Freie Universität Berlin
FB Mathematik und Informatik

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Network Design and Operation (WS 2015)

Excercise Sheet 9

Submission: Mo, 04. January 2016, tutorial session

Exercise 1. **4+1+4+1 Points**

Let $G = (U \cup V, E)$ be a bipartite graph and $M \subseteq E$ a matching. Show that M is maximal (w.r.t. inclusion) if and only if there is no augmenting path in G if and only if there is no augmenting path in any maximal forest of alternating trees in G.

Exercise 2. **10 Points**

Show that the Hungarian method has a running time of $O(|V|^3)$.

Exercise 3. **7+3 Points**

Let $f : \mathbb{R}^m \mapsto \mathbb{R}$ be a concave function that is differentiable in $\lambda_0 \in \mathbb{R}^m$. Show that

$$\partial f(\lambda_0) = \{f'(\lambda_0)\}.$$

Exercise 4. **10 Points**

Show that the Lagrangean relaxation of the optimization problem

$$\text{(P)} \quad \min c^{\mathrm{T}}x,\ Dx \geq d, x \in X$$

is reasonably defined as

$$\max_{\lambda \geq 0} \min c^{\mathrm{T}}x - \lambda T(Dx - d), x \in X.$$

Hint: Introduce slacks variables.

Exercise 5. **Tutorial Session**

Consider the Dutch intercity network in Fig.1. Data for this network is given in the following files:

- `edges.dat` edges of the graph. They are directed and the data contains forward and backward directions (useful for `Zimpl`).
- `times.dat` travel times for each edge (useful for `Zimpl`).
- `costs.dat` costs each edge (useful for `Zimpl`).

a) Formulate the shortest path problem as an integer program $\min c^{\mathrm{T}}x, x \in X$.
b) Use `Zimpl` and `SCIP` to compute a quickest path from Apeldoorn to Rotterdam in the Dutch network (use `times.dat` as the objective).

c) Use `Zimpl` and `SCIP` to compute a cheapest path from Apeldoorn to Rotterdam in the Dutch network (use `cost.dat` as objective).
d) Upgrade your model to solve constrained shortest path problems of the form $\min c^{\mathrm{T}}x, w^{\mathrm{T}}x \leq \omega, x \in X$, namely, to compute a quickest path subject to a budget constraint.
e) Compute the quickest path from Apeldoorn to Rotterdam with a maximal cost of 15.000, (i.e., values in `times.dat` are the objective coefficients and values in `cost.dat` the weight coefficients).
f) Solve the LP-relaxation of your constrained shortest path model.
g) Formulate the Lagrangean relaxation $\max_{\lambda \geq 0} \min_{x \in X} c^{\mathrm{T}}x - \lambda(\beta - w^{\mathrm{T}}x)$ of the constrained shortest path problem w.r.t. the budget constraint.
h) Let $X = \{x_1, \ldots, x_k\}$ be the set of paths from Apeldoorn to Rotterdam. Solve the Lagrangean relaxation of the constrained shortest path problem by adding paths to X one at a time in the following way.
   i) Start with $X' \subseteq X$ containing the cheapest path only.
   ii) Compute $\lambda' := \operatorname{argmax} f'(\lambda) := \operatorname{argmax}_{\lambda \geq 0} \min_{x \in X'} c^{\mathrm{T}}x - \lambda(\omega - w^{\mathrm{T}}x)$.
   iii) Check if $f'(\lambda') = f(\lambda')$ by computing $f(\lambda') = \min_{x \in X} c^{\mathrm{T}}x - \lambda'(\omega - w^{\mathrm{T}}x)$.
   iv) If $f'(\lambda') = f(\lambda')$, stop.
   v) Otherwise, add $\operatorname{argmin}_{x \in X} c^{\mathrm{T}}x - \lambda'(\omega - w^{\mathrm{T}}x)$ to X' and repeat.

   This procedure is known as Kelly's cutting plane method.

Ah	Arnhem	Lls	Lelystad Centrum
Apd	Apeldoorn	Lw	Leeuwarden
Asd	Amsterdam CS	Mt	Maastricht
Asdz	Amsterdam Zuid WTC	Odzg	Oldenzaal Grens
Asn	Assen	Rsdg	Rosendaal Grens
Bd	Breda	Rtd	Rotterdam CS
Ehv	Eindhoven	Shl	Schiphol
Gn	Groningen	Std	Sittard
Gv	Den Haag HS	Ut	Utrecht CS
Gvc	Den Haag CS	Zl	Zwolle
Hgl	Hengelo	Zvg	Zevenaar Grens
Hr	Heerenveen		

Table 1: Station names and abbreviations in the Dutch high-speed railway network.

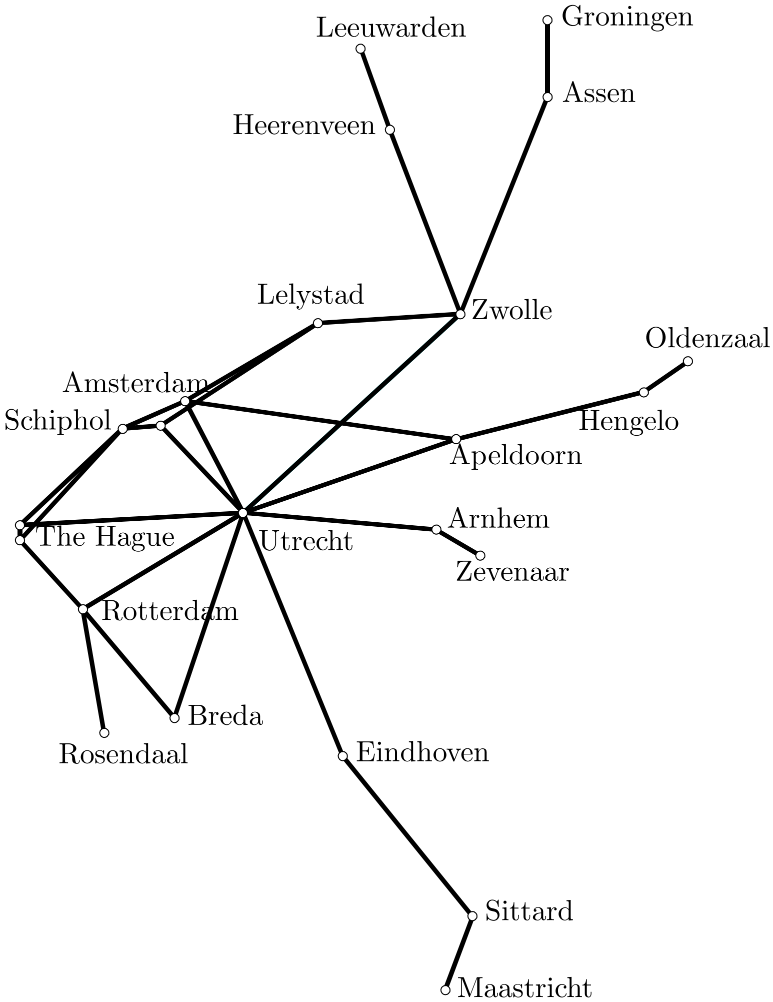

Figure 1: The Dutch high-speed railway network.