Exercise 12: Implementing the Lin-Kernighan heuristic for the TSP

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Outline

1. The Traveling Salesman Problem

2. The Lin-Kernighan heuristic
The Traveling Salesman Problem

Given
- Complete undirected graph $G = (V, E)$
- Metric edge costs $c_e \geq 0$ for all $e \in E$.

Problem
- Find a Hamiltonian cycle with minimal cost.
Lin-Kernighan (LK)

- **Most famous and best** local search approach for the sym. TSP.
- Best **exact** solver for the TSP is **Concorde** (Applegate, Bixby, Chvátal, Cook).
- Best LK-Code today: Keld Helsgaun.
- **Concorde + Code of Helsgaun, 2006:** p1a85900 solved (**world record**). (Applegate, Bixby, Chvátal, Cook, Espinoza, Goycoolea, Helsgaun)
**k-opts**

- *k-opt neighborhood for tour \( x \): \( \mathcal{N}_k(x) \) consists of all tours, which can be constructed from \( x \) by deleting and adding \( k \) edges.*
$k$-opts

- $k$-opt neighborhood for tour $x$: $\mathcal{N}_k(x)$ consists of all tours, which can be constructed from $x$ by deleting and adding $k$ edges.

Observations

- Two hamiltonian cycles only differ in $k$ edges ($2 \leq k \leq n$), i.e.: $x_{opt} \in \mathcal{N}_k(x)$ for every $x$.
- **Problem 1:** $k$-optimality in $\mathcal{N}_k$ can only be tested in $O(n^k)$.
- **Problem 2:** $k$ is unknown.
- **Approach:** Choose an **efficient searchable** neighborhood such that $k$ can be chosen dynamically.

⇒ Sequential $k$-opt moves.

**Definition:** Sequential $k$-opt move

A $k$-opt move is called sequential if it can be described by a path alternating between deleted and added edges.
k-opts

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$\Rightarrow$ **Sequential $k$-opt moves.**
k opts

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  $\implies$ **Sequential** $k$-opt moves.

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Example

Sequential 6-opt

Double-Bridge-Move (4-opt)
Lin-Kernighan

Flip operations

Operation: flip(next(a), prev(b))

Gain $g_t$ of flip $t$:

$$g_t = c(a, \text{next}(a)) + c(\text{prev}(b), b) - c(\text{next}(a), b) - c(a, \text{prev}(b))$$
Lin-Kernighan

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Lin-Kernighan

- Choose a fix starting node $a$ and construct the alternating path by a sequence of flip operations of the form $\text{flip}(\text{next}(a), \text{prev}(b))$.
- The goal within this construction is to obtain

$$\sum_{i=1}^{k} g_{t_i} > 0.$$
Example

\[ v_1 v_2 - v_8 v_7 - v_4 v_3 - v_{10} v_9 - v_{12} v_{11} - v_6 v_5 - \]

\[ \text{flip}(v_2, v_4) \]
Example

\[ v_1 v_4 - v_7 v_8 - v_2 v_3 - v_10 v_9 - v_{12} v_{11} - v_6 v_5 - \]

\(\text{flip}(v_4, v_6)\)
Example

\[ v_1 v_6 - v_{11} \ v_{12} - v_9 \ v_{10} - v_3 \ v_2 - v_8 \ v_7 - v_4 \ v_5 - \]

\[ \text{flip}(v_6, v_8) \]
Example

\[ v_1 v_8 - v_2 v_3 - v_{10} v_9 - v_{12} v_{11} - v_6 v_7 - v_4 v_5 - \]

\[ \text{flip}(v_8, v_{10}) \]
Example

\[v_1 v_{10} - v_3 v_2 - v_8 v_9 - v_{12} v_{11} - v_6 v_7 - v_4 v_5 -\]

\[\text{flip}(v_{10}, v_{12})\]
Example

\[v_1 v_{12} - v_9 v_8 - v_2 v_3 - v_{10} v_{11} - v_6 v_7 - v_4 v_5 -\]
Example
Implementation details

- Backtracking
- Neighborhood graph
  - $k$-Nearest graph
  - $\alpha$-Nearest graph
  - Delaunay triangulation
- Mak-Morton-Moves
- alternateStep()
- Swaps
- Bentley-Marking-Scheme
- Kicking-Strategic
  - Double-Bridge-Moves
  - ...

Neighborhood graphs for the choice of the edge \( \{a, \text{next}(a)\} \)

Figure: Neighborhood graph: 15 nearest neighbors
Neighborhood graphs for the choice of the edge \( \{a, \text{next}(a)\} \)

Figure: Neighborhood graph: Delaunay triangulation