COMMENT ON 'ANGLES ARE INHERENTLY NEITHER LENGTH RATIOS NOR DIMENSIONLESS'

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Abstract. We address the issue of angular measure, which is a contested issue for the International System of Units (SI), from a mathematical point of view. Our construction, rooted in the traditional way of measuring a plane angle subtended by a circular arc, shows that angles are intrinsically mathematical objects, and so are the methods of assigning measures to them, a point that has yet to be discussed in the metrology community. As such, angular measure is a quantity of dimension number.

The paper by Quincey, Mohr, and Phillips (*Metrologia* 2019) argues for the opposite viewpoint. In our comment, we counter these conclusions by considering the problem from a *mathematical* viewpoint.

We distinguish between the *angular magnitude*, defined in terms of congruent angles, and the (numerical) *angular measure* be assigned to congruent angles. The mathematical approach is the traditional way of measuring a plane angle subtended by a circular arc as the quotient of the lengths of the arc and its radius, a pure number. We argue that angles mathematically are intrinsically different from line segments, as there are angles of special significance (such as the right angle, or the straight angle), while there is no distinguished line segment. This is further underlined by the observation that, while units such as the meter and kilogram have been refined over time due to advances in metrology, no such refinement of the radian is conceivable, as it is a mathematically constructed quantity, independent of the physical constants employed when defining the units in SI. We conclude that the angular measure is a pure number and thus its unit is the number one, i.e., "radian" is just an alias for the number one. Therefore, since the number one is the neutral element of any multiplicative group, the "radian" cannot be a base unit in any system of units.

1. Background

There has been a long discussion within the metrology community regarding angular measures, see, e.g., [1, 3, 4, 6, 7, 9, 11, 5, 10, 8], and references therein. Should they be considered as quantities of dimension number, or should the radian even become another (eighth) base unit? Here we argue from a mathematical point of view against adopting the radian as a base unit in the SI. In particular, we seek to counter the arguments put forward by Quincey et al. in [11].

Our argument is based on insights from the study of classical Euclidean geometry, as rendered in modern axiomatic form by Hilbert. We provide a more detailed account in [2]; here we take a somewhat different approach, glossing over all technical details. We concentrate on *plane* geometry, as that is all that is needed to make the points we wish to make, while remaining technically much easier than working in three dimensions.

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First thing to notice, is that Euclidean geometry has no concept of measuring lengths, nor ways to measure angles. It does, however, have the concept of *congruence*: Congruent line segments have by definition the same length, and, similarly, congruent angles have the same angular measure.

As congruence is an equivalence relation, both in the case of line segments and of angles, we can turn this observation into definitions: A *linear magnitude* is a congruence class of line segments, while a *angular magnitude* is a congruence class of angles.

Here we take a moment to discuss terminology. In everyday language, the term *angle* is ambiguous. For the purpose of the present discussion, however, we need to use different names for the different meanings of the word. We will reserve the word *angle* itself for the concept defined in geometry, namely, the union of two distinct, but non-opposite, rays with a common end point (the apex of the angle). Since we often do not wish to distinguish between different but congruent angles, we use *angular magnitude* as described in the previous paragraph. And finally, we use *angular measure* to refer to the measure of an angle as a real number. Because angular measure must be the same for congruent angles, it is really a function of angular magnitude. All three concepts may be called "angle" when a higher level of precision is not called for.

For mathematical quantities it is indispensable to define how one can operate with them. By the simple expedient of placing line segments end to end, we define addition of linear magnitudes. In particular, we define any positive integer multiple of a given length by repeated addition, and we can also divide any length by a positive integer. Thus we can define any positive *rational* multiple of a given length, and finally, by a limit procedure, any positive *real* multiple of a given length. We write tx when t is a positive real number and x is a linear magnitude. Given two linear magnitudes x and y, there is a unique t so that $x = ty$, and we write $t = x/y$. Thus we define the *ratio* of two linear magnitudes: We can use one to measure the other.

If needed, we choose a particular line segment as unit length, and use it to measure all other lengths. This unit length is completely arbitrary however: Nothing in the axioms of geometry favours any line segment over another.

Similarly, we can define the sum of angular magnitudes. Angles, as defined here, are all smaller than a straight angle, so we cannot add angles if their sum is too large. Or to put it more accurately: Two angles can be added if and only if either is smaller than the supplement of the other. (Extending this to angles of arbitrary size is fairly simple, but doing so is not necessary for our current purposes.) Again, we can define integer multiples of any angle (with the constraint already mentioned), and we can divide an angle by repeated bisection. Thus we can define $t\alpha$, where α is a angular magnitude and t is a (not too large) dyadic rational, and again, a limiting procedure allows us to extend this to real t, i.e., to define the ratio of two angular magnitudes as a real number.

Up to this point linear magnitudes and angles have been treated similarly. However, there exist special angles, while there are no special lengths. A prime example is the right angle, defined as the unique angle that is congruent to its own supplement.

Furthermore, consider scaling transformations: A homothety is a scaling transformation such that a homothetic image of a geometric figure is similar to the original, but with all lengths multiplied by the same constant. However, it is impossible to scale a triangle such that all angles become twice those of the original.

Since scaling changes lengths but not angular measures, to choose a unit is mandatory for length measurement, but is not necessary for angular measurement.

Moreover, since angular measures are pure numbers, their unit is the number one in any system of units.

Despite the differences outlined above, angular magnitudes and linear magnitudes are closely interrelated. Two similar triangles have the same angles, which can be characterized in terms of length ratios.

Given the above discussion about the nature of angles, we have to be more precise: To each angular magnitude α we associate a number, called its angular measure, $\vartheta(\alpha) = s/r$, with s being the linear magnitude of the arc and r the linear magnitude of the radius of an arc subtended by a concrete representative of α.

The common conflation of identifying $\vartheta(\alpha)$ and α is the main source of much confusion regarding angular measure. In practical computations and measurements, this does not cause any problems, but they are conceptually different. In the present letter, we focus exclusively on the mathematical aspects of angular measures, and not on the practical, accurate measurement of angles. Our goal is to participate from a mathematical point of view in the ongoing discussion regarding a base unit for angular measures. In light of our findings, we conclude that the angular measure is a pure number and thus its unit is the number one, i.e. "radian" is just an alias for the number one, as it is stated already in the SI. Therefore, since the number one is the neutral element of any multiplicative group, the "radian" cannot be a base unit in any system of units (for details see [4]).

2. The paper by Quincey et al.

The discussion of angular measure in the metrology community has been intense, with more than 20 papers devoted to the issue in *Metrologia* only. In our opinion, part of the reason for this extensive discussion is the mixing of physical and mathematical concepts of angles and their measure. Our point of view is that *angles are intrinsically mathematical objects*, and so is the method of assigning a measure to them. The procedures of performing precise angular measurements are a different matter, outside the scope of this comment.

In the letter by Quincey et al. [11], the authors argue that angles are not inherently length ratios, nor dimensionless. They list four "misconceptions that arise in the discussion of angles". We agree that the first two are at least somewhat misleading, disagree regarding the third, and express no opinion on the fourth.

We address the first two together: *"Misconception: mathematics tells us that angle is a length ratio since $s = r\theta$"*, and *"Misconception: an angle is defined as the ratio of an arc length to a radius"*. Obviously, by "angle", the authors mean what we call angular measure. Indeed, the first misconception relies on circular reasoning. But for sure, mathematics *does* tell us that the angular measure as we have defined it here, i.e., measuring angles in terms of radians, is the most natural way to measure angles, not only because it leads to the equation $s = r\theta$ with no extra constant of proportionality, but also because of the related fact that, when the trigonometric functions are expressed in terms of radians, we have the derivatives $\sin' x = \cos x$ and $\cos' x = -\sin x$, again with no constant of proportionality needed. These equations express a fundamental truth about the group of rotations, and hence an extra constant in these equations would be most unfortunate. However, the choice of measuring angles in radians is merely a convention, and like any convention, it is not subject to mathematical proof. Our point is rather that angles and angular measure are purely *mathematical* objects, which are applied to the physical world. As such, angular measure can only be reasonably expressed as pure numbers. Moreover, mathematics does indeed tell us that angular measure depends in fundamental ways on length ratios.

We would put it even more strongly, and claim that angles *cannot even be understood* without reference to length ratios.

When we identify angles of the same measure by translating and rotating one of them until it coincides with the other, we fundamentally require that length ratios do not change in this process.

The third "misconception" is this: *Angle measurement requires length measurement*. We believe our discussion above disposes of this argument. It is certainly true that one can use a compass and straightedge to repeatedly bisect a right angle, thereby essentially creating a protractor with which one can measure an arbitrary angle to any desired precision. However, the defining characteristic of a compass is its ability to keep the distance between its points constant as one moves it about: It is a distance measuring device, albeit a rather limited one. The same is true of a physical protractor. To be useful, the protractor must be stiff, which by definition means that the distance between any two points of the protractor does not change as we move the protractor around.

We are of course aware that compasses and protractors are not very precise instruments for the measurement of angles. We would have no reason to deal with them here, except in response to their mention in [11]. However, any high precision optical device used to measure angles is in principle not different, since it relies on a precise knowledge of distances between the optical components of the instrument. Moreover, even a cursory glance reveals that physical rigidity is of paramount importance for the accurate operation of such devices.

3. Conclusion

Let us look at the previous discussion from a general point of view. Nobody will question that points and lines are geometric objects. Angles, introduced as the union of two rays with a common apex, are geometric objects as well. There is a practical and theoretical need to associate numerical measures with geometric objects. For lines, the common way is to choose a length unit (such as meter) with which, for every line, the length of a line segment can be measured.

At first glance, it seems reasonable to treat angles in exactly the same way, but due to the lack of scaling groups, this is not necessary. We have shown [2] that the very identification of angles of the same measure, and hence the notion of angular magnitudes, relies crucially on the concept of length. However, since length units have no influence on angular measures, we must conclude that angular measure have to be considered as a function of length ratios associated with pairs of linear segments.

To each angular magnitude α we have assigned an *angular measure* $\vartheta(\alpha)$, for which we can write in the conventional manner $\vartheta(\alpha) = s/r$. We define the radian as the angular measure of the angular magnitude α for which $\vartheta(\alpha) = 1$.

Note that the conventional notation $\alpha = s/r$ is, strictly speaking, a category error, since a magnitude is not a number. It is, however, quite common to conflate the two concepts, i.e., not to distinguish between α and $\vartheta(\alpha)$. This is one of the main reasons of misunderstandings. In the vast majority of cases this is harmless, so long as the context makes the meaning clear.

Also the statement expressing the angular unit 'rad' by the quotient m/m appears in the SI brochure. While this seems to be natural from a practical point of view, it does conflate the angular magnitude and its numerical representation. However, our main point here is that mathematically, one cannot justify the addition of 'rad' as a base unit in the SI system, but to continue to use the "radian" as an alias for the number one, as it is correctly done for a long time in the SI. The "radian" was introduced in 1873 by James Thomson, brother of Lord Kelvin, in order to express that the "circular measure" of angle is meant. It was never intended to be the name of a unit, but rather to simplify the communication.

When dealing with trigonometric functions, a commonly used expression such as $\sin\alpha$ would be a category error, the correct expression being instead $\sin\vartheta(\alpha)$.

We introduced the notion of angular magnitude and the meaning of ϑ only for the purpose of the present discussion. However, requiring scientists and engineers to maintain the distinction between angular magnitudes and their measure in radians would impose an undue burden on them.

At this point, we wish to make a point regarding the fundamental nature of angles versus lengths and other physical quantities. Since the meter was introduced in 1793, improvements in the science of metrology has vastly increased our ability to measure lengths accurately, in turn leading to the need to refine the very definition of the meter in order to keep up with the technology. No such claim can be made for angles. In fact, even though we can certainly measure angles much more accurately today than we could three centuries ago, no conceivable technological advance can lead to a need to refine the definition of the radian, or a right angle. This simple observation supports the notion that angle is a *mathematical* concept rather than a topic of the physical sciences.

Although the discussion here has been confined to planar angles, all conclusions apply equally to the concepts of "angle of rotation" and "phase angle", which have not been discussed here in order to concentrate on the essential points.

A different argument for introducing a base unit for angles was presented in [10]. Here the starting point is to deduce from conservation principles the number of naturally independent quantities, and the minimum number of base quantities within a unit system. We clearly agree with the general principle of expressing the laws of physics using equations that are invariant under the change of chosen units, but this requirement is already guaranteed for angular measures, since they are pure numbers. We will focus on this aspect below.

The authors' argument is concerned with the unit for angular momentum, which in the current SI equals the unit for action. They discuss the tendency of theoreticians to remove dimensional constants from the equations, the radian being one example and the speed of light another, and its dire consequences. Regarding the not uncommon practice of setting the speed of light to 1, they state: "According to Noether's theorem, this is equivalent to removing the distinction between energy and momentum, and hence removing one of the conservation laws." [10, p. 3].

However, this statement is incorrect, as conservation laws are unaffected by the choice of units. We could do away with all units, and the equations of physics would still remain valid, admitting the same conservation laws as before. Nevertheless, units are clearly needed not only for measurements, but are essential for all engineering and applied sciences.

The authors conclude [10, p. 7]: "A great deal of time and energy spent discussing unit systems could be saved by, firstly, appreciating that they should be built on foundations of basic physics, not mathematics or arbitrary choices, and secondly, recognizing the difference between complete equations and unit-specific equations."

It has been well known for centuries that units are necessary for unique measurements, but may be (and have always been) chosen by convention. There is only one exception: the unit of the numbers, the number one, is fixed by mathematics and no convention can change this. Thus pure numbers, as angular measures are, have only one unit, the number one, and therefore "rad" is just an alias for this (mathematically determined) unit. This is the current situation in the SI and should not be changed. The number one is already there as an (implicit) unit of every system of units! What is wrong, is to state rad = m/m in the SI brochure and to say that "the rad is a derived unit".

We have argued that angles are geometric objects, and thus do not belong to the realm of physics. Angular measures are far from built on "arbitrary choices", but rather on two millennia of mathematical development.

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