Martin Weiser

Peter Deuflhard

The Central Path towards the Numerical Solution of Optimal Control Problems

Cost functional: $J(u,y)=\int_0^1 f(u(t),y(t))\ dt$

Equality constraints: $c(u,y)=\begin{bmatrix} \dot{y}-\phi(u,y) \\ r(y(0),y(1)) \end{bmatrix}$ differential equation; boundary conditions

Inequality constraints: $g_u(u)\geq 0$ control constraints

$g_y(y)\geq 0$ state constraints

Optimization problem:

$$\min J(x)$$
$$\text{subject to} \quad c(x)=0$$
$$g(x)=0$$

$x=[u,y]$

Indirect methods:
- highly accurate solutions
- small convergence domain
- require immense amount of preparatory work (manpower)

Derive necessary conditions separately for:
- control constrained subarcs
- singular control subarcs
- state constrained subarcs of different order
- touch points
- interior point conditions

Guess switching structure
Update switching structure if incorrect

Direct methods:
- less accurate solutions
- larger convergence domain
- almost no preparatory work

Aim:
- reduce (wo)manpower required
- obtain highly accurate solutions nevertheless

KKT conditions

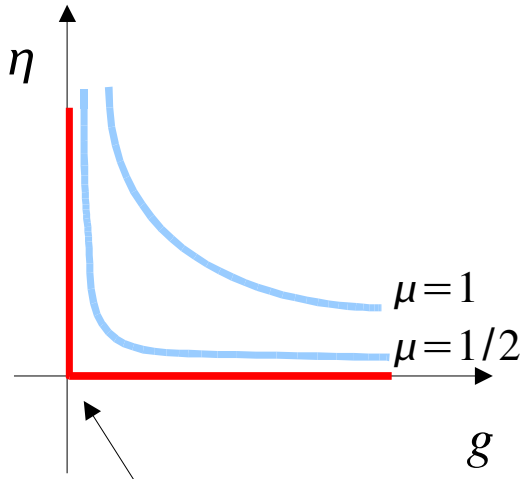
$$\exists \lambda, \eta:$$

$$J'(x) - c'(x)^* \lambda - g'(x)^* \eta = 0$$

$$-c(x) = 0$$

$$\langle \eta, g(x) \rangle = 0$$

$$\eta, g(x) \geq 0$$



nondifferentiable

regularization by complementarity formulation

$$F(x;\mu) = \begin{bmatrix} J'(x) - c'(x)^* \lambda - g'(x)^* \eta \\ -c(x) \\ \psi(\eta, g(x); \mu) \end{bmatrix}$$

with complementarity function

$$\psi(a,b;\mu) = \sqrt{a^2 + b^2 + 2\mu} - a - b$$

Homotopy in duality gap parameter

$$\mu \to 0$$

defines the central path.

The Two Norm Discrepancy

(Maurer '81)

Nonlinear constraints and complementarity function are continuously differentiable only if

$$u, y, \lambda, \eta \in L_\infty$$

But state constraints Lagrange multipliers often contain Dirac functionals!

$\longrightarrow$ convergence in L_∞ at most for purely control constrained problems

The sufficient second order condition is formulated for the time being in the weaker L_2 – norm:

$$\langle \partial_x^2 L(v^*)\xi, \xi \rangle \geq \alpha \|\xi\|_2^2 \quad \text{for} \quad \xi \in \ker c'(x^*) \cap \{\xi | \langle \lambda^*, g'(x^*)\xi \rangle \geq \beta \|\xi\|_2\}$$

where $L(v) = J(x) - \langle \lambda, c(x) \rangle - \langle \eta, g(x) \rangle$ is the Lagrange functional

The Central Path: Theory I

Let $v=[x,\lambda,\eta]$ denote all variables.

Theorem (Existence of the central path on open inverval $\mu\in(0,\mu_0]$)

Assume

- f,ϕ,r,g are $C^2(D)$
- linearized state equation is solvable
- strengthened Legendre–Clebsch condition:

$$\xi(t)^T H(t)\xi(t)\geq \text{const}\,|\xi(t)|^2 \quad \text{for} \quad \xi\in\ker c'(x)$$

with

$$H=f''(x)-\phi''(x)^T\lambda-g''(x)^T\eta + g'(x)^T\partial_\eta\psi(g,\eta)^{-1}\partial_g\psi(g,\eta)g'(x)$$

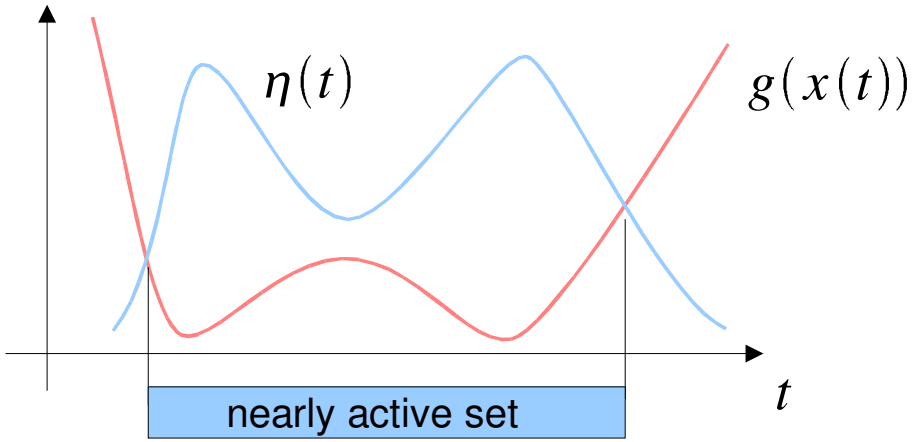
- there exists a starting point v_0 with $F(v_0;\mu_0)=0$

Then there exists a path $v(\mu)$ with $F(v(\mu);\mu)=0$

Nearly Active Constraints

Nearly active set $\Omega_\rho(x) = \{t \in [0,1] : g(x(t)) \le \rho\, \eta(t)\}$

$\Omega_\rho^c = [0,1] \setminus \Omega_\rho$



splitting of inequality constraints $g(x) = \begin{bmatrix} g_\rho(x) \\ g_\rho^c(x) \end{bmatrix} \qquad \eta = \begin{bmatrix} \eta_\rho \\ \eta_\rho^c \end{bmatrix}$

Assume $\partial_x^2 L(v)$ spd on $\ker c'(x) \cap \ker g_\rho'(x)$ ← similar to sufficient second order condition

Theorem (Existence of the central path on closed inverval $\mu \in [0, \mu_0]$)

Assume

- only control constraints
- control enters linearly
- the operator $C = \begin{bmatrix} c'(x) \\ g_\rho'(x) \end{bmatrix}$ of nearly active constraints

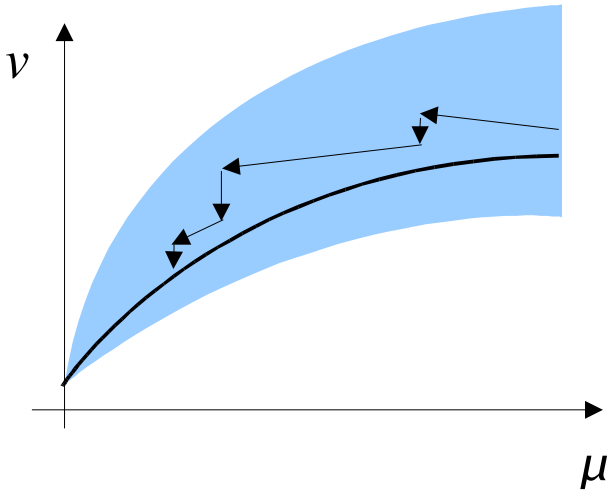
  satisfies the inf–sup–condition ← constraints qualification
- the modified Hessian $H = \partial_x^2 L + (g_\rho^{c\,\prime})^* (\partial_\eta \psi_\rho^c)^{-1} \partial_g \psi_\rho^c g_\rho^{c\,\prime}$

  is positive definite on $\ker C$
- the central path exists on $(0, \mu_0]$

Then the path can be continuously extended (w.r.t. L_2–norm) to include a KKT point $v(0)$

The Central Path: Inexact Continuation

Employ a Newton type predictor–corrector method in function space



Stepsize selection scheme (Deuflhard '78) needs extension to function space

inexact Newton corrector

$$F'(v^k)\,\delta v^k = -F(v^k) + r^k$$
$$v^{k+1} = v^k + \delta v^k$$

inexact tangential predictor

$$F'(v;\mu)\,p = -\partial_\mu F(v;\mu) + r$$

Reduction of discretization error by adaptive mesh refinement.

Theorem (Inexact Newton Method)

Lipschitz condition

Assume $\|(F'(v)-F'(w))(v-w)\|_z \le \omega \|F'(v)(v-w)\|_v^2$ for collinear v,w,z

Lipschitz continuity of local norm

$\|r\|_w \le (1+\gamma_v \|F'(v)(w-v)\|_v)\|r\|_v$ for all r,v,w

$$\frac{\omega}{2}(1+\delta_k)^2\|F(v^k)\|_{v^k}+(1+(1+\delta_k)\gamma_v\|F(v^k)\|_{v^k})\delta_k \le \Theta < 1$$

where $\delta_k = \dfrac{\|r^k\|_{v^k}}{\|F(v^k)\|_{v^k}}$ for all k

accuracy requirement for inner iteration

Then the residuals converge q–linearly to zero:

$$\|F(v^{k+1})\|_{v^{k+1}} \le \Theta \|F(v^k)\|_{v^k}$$

Termination criterion for inner iteration
requires unknown constants ω, γ

→ derive efficiently computable estimates to be inserted in accuracy matching condition

$$[\omega]_k = \frac{2}{(1+\delta_k)^2 \|F(v^k)\|_{v^k}} \left(\frac{\|F(v^{k+1})\|_{v^k}}{\|F(v^k)\|_{v^k}} - \delta_k \right) \leq \omega$$

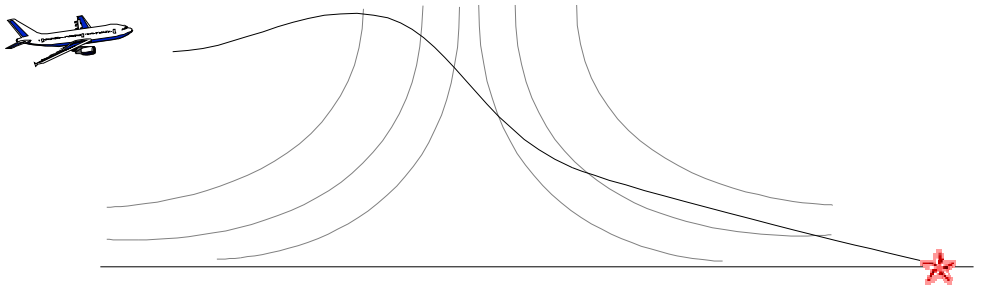
$$[\gamma_x]_k = \frac{\left| \|F(v^{k+1})\|_{v^{k+1}} - \|F(v^{k+1})\|_{v^k} \right|}{\|F(v^k)\|_{v^k}} \leq \gamma$$

For reliable estimates use problem–adapted affine invariant norm.
(Deuflhard ’79, Volkwein/Weiser ’00)

Windshear Situation

Descending column of air leads to subsequential encounter of

- headwind
- downburst
- tailwind



Low altitude windshear is a dangerous situation due to

- loss of altitude in the downburst region
- loss of lift in the tailwind region

Flight crew's typical reaction: decrease engine power and/or flight path angle to compensate additional lift in headwind region

Fatal crash possible if settings remain unchanged during downburst and tailwind

At least 30 aircraft accidents attributed to windshear
(New Orleans '82, Denver '85)

If initial altitude is sufficiently high, abort landing is safer than penetration landing

Abort Landing in Presence of Windshear: Results

Extremely difficult problem due to

- high nonlinearity
- third order state constraint
- complicated switching structure
- multiple touch points
- state constrained subarcs
- control constrained subarcs
- singular subarcs

mathematical model
due to Miele et al. ’87

Indirect method requires an immense amount of
preparatory analytical work.

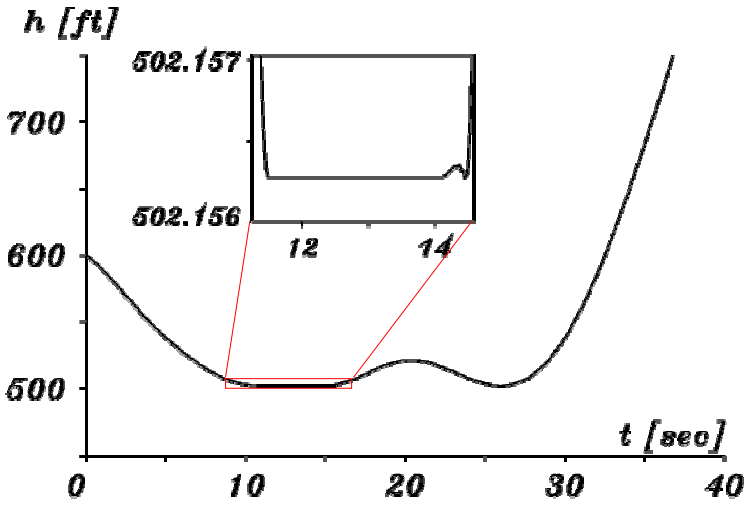
Direct function space method obtains competitive solution with
virtually no preparation.

Abort Landing in Presence of Windshear: Results

(Bulirsch/Montrone/Pesch '91)
(Weiser/Deuflhard '01)

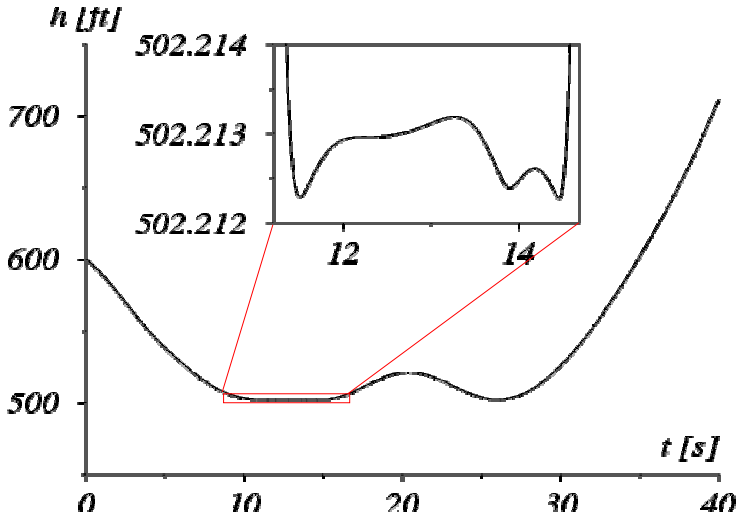
Altitude above ground vs. time.

Solution features a boundary arc and two touch points.



indirect method

After the last touch point of minimal altitude there is a continuum of optimal solutions.



direct function space method

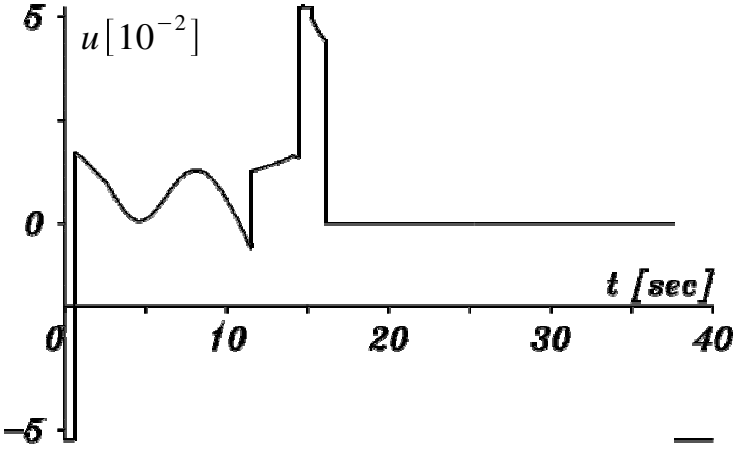
Abort Landing in Presence of Windshear: Results

(Bulirsch/Montrone/Pesch '91)
(Weiser/Deuflhard '01)

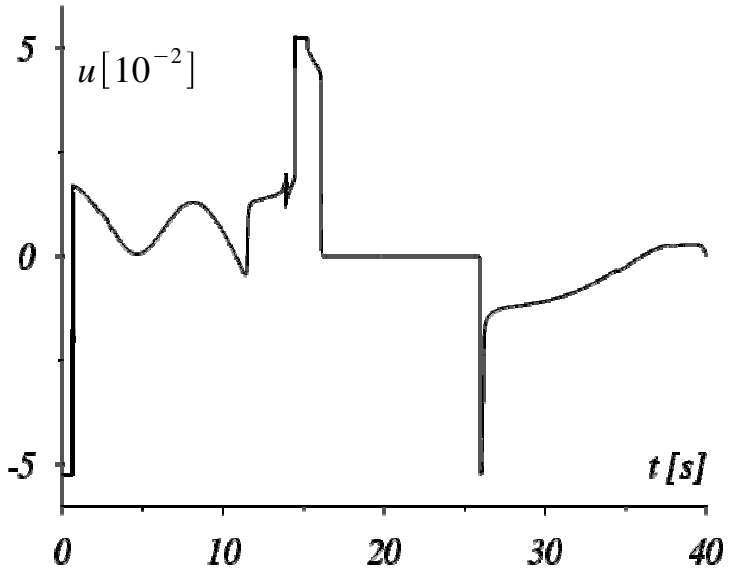
Angle of attack rate (control) vs. time.

Solution features a complicated switching structure and constrained as well as singular subarcs.

After the last touch point of minimal altitude there is a continuum of optimal solutions.



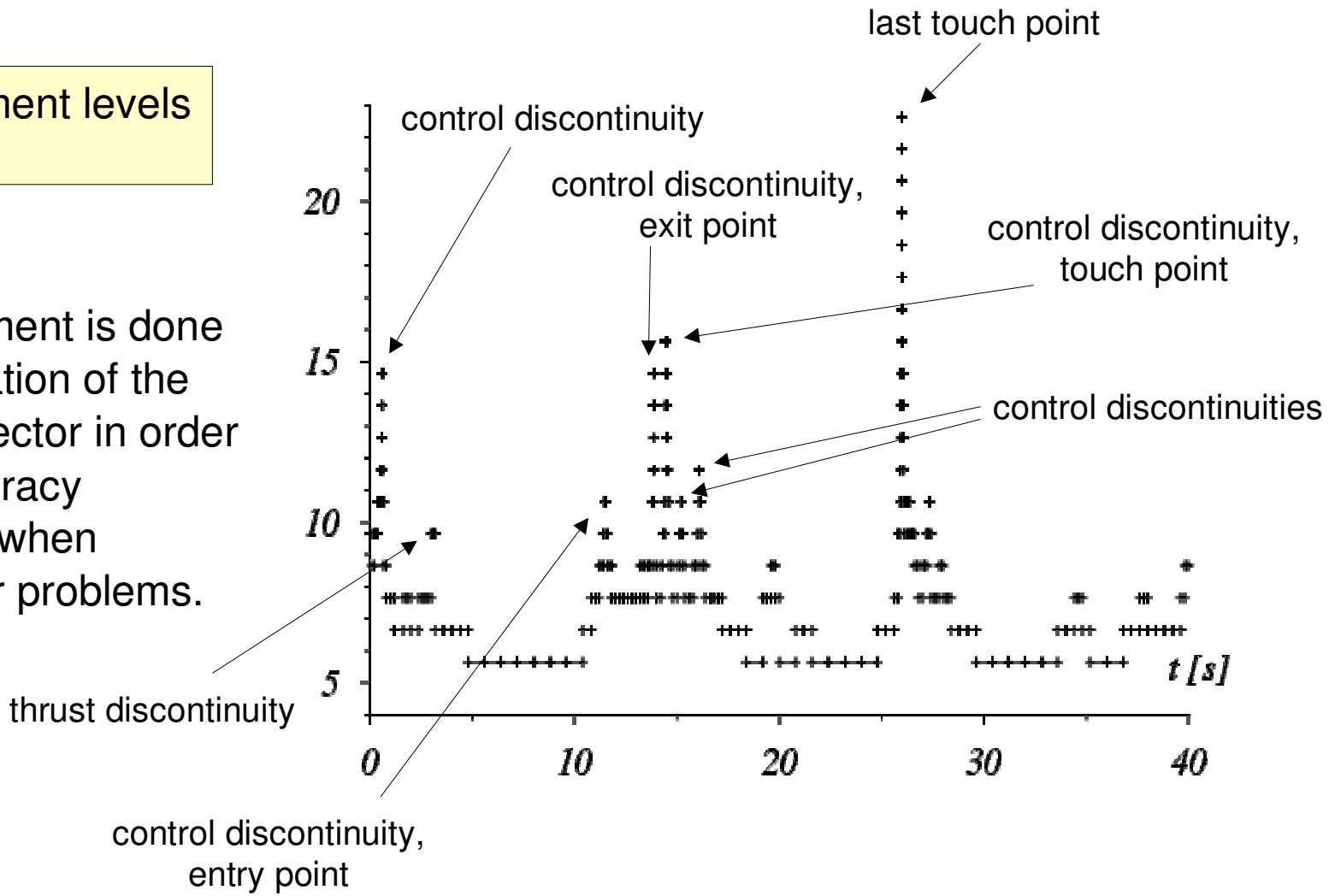
indirect method



direct function space method

Mesh refinement levels versus time.

Mesh refinement is done by inner iteration of the Newton corrector in order to meet accuracy requirement when solving linear problems.



Function Space Central Path Methods:

- substantial reduction of (wo)manpower required for solving intricate optimal control problems to high accuracy
- sound theory only for limited situations
- numerical implementation currently not (yet) competitive

Thank You!

... for your attention!