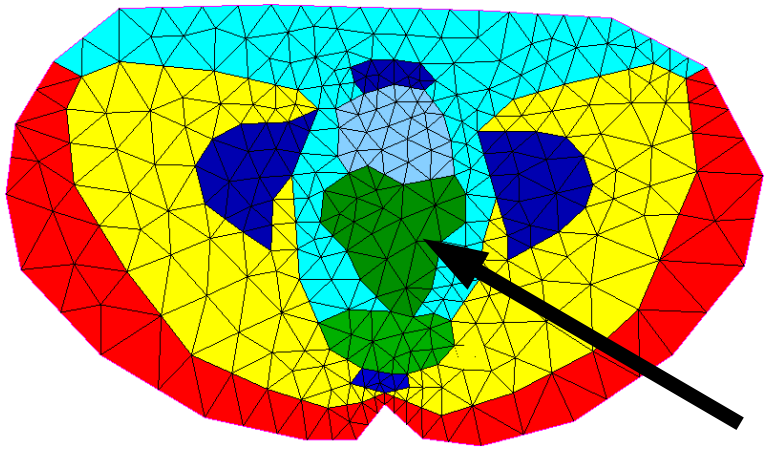
subject to

$$-\mathrm{div}(\kappa \nabla T) + cW(T - T_a) = u$$

$$a \le u \le b$$



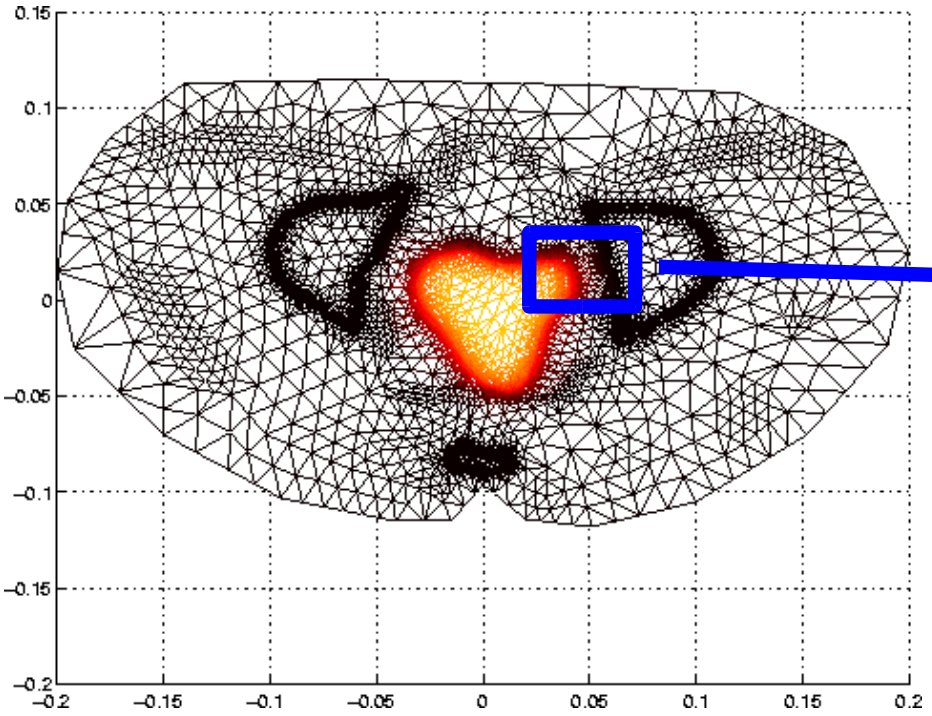
control

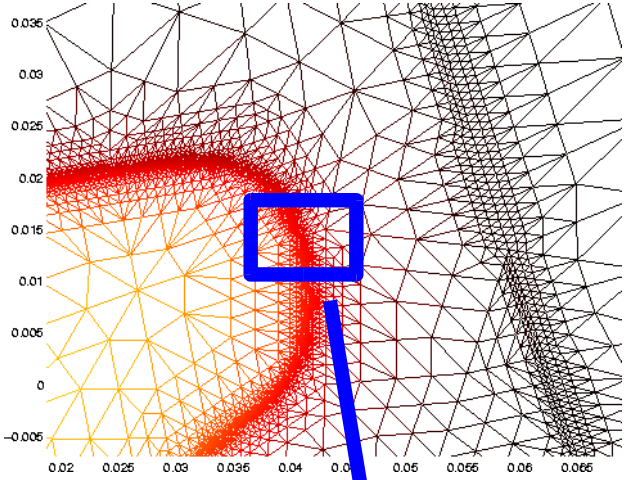


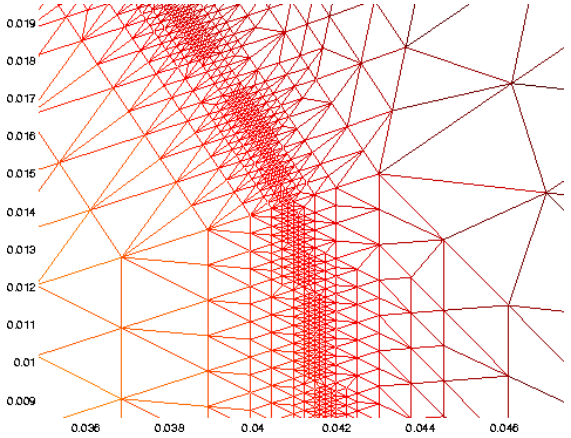
tumor

cross section of pelvic region

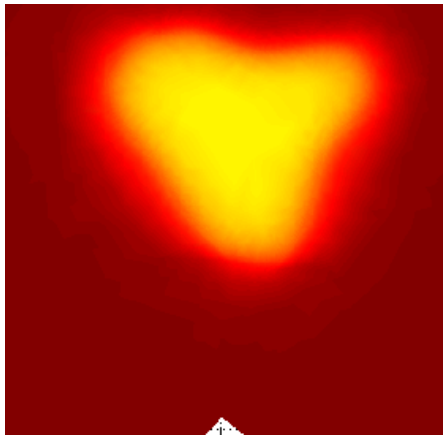
Numerical Experiment



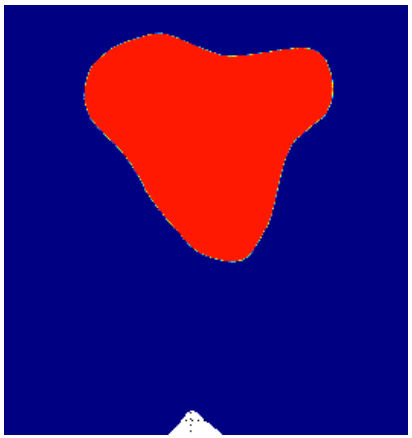




temperature



control



Primal-Dual IP

$$
\begin{aligned}
y - y_d + L\lambda &= 0 \\
\alpha u + \lambda - \underline{\eta} + \overline{\eta} &= 0 \\
Ly + u &= 0 \\
\underline{\eta}(1+u) &= \mu \\
\overline{\eta}(1-u) &= \mu \\
\underline{\eta}, \overline{\eta}, 1+u, 1-u &\geq 0
\end{aligned}
$$

Primal IP

$$
\begin{aligned}
y - y_d + L\lambda &= 0 \\
\alpha u + \lambda - \frac{\mu}{1-u} + \frac{\mu}{1+u} &= 0 \\
Ly + u &= 0 \\
1+u, 1-u &\geq 0
\end{aligned}
$$

Control Reduced Primal IP

$$
\begin{aligned}
y - y_d + L\lambda &= 0 \\
Ly + u(\lambda;\mu) &= 0
\end{aligned}
$$

$u(\lambda;\mu)$: unique solution of cubic equation

$$
\min_{u \in L_2,\, y \in H_0^1} \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_{L_2}^2 - \mu \int_\Omega (\ln(1+u) + \ln(1-u))\, d\xi
$$

subject to $\quad Ly + u = 0$

triangulation of Ω: T

piecewise polynomial functions: $P_{p,h} = \{\phi \in L^2 : \phi|_T \in P_p \; \forall T \in \mathrm{T}\}$

ansatz space: $V_h^p = P_{p,h} \cap H_0^1$

weak formulation:

$$\langle y_h - y_d + L\lambda_h, \phi \rangle = 0$$
$$\langle L y_h - P_h u(\lambda_h), \phi \rangle = 0 \qquad \forall \phi \in V_h^p$$

exact integration

numerical integration: linear projector P_h

$$\|P_h u - u\|_{L^2} \le c \|u\|_{H^2, \mathrm{T}}$$

Results

Artificial Example

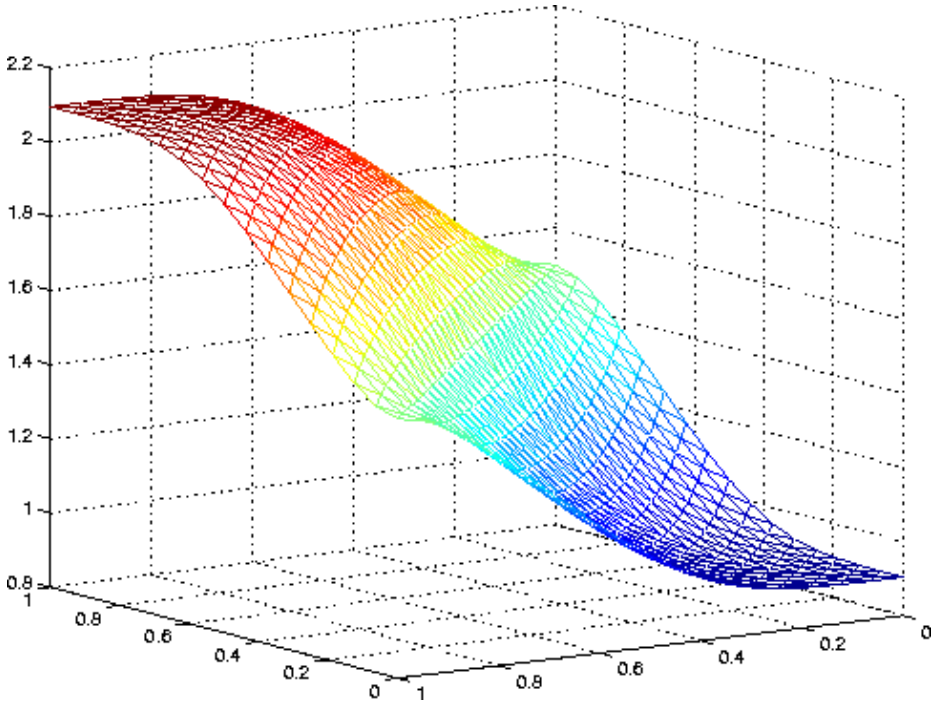
$$\min \frac{1}{2}\|y-y_d\|^2+\frac{\alpha}{2}\|u\|^2$$

$$\begin{aligned} -\Delta y &= u && \Omega \\ \partial_n y &= 0 && \partial\Omega \end{aligned}$$

$$|u|\le 6 \qquad \Omega=[0,1]^2$$

$$y_d=\begin{cases} 1, & x_1+x_2<1 \\ 2, & \text{otherwise} \end{cases}$$

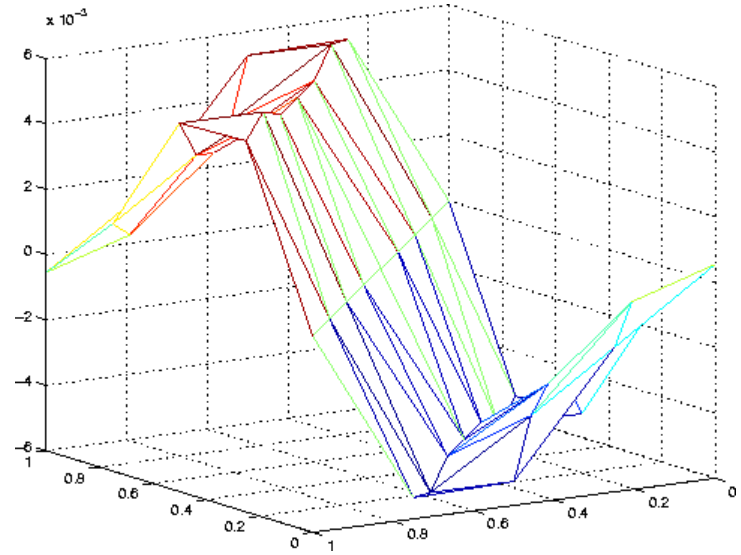
$$\alpha=5\cdot 10^{-4}$$

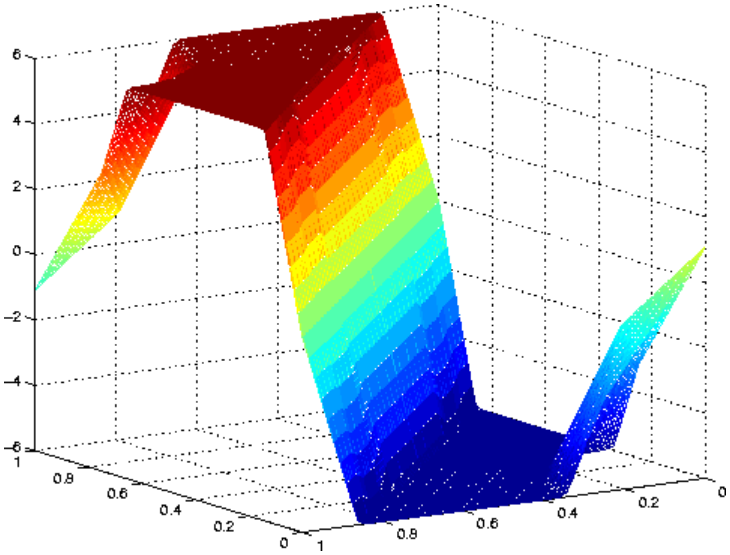


state at $\mu=2^{-18}$

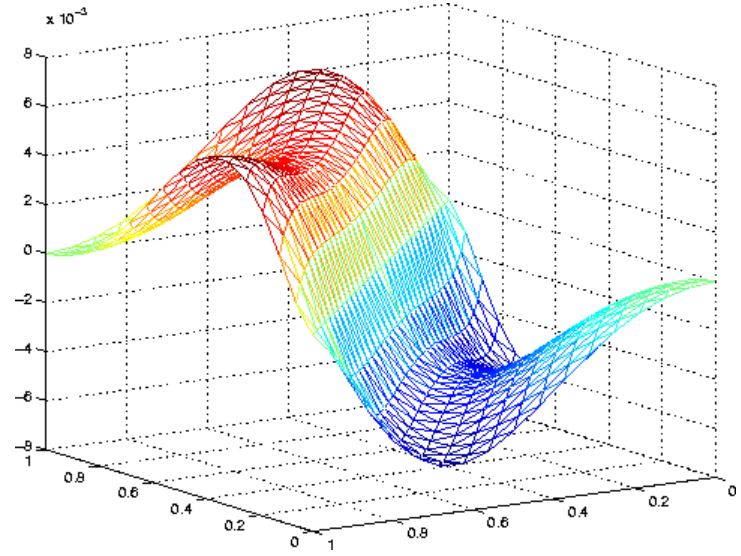
Illustrative Example

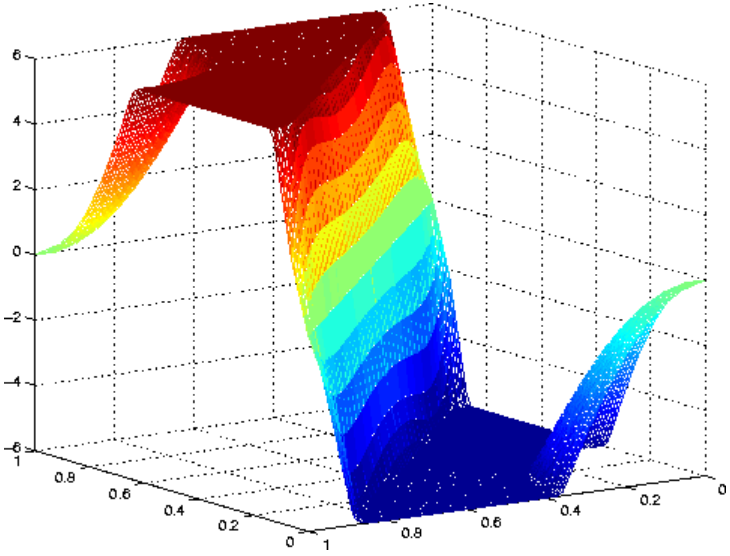
$h = \frac{1}{4}$



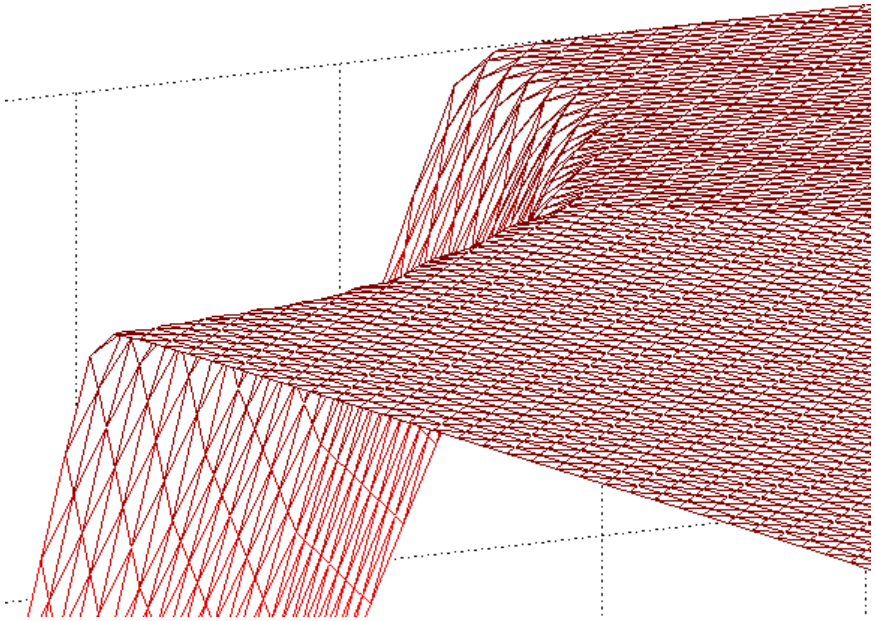


$h = \frac{1}{16}$

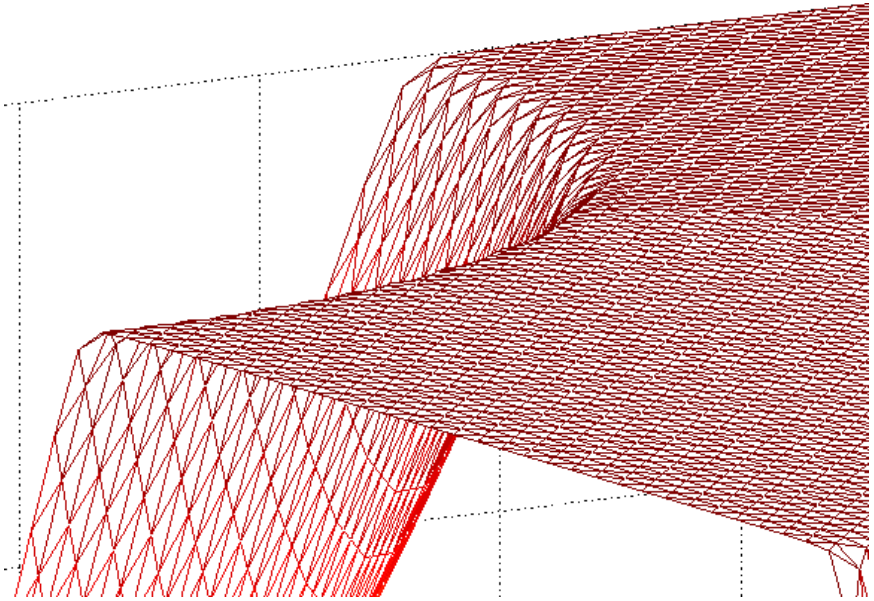




Zoom of control

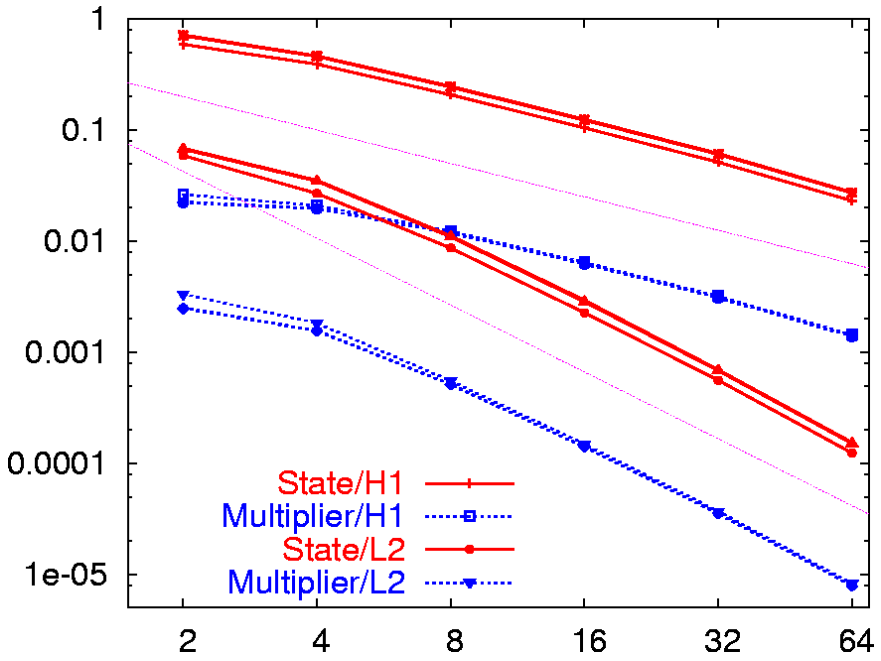


$$h = \frac{1}{4}$$

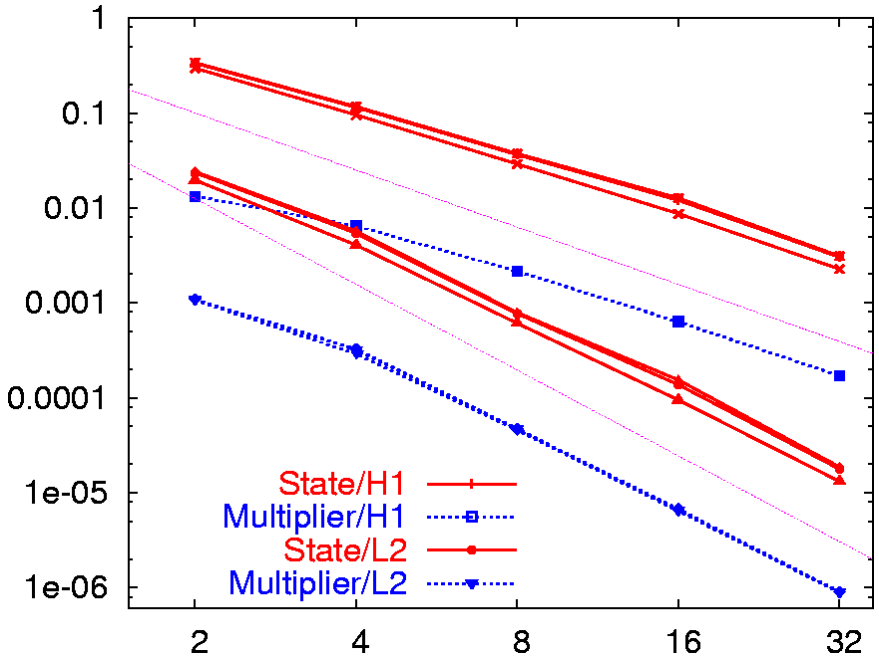


$$h = \frac{1}{16}$$

Illustrative Example



linear finite elements



quadratic finite elements