The Mathematics of László Lovász

Martin Grötschel and Jaroslav Nešetřil

Abstract This is an exposition of the contributions of László Lovász to mathematics and computer science written on the occasion of the bestowal of the Abel Prize 2021 to him. Our survey, of course, cannot be exhaustive. We sketch remarkable results that solved well-known open and important problems and that – in addition – had lasting impact on the development of subsequent research and even started whole new theories. Although discrete mathematics is what one can call the Lovász home

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turf, his interests were, from the beginning of his academic career, much broader. He employed algebra, geometry, topology, analysis, stochastics, statistical physics, optimization, and complexity theory, to name a few, to contribute significantly to the explosive growth of combinatorics; but he also exported combinatorial techniques to many other fields, and thus built enduring bridges between several branches of mathematics and computer science. Topics such as computational convexity or topological combinatorics, for example, would not exist without his fundamental results. We also briefly mention his substantial influence on various developments in applied mathematics such as the optimization of real-world applications and cryptography.

1 Introduction

László Lovász was born in 1948 in Budapest. Laci, as he is called by his friends, attended the Fazekas Mihály Gimnázium in Budapest, a special school for mathematically gifted students and a fertile ground of world-class mathematicians. Katalin Vesztergombi, his wife since 1969, was one of his classmates. Laci's outstanding talent became visible at very young age. He won, for example, several mathematics competitions in Hungary and also won three gold medals in the International Mathematical Olympiad.

Lovász studied mathematics at Eötvös Loránd University (ELTE). He received – with Tibor Galai as his mentor – his first doctorate (Dr. Rer. Nat.) degree from ELTE in 1971, the Candidate of Sciences (C. Sc.) degree in 1970 and his second doctorate (Dr. Math. Sci.) degree in 1977 from the Hungarian Academy of Sciences. Of great influence for his scientific growth was the outstanding Hungarian combinatorial

Fig. 1 Lovász and Erdős at dinner in 1977 (Photo: Private)

Fig. 2 In a Tokyo subway station on the way to the Kyoto Prize ceremony: Laci, Kati, and son Laci M. Lovász, András Frank in the back (Photo: Private)

school (e.g., T. Galai, A. Hajnal, A. Rényi, M. Simonovits, V. T. Sós, P. Turán, and foremost P. Erdős).

In 1971 Lovász started his professional career as a research associate at ELTE. From 1975 to 1982 he was Docent, later Professor and Chair of Geometry at József Attila University, Szeged; 1983–1993 Chair of Computer Science at ELTE; 1993–1999 Professor of Computer Science at Yale University; and 1999–2006 Senior Researcher, Microsoft Research, Redmond. In 2006 Lovász returned to his hometown Budapest as a Professor and Director of the Mathematical Institute at ELTE from which he retired in 2018. In 2020 he joined the Alfréd Rényi Institute of Mathematics. Lovász served the International Mathematical Union as its President from 2007 to 2010 and the Hungarian Academy of Sciences as its President from 2014 to 2020 during demanding times.

Among the institutions Lovász visited for extended periods of time are Vanderbilt University, University of Waterloo, Universität Bonn, University of Chicago, Cornell University, Mathematical Sciences Research Institute in Berkeley, Princeton University, Princeton Institute for Advanced Study, and ETH Zürich. Five universities bestowed special professorships upon him, he received six honorary degrees and countless high-ranking honors and distinctions, including the Kyoto Prize 2010, see Fig. 2.

Like every scientific discipline, mathematics has become a field with a large number of specializations. The Mathematics Subject Classification (MSC 2020) with its 63 first-level areas and 6,006 specific research areas is a witness of this development. Today, no mathematician has a full understanding of all the mathematical branches. But there are still a few people with broad mathematical knowledge, deep command of their fields of special interest, and the ability to build bridges by transferring results and techniques between fields to expand the mathematical toolboxes and open up new research areas. One of these rare persons is László Lovász. In fact, quite fittingly, two volumes published in his honor at special occasions were entitled *Building Bridges*, see [65] and [12].

Laci's mathematical roots are in combinatorics. But he vastly expanded his reach by employing combinatorial methods in other mathematical fields and bringing, in return, tools from geometry, topology, algebra, analysis, probability theory, information theory, optimization, and even ideas from physics into combinatorics. His deep interest in algorithms led to major advances in modern complexity theory. In his work, Lovász established profound connections between discrete mathematics and computer science. This is reflected in the statement that the Norwegian Academy of Science and Letters issued in its announcement of the award of the Abel Prize 2021 to him and Avi Wigderson

> for their foundational contributions to theoretical computer science and discrete mathematics, and their leading role in shaping them into central fields of modern mathematics.

At the end of the 1960s and the beginning of the 1970s, graph theory, discrete mathematics, combinatorics, and theoretical computer science were considered peripheral fields of mathematics. This changed completely during Lovász's lifetime. They became central parts of modern mathematics for many reasons. The tremendous development of computer technologies is the most obvious one. Essential factors were also the high quality of the research and the results in these areas and their wide applicability. The solutions of many problems arising in industry, society, other sciences, even in other fields within mathematics critically depend on theories and algorithms invented in discrete mathematics. Many mathematicians and computer scientists contributed to this. László Lovász undoubtedly was and still is one of the key players in this development.

There are other aspects that make László Lovász special. Mathematicians are often divided into "problem solvers" and "theory builders". Graph theory is, in particular, a field to which problem solvers are drawn. Theory builders often see deep and unusual connections, but often leave the difficult exploration of details to others. As we will demonstrate, Lovász is a member of this rare breed of people who possess both talents. Moreover, he brought his talents to bear not only in one field of mathematics, he has also fertilized and inspired significant developments in a wide range of other areas. If asked to formulate the essence of his contributions in few words, we could use the following three:

Depth: Lovász solves many important and widely known problems in a competitive environment. He isolates seemingly special topics and develops them into broad and important calculi.

Elegance: His solutions are often surprisingly (and sometimes seemingly) simple. At the same time, they often are mathematically beautiful and suggest fundamentally new ways to address a problem.

Inspiration: Many of his solutions are the basis of further active research and even the foundations of whole new areas.

László Lovász published eleven books and more than 300 articles. There is no way to survey his contributions in an article like this. We have chosen to sketch some of the publications and topics that we consider highlights, are not too difficult to explain, had significant impact, moved the frontier of knowledge in the interface of mathematics and computer science substantially, and are of lasting value.

2 Logic and Universal Algebra – Homomorphisms and Tarski's Problem

L. Lovász. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungarica* 18:321–328, 1967.

L. Lovász. On the cancellation law among finite relational structures. *Periodica Mathematica Hungarica* 1:145–156, 1971.

M. Freedman, L. Lovász, L. Schrijver. Reflection positivity, rank connectivity, and homomorphisms of graphs. *Journal of American Mathematical Society* 20(1):37–51, 2007.

Up to the 1960s graph theory was mainly concerned with graphs as objects. Graph parameters were introduced and the structural properties of graphs having these properties were investigated. László Lovász made, as we will outline, very significant contributions to this kind of research, but he left his first fundamental mark, when he was 19 years old, in the more general context of universal algebra.

Intending to step out of the object orientation of graph theory, Lovász got interested in operations with graphs and their algebraic properties. We all know that, for nonzero real numbers a, b, and c, the equation $ac = bc$ implies $a = b$. Suppose we have three graphs A, B, and C, and suppose we have defined a product "$\times$" for which $A \times C = B \times C$ holds, can we infer that $A = B$? Such a question makes only sense if equality "=" is replaced by "isomorphic" and the concrete issue to be addressed is: Under what conditions, does such a "cancellation law" hold?

Questions of this type were asked by Alfred Tarski, in the context of finite relational structures, to students in Berkeley in the 1960s. Lovász points this out in the following quote, extracted from his article [97], where he states the question and announces his solution:

> Our main concern will be in the direct product of finite structures (i.e. $\langle H, R\rangle$ with finite domain H). In [1] the question was discussed under what conditions it is true that any two direct factorizations of a structure have a common refinement. It was mentioned that if the structures A, B have this "refinement-property" then e.g. $A^2 \cong B^2$ implies $A \cong B$. We shall prove a general theorem from which it follows that for finite A, B the last implication always holds. On the other hand, it is easy to see that not all finite structures have the refinement-property (or the unique prime factorization-property).

Fig. 3 Quote from [97]

Reference [1] in the quote above is the paper [25] of Chang, Jónsson, and Tarski of 1964, see also [130].

A finite *graph* G with *vertex set* $V(G)$ and *edge set* $E(G)$ is such a relational structure where $V(G)$ is the ground set and the edges uv define the (binary) relations between vertices u and v. A standard product in graph theory is the *direct* (also named categorial or tensor) *product* $G_1 \times G_2$ of two graphs G_1 and G_2. Its vertex set is $V(G_1) \times V(G_2) = \{(u,v) \mid u \in V(G_1), v \in V(G_2)\}$ and its edge set $E(G_1 \times G_2)$ is defined to be the set of all pairs of vertices $(u_1,u_2),(v_1,v_2) \in V(G_1) \times V(G_2)$ with $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. The question to be addressed is: Given two graphs G and H and a third graph F, can one conclude that G and H are isomorphic if the direct product $F \times G$ is isomorphic to $F \times H$? This particular question and most of the related problems for finite relational structures were unsolved, despite considerable effort. The earlier solution approaches taken were usually elementary, trying to reduce the problem to known invariants.

Lovász devoted to these problems three of his early papers written in 1967, 1971, 1972. His approach was radically different: He invented a new invariant which solved these problems for the direct product in full generality. His results completely changed this area.

The Lovász argument is easy and can be given here in full. Interestingly, young Lovász formulates his results very generally for finite *relational structures*, i.e., objects of the form $\boldsymbol{A} = (X_{\boldsymbol{A}}, (R_{\boldsymbol{A}}; R \in L))$ where R_A is a subset of $X^{\,a(R)}$ ($a(R)$ is the arity of the relational symbol R; L is the fixed set of symbols usually called language). Shortly, we speak about *L-structures*.

A *homomorphism* $f : \boldsymbol{A} \longrightarrow \boldsymbol{B} = (X_{\boldsymbol{B}}, (R_{\boldsymbol{B}}; \boldsymbol{R} \in L))$ is a mapping $f : X_{\boldsymbol{A}} \longrightarrow X_{\boldsymbol{B}}$ such that for every $R \in L$ holds $(x_1,\dots,x_{a(R)}) \in R_{\boldsymbol{A}} \Rightarrow (f(x_1),\dots,fx_{a(R)} \in R_{\boldsymbol{B}}$. The *product* $\boldsymbol{A} \times \boldsymbol{B}$ is defined as $X_{\boldsymbol{A}\times\boldsymbol{B}} = X_{\boldsymbol{A}} \times X_{\boldsymbol{B}}$ where $\boldsymbol{R}_{\boldsymbol{A}\times\boldsymbol{B}}$ is the set of all tuples $((x_1,y_1),\dots,(x_{a(R)},y_{a(R)}))$ where $(x_1,\dots x_{a(R)}) \in R_{\boldsymbol{A}}$ and $(y_1,\dots,y_{a(R)}) \in R_{\boldsymbol{B}}$.

Note that the projections $\pi_{\boldsymbol{A}} : X_{\boldsymbol{A}\times\boldsymbol{B}} \longrightarrow X_{\boldsymbol{A}}$ and $\pi_{\boldsymbol{B}} : X_{\boldsymbol{A}\times\boldsymbol{B}} \longrightarrow X_{\boldsymbol{A}}$ are homomorphisms. Up to an isomorphiosm, projections uniquely determine the above product. (The whole theory may be restated in categorical terms as worked out in papers by Lovász [102] and Pultr [139].)

Denote by $\hom(\boldsymbol{A},\boldsymbol{B})$ the number of homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. The key of Lovász's argument is the following statement:

Theorem. *Finite L-structures $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic if and only if for every other finite structure $\boldsymbol{C}$ holds:* $\hom(\boldsymbol{C},\boldsymbol{A}) = \hom(\boldsymbol{C},\boldsymbol{B})$.

In other words (and today's setting), if we take a fixed enumeration $F_1, F_2, \dots F_n, \dots$ of all non-isomorphic finite graphs then the vector $L(\boldsymbol{A}) = (\hom(F_i,\boldsymbol{A});\ i = 1,\dots)$ is the isomorphism invariant, expressed equivalently: $\boldsymbol{A} \cong \boldsymbol{B}$ if and only if $L(\boldsymbol{A}) = L(\boldsymbol{B})$.

Hell and Nešetřil [76] (and others) call this invariant $L(\boldsymbol{A})$ *Lovász vector*.

This setting is very suitable for the Tarski problem. For example, one immediately obtains that for finite structures $\boldsymbol{A}^k \cong \boldsymbol{B}^{\mathbf{k}}$ holds if and only if $\boldsymbol{A} \ \cong \boldsymbol{B}$. This follows readily from $\hom(\boldsymbol{C},\boldsymbol{A}^k) = (\hom(\boldsymbol{C},\boldsymbol{A}))^k$.

For brevity we mention another consequence for the special case of graphs. If $\boldsymbol{C}$ is a nonbipartite graph then $\boldsymbol{A} \times \boldsymbol{C} \cong \boldsymbol{B} \times \boldsymbol{C}$ if and only if $\boldsymbol{A} \cong \boldsymbol{B}$. (Note that for bipartite graphs $\boldsymbol{C}$, cancelation need not hold as already for circuits we have $2\boldsymbol{C}_3 \times K_2 \cong \boldsymbol{C}_6 \times K_2$.)

The above theorem is very general and yet the proof is easy. In the nontrivial direction we prove by induction on the cardinality $|X_{\boldsymbol{C}}|$ of the ground set $X_{\boldsymbol{C}}$ that, if $\hom(\boldsymbol{C},\boldsymbol{A}) = \hom(\boldsymbol{C},\boldsymbol{B})$, then also the number of injective homomorphisms coincides, i.e., $\mathrm{inj}(\boldsymbol{C},\boldsymbol{A}) = \mathrm{inj}(\boldsymbol{C},\boldsymbol{B})$.

In the inductive step we have $\hom(\boldsymbol{C},\boldsymbol{A}) = \sum_\theta \mathrm{inj}(\boldsymbol{C}/\theta,\boldsymbol{A})$ where θ is an equivalence on $X_{\boldsymbol{C}}$. Thus by induction assumption we have $0 = \hom(\boldsymbol{C},\boldsymbol{A}) - \hom(\boldsymbol{C},\boldsymbol{B}) = \mathrm{inj}(\boldsymbol{A},\boldsymbol{A}) - \mathrm{inj}(\boldsymbol{A},\boldsymbol{B}) = \mathrm{inj}(\boldsymbol{B},\boldsymbol{B}) - \mathrm{inj}(\boldsymbol{B},\boldsymbol{A})$. But obviously $\mathrm{inj}(\boldsymbol{A},\boldsymbol{A}) > 0$ and $\mathrm{inj}(\boldsymbol{B},\boldsymbol{B}) > 0$, and thus, we have that there are injective homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ and also from $\boldsymbol{B}$ to $\boldsymbol{A}$. Now as $\boldsymbol{A}$ and $\boldsymbol{B}$ are finite structures we have that $\boldsymbol{A} \cong \boldsymbol{B}$.

Lovász recognized in the Tarski problem a magnificent pearl. His theorem turned out to be very useful. It found many applications and inspired further research. This continues until today, see the articles by Lovász and Schrijver [118], Dvořák [38] and Dawar et al. [34], for example.

The papers [97] and [99] of Lovász belong to the first occurrences of homomorphisms in graph theory. Their successful utilization led to a rich calculus (see, e.g., the books by Hell and Nešetřil [76] and Lovász [112] and the article by Borgs et al. [21]). We outline important parts of this approach.

Lovász already defined in [97] exponential structures $\boldsymbol{A}^{\boldsymbol{B}}$ (and exponential graphs G^H). These played very recently a decisive role in the disproof of the Hedetniemi conjecture which claimed that $\chi(G \times H) = \min(\chi(G),\chi(H))$, see Shitov [147], Wrochna [159], Tardif [156], and Zhu [161].

Another application of homomorphism counting was provided by Lovász in [103] which deals with the following problem: When can one recognize a given finite structure from the collection of all its proper substructures? The special case for undirected graphs is a classical *conjecture of Ulam*, see [157], which may be formulated in our setting as follows:

Do the homomorphism numbers $\hom(F,G)$ *for all graphs F with fewer edges than G determine the graph G?*

This conjecture is known to be true for special classes of graphs (such as trees and maximal planar graphs), and the proofs usually consist of a complicated case analysis. In [103] Lovász gave the first general result: The conjecture is true for graphs that have more edges than their complement (i.e., more than half of all edges).

The proof, although not directly linked to the above theorem proceeds again by clever homomorphism counting. Shortly after, this proof has been extended by Müller in [132] (again by homomorphism counting) to graphs with $n\log n$ edges. This is still the best result.

Counting of homomorphisms and the investigation of their structure are cornerstones of further areas of mathematics and theoretical computer science. We just indicate three examples, where they play important roles: Tutte polynomials and their variants, see [46]; constraint satisfaction problems (which can alternatively be viewed as existence theorems for general relational structures), see [52]; and partition functions in statistical physics, see [23, 24, 112].

Let us finally elaborate on partition functions, the last item mentioned above. The concept of graph homomorphisms can be extended to graphs with loops and weights assigned to vertices $\propto_v (G)$ and edges $\beta_{uv}(G)$. For unlabelled graphs F and labelled graphs G, one can define naturally the weight of a mapping $\varphi : V(F) \to V(G)$ and then the total weight of $\hom(F,G)$. Allowing weights on the vertices and edges greatly extends the expressive power of (weighted) homomorphisms. For example, the number $\hom(F,G)$ can express the number of colorings (leading to chromatic and Tutte polynomials), the counting of stable sets (corresponding to the so-called *hard core model* in statistical physics) and also the counting of nowhere zero flows and B-flows (i.e., flows attaining values from a given set B only). All these are parameters of the form $\hom(-,H)$. Freedman, Lovász, and Schrijver [54] provided a structural characterization for all such parameters as follows:

Theorem. *Let f be a (real) graph parameter defined on multigraphs without loops. Then f is equal to* $\hom(-,H)$ *for some weighted graph H on q vertices if and only if $f(K_0) = 1$, the f connection matrix $M(f,k)$ is reflection positive, and its rank satisfies $r(M(f,k)) \leq q^k$ for all $k \geq 0$.*

(Briefly: Above, K_k is the complete graph on k vertices; the connection matrix $M(f,k)$ is defined by values of the parameter f for amalgams of k-multilabeled multigraphs; reflection positivity means that, for all k, such matrices are positive semidefinite.)

This theorem led to many similar results for other classes of graphs and for other types of homomorphism numbers (e.g., in a dual setting with $\hom(F,-)$ instead of $\hom(-,H)$, see [119]). In terms of statistical physics, this theorem can be viewed as a characterization of partition functions of vertex coloring models.

Lovász wrote extensively on this topic and devoted – ten years ago – a monograph [112] to this subject, where the topics indicated here are treated in depth.

3 Coloring Graphs Constructively (on a Way to Expanders)

L. Lovász. On chromatic number of finite set-systems. *Acta Mathematica Academiae Scientiarum Hungaricae*, 19:59–67,1968.

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors which suffice to color all vertices of G such that no two adjacent vertices get the same colour. Alternatively, using the notion of the preceding section, $\chi(G)$ is smallest k for which $\hom(G,K_k) > 0$.

The chromatic number belongs to the most frequently studied combinatorial parameters. Reasons for such an attention are that the question of how to color the countries on a map can be easily explained to everyone and that the mathematical modelling of this question can be employed as an appealing introduction to graph theory. The "colorful story of the 4-color conjecture" can be used to shed some light on the rich history of mathematics and the difficulty of finding proofs for problems that appear to be easy. Coloring the vertices of a graph captures the substance and the difficulty of many problems. In a multiple sense, the chromatic number is a difficult concept.

Just consider the easiest question: Are there graphs with large chromatic number? Of course, complete graphs K_n satisfy $\chi(K_n) = n$. But are there any other essentially different graphs?

The answer is yes and a classical result, rediscovered several times, states that, for every $k \geq 1$, there are graphs G_k for which $\chi(G_k) = k$ and G_k does not contain K_3 (i.e., the triangle) as a subgraph. This result and its many ramifications, for instance in extremal graph theory, are still in the current focus of coloring research. In fact, any new constructive proof of the existence of such graphs G_k is interesting and attracts great attention. Here is perhaps the simplest proof of this fact: Let us define, for any integer $n \geq 4$, the graph $G = (V, E)$ where V is the set of integer pairs $\{ij\}, 1 \leq i < j \leq n$, and $\{ij, kl\} \in E$ if $i < j = k < l$. Such a graph G is called a *shift graph*. G has no triangles, and it can be shown that $\chi(G) = [\log n]$.

But this is not the end of the story. Graphs may have high chromatic number and very low edge density. P. Erdős showed in [47] that there exist graphs which have arbitrarily large chromatic number and which are locally trees and forests.

Theorem. *For every k, l there exists a graph $G_{k,l}$ such that $\chi(G_{k,l}) \geq k$ and $G_{k,l}$ does not contain circuits of length $\leq l$. (So the shift graph above is a graph of type $G_{k,3}$.)*

Erdős' proof was a landmark. It constitutes one of the key applications of the probabilistic method in graph theory, see, e.g., [6]. The proof shows that the probability of the existence of such graphs $G_{k,l}$ is positive, but does not give any hint how to construct concrete examples of graphs of type $G_{k,l}$. The construction of such graphs has been a longstanding problem with very slow progress (for the historic development and related issues, see, e.g., the Nešetřil article [133]).

The first constructive proof of the theorem above was found by Lovász in one of his early papers [98]. It was one of the highlights of the 1969 conference in Calgary; and through his proof, Lovász again changed the setting of the problem as he constructed the graphs $G_{k,l}$ as special cases of a more general theorem about hypergraphs. His complicated construction was later simplified, the Nešetřil–Rödl construction is perhaps the simplest [135].

But various problems remained.

One of them is the question whether one can provide a construction that uses only graphs. The answer is positive. I. Kříž [90] and more recently N. Alon et al. [3] came up with such constructions, and Ramanujan graphs have to be mentioned here as well.

The existence of graphs $G_{k,l}$ with a large chromatic number and no short circuit is a phenomenon of finite (and of countable) graphs. For graphs with an uncountable number of vertices and uncountable chromatic number, an analogous result does not hold. This was shown by Erdős and Hajnal [48]:

If the chromatic number of a graph is uncountable then it contains every bipartite graph.

A consequence of this result is that such a graph contains every circuit of even length, for example the circuit $\boldsymbol{C}_4$ of length four.

Graphs $G_{k,l}$ are what can be called difficult examples. They also play an important role in Ramsey theory, extremal combinatorics, topological dynamics, and model theory, to name just a few. In all these areas they are used as examples of complex yet locally simple structures; they are prototypes of local-global phenomena.

It took some time to understand why the construction of graphs $G_{k,l}$ matters, why it is important to know such graphs explicitly. This led to an explosion of theoretical developments combining group theory, number theory, geometry, algebraic graph theory, and, of course, combinatorics. The key notions are now familiar to every student of theoretical computer science: expanders, Ramanujan graphs and sparsification, see Margulis [127], Lubotzky, Phillips, and Sarnak [125], and Spielman and Teng [153].

An expander graph, for instance, is a finite, undirected multigraph (parallel edges are allowed) in which every subset of the vertices that is not "too large" has a "large boundary". There are various formalizations of these notions. Each of them gives rise to a different notion of expanders, e.g., edge expanders, vertex expanders, and spectral expanders. Expander graphs have found applications in the design of algorithms, error correcting codes, pseudorandom generators, sorting networks, robust computer networks and hash functions in cryptography. They also played a role in proofs of important results in computational complexity theory, such as the PCP theorem.

The construction and structure of graphs similar to $G_{k,l}$ continues to be one of the key problems of finite combinatorics and has a character of a saga (see, e.g., Hoory et al. [78] and Nešetřil [133]).

Coloring of graphs and hypergraphs has been a permanent theme of Lovász, and thus, it is mentioned in most sections of our survey. For example, one of the motivations of the next section was the study of 3-chromatic linear hypergraphs, i.e., hypergraphs in which edges meet in at most one vertex, or equivalently, hypergraphs without cycles of length 2.

4 The Lovász Local Lemma

P. Erdős, L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets. Coll. Math. Soc. J. Bolyai*, North Holland:609–627, 1975.

A *hypergraph* is a collection of sets. The sets are called *edges*, the elements of the edges are *vertices*. The *degree* of a vertex is the number of edges containing it. A hypergraph is called *r-uniform* if every edge has r vertices. The *chromatic number* *of a hypergraph* is the least number k such that the vertices can be k-colored so that no edge is monochromatic.

Lemma. *Let G be a (finite) graph with maximum degree d and vertices $v_1, \dots, v_n$. Let us associate an event A_i with v_i $(i = 1, \dots, n)$ and suppose that A_i is independent of the set*

$$\{A_j \colon (v_i, v_j) \in E(G)\}\,.$$

Also suppose

$$P(A_i) \leqslant \frac{1}{4d}\,. \tag{3}$$

Then

$$P(\bar{A}_1 \dots \bar{A}_n) > 0\,. \tag{4}$$

Proof. We prove more, namely that

$$P(A_1 \mid \bar{A}_2 \dots \bar{A}_n) \leqslant \frac{1}{2d}\,. \tag{5}$$

This formula makes sense because we may assume by induction

$$P(\bar{A}_2 \dots \bar{A}_n) > 0\,.$$

Then (5) obviously implies (4).

We prove (5) by induction on n. For $n = 1$ it is trivial. Let $v_2, \dots, v_q$ be the points adjacent to v_1, $(q \leqslant d + 1)$. Then we have

$$P(A_1 \mid \bar{A}_2 \dots \bar{A}_n) = \frac{P(A_1 \bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n)}{P(\bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n)}\,.$$

Here, by (3)

$$P(A_1 \bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n) \leqslant$$
$$\leqslant P(A_1 \mid \bar{A}_{q+1} \dots \bar{A}_n) = P(A_1) \leqslant \frac{1}{4d}\,,$$

and on the other hand

$$P(\bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n) =$$
$$= 1 - P(A_2 + \dots + A_q \mid \bar{A}_{q+1} \dots \bar{A}_n) \geqslant$$
$$\geqslant 1 - \sum_{i=2}^{q} P(A_i \mid \bar{A}_{q+1} \dots \bar{A}_n) \geqslant 1 - (q-1)\frac{1}{2d} \geqslant \frac{1}{2}$$

by the induction hypothesis. Thus

$$P(A_1 \mid \bar{A}_2 \dots \bar{A}_n) \geqslant \frac{1}{4d} \Big/ \frac{1}{2} = \frac{1}{2d}\,.$$

This proves the lemma.

Fig. 4 Extracted from [49]

Graphs with chromatic number at least 3 are simple to characterize: they must contain an odd circuit. But for hypergraphs, even the characterization of 3-chromatic 3-uniform hypergraphs is difficult (it is an $\mathcal{NP}$-complete problem). Lovász and Woodall had independently shown that every 3-chromatic r-uniform hypergraph contains a vertex of degree at least r. Erdős and Lovász [49] aimed at generalizing this result in various ways. One of the key results of their article is the following:

Theorem. *A $(k+1)$-chromatic r-uniform hypergraph contains an edge which is intersected by at least $k^{r-1}/4$ other edges. Thus, the degree of at least one vertex is larger than $k^{r-1}/(4r)$.*

To prove this theorem, the authors employed probability theory. As pointed out by Erdős, Lovász contributed to the proof a substantial new result of elementary probability. This was later called the *Lovász Local Lemma*.

The motivation for this lemma comes from a well-known observation of elementary probability:

If $X_1, \ldots, X_n$ are random events which are pairwise independent and if the probability of each event X_i is smaller than 1, then the probability that none of the events X_i occurs is positive. The Lovász Local Lemma is a quantitative refinement of this observation for variables which are dependent.

Fig. 4 shows the formulation of the Lovász Local Lemma as stated and proved in the original article [49]. Indeed, it is "just a lemma".

Crystal clear: Not only when the events are independent, but if the dependence graph G has a small degree ($\leq d$) then also none of the events occurs with positive probability. The adjective local in the name of the lemma refers to the situation that each event is dependent only on a small number d of others.

It is hard to overestimate the general importance of this result that just turned up as a "supporting observation" for a proof in the chromatic theory of hypergraphs. It appears again and again in multiple applications, ramifications, and forms. It is not possible to cover here all the applications in Ramsey theory (see Spencer [152]), extremal combinatorics (see Alon and Spencer [6]), number theory, and elsewhere (see, e.g., Ambainis et al. [7], He et al. [75], and Szegedy [154]). It was also discovered, see [145], that the Lovász Local Lemma closely relates to important results of Dobrushin in statistical physics [37]. In fact, the proper setting of the Dobrushin results is in the context of graph limits, see [112], which we discuss in Section 17.

One of the motivations for [49] is the following number theoretic problem which goes back to Ernst Straus (who was an assistant of Albert Einstein): Is there a function $f(k)$ such that, if S is any set of integers with $|S| = f(k)$, then the integers can be k-colored so that each color meets every translated copy of S (i.e., every set of the form $S+a = \{x+a \mid a \in S\}$)? Lovász and Erdős, already in their paper [49], made use of the Lovász Local Lemma to prove the following more geometric generalization of the question asked by Straus:

For every k, there exists a function $f(k)$, such that $f(k) \leq k \log k$ and for every set S of lattice points in the n-dimensional space E^n with $|S| > f(k)$ there exists a k-coloring of all lattice points such that each translated copy of S contains points of all k colors.

A side remark: There are many variants of coloring problems, and some of them are surprisingly difficult. For example, during a conference in Boulder in 1972 Paul

Erdős, Vance Faber, and László Lovász asked whether the vertices of any n-uniform linear hypergraph with n edges can be colored by n colors such that the vertices of any edge get all n colors. This question has many reformulations and turned out to be more difficult than originally thought (even by the authors as Erdős originally offered \$50 for a solution and eventually increased the prize to \$500). About 50 years later the Erdős–Faber–Lovász conjecture was shown to be true for large values of n by D. Y. Kang, T. Kelly, D. Kühn, A. Methuku, and D. Osthus [84].

Nowadays, the Lovász Local Lemma is a "standard trick" which is often taught in basic courses. And it is a very effective trick, as Joel Spencer once remarked: "*Using the Local Lemma one can prove the existence of a needle in a haystack*."

But the Lovász Local Lemma delivers only existence. The above proof does not yield a method how to find that needle. We only know that certain things exist with positive probability. Only much later a constructive proof was found by Marcus and Tardos [126]. (Remark: A constructive proof for the above Straus' problem is in Alon et al. [4]; see also J. Beck [14].) Recently, Harvey and Vondrák [72] found another constructive approach to the Lovász Local Lemma.

Investigations of infinite versions (Borel and measurable) of the Lovász Local Lemma started also very recently by A. Bershteyn, G. Kun, O. Pikhurko, and others, see, e.g., [18].)

The Lovász Local Lemma became what one can truly call *a combinatorial principle*. This is László Lovász at its best: Maybe no other Lovász-contribution is so profoundly simple and yet useful and elegant.

5 Coloring Graphs via Topology

L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory A* 25:319–324, 1978.

Combinatorial questions are often easy to formulate; some have also an elementary solution. But in many cases, the elementary nature of combinatorial problems is just the top of an iceberg, and the hidden complexity must be discovered and tamed before a solution can be found.

A beautiful example of this is the following elementary problem posed in 1955 by Martin Kneser [89] who was working on quadratic forms. In today's language:

Let X be a set with n elements, $n \geq 2k > 0$. Denote by $\binom{X}{k}$ the set of all k-element subsets of X. Then, for every coloring of the sets in $\binom{X}{k}$ by fewer than $n-2k+2$ colors, there are two disjoint sets of the same color.

This problem can be reformulated as a graph theory question as follows. Let $KG(n,k)$ denote the graph (called *Kneser graph*) whose vertices are all k-element subsets of set $X = \{1,2,\dots,n\}$, and in which two vertices are joined by an edge if the corresponding k-element subsets are disjoint. For example, $KG(n,1)$ is the

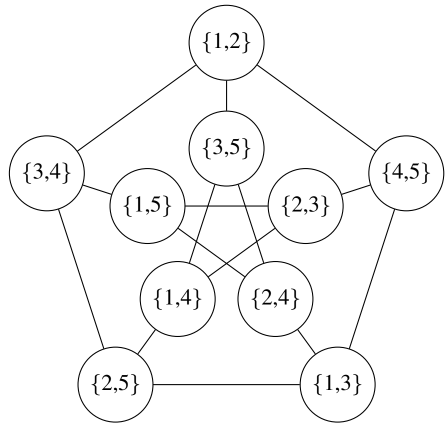


Fig. 5 $KG(5.2)$ = The Petersen graph

complete graph K_n and $KG(5,2)$ is the famous Petersen graph (the “universal” counterexample to many conjectures in graph theory) shown in Fig. 5.

Kneser’s question reads now: *Does the Kneser graph $KG(n,k)$ have chromatic number $n-2k+2$?*

It is easy to see that $\chi(KG(n,k)) \leq n-2k+2$. However, to find the fitting lower bound for the chromatic number proved to be much harder.

Lovász [98] solved this problem in a surprising way using methods of algebraic topology. The general idea is the following. Lovász associates with any graph G a topological space and establishes a connection between a topological invariant of this space with the chromatic number of G. He then infers properties of the chromatic number of G from properties of the topological invariant of the associated topological space. That this is possible and that topology can yield solutions of difficult graph theory questions was completely unexpected. Lovász’s success with this approach was the starting point of a new field: *topological combinatorics*. We briefly sketch the main steps of Lovász’s solution of Kneser’s problem here.

Lovász proceeds as follows: Given a graph $G = (V,E)$, the neighborhood of a vertex v is composed of all vertices adjacent to v in G. The *neighborhood complex* $N(G)$ of G consist of all the vertices V of the graph G; the simplices of $N(G)$ are sets of vertices with a common neighbor in the graph. Homomorphisms between graphs lead to continuous mappings of neighborhood complexes. From the topological connectivity of $N(KG(n,k))$ it is possible to construct an antipodal continuous mapping between spheres ($N(K_{m+2})$ is an m-dimensional sphere) and one can then apply the Borsuk–Ulam theorem. Thus, Lovász obtained:

Theorem. *If the neighborhood complex $N(G)$ of a graph G is (topologically) k-connected then $\chi(G) \geq k+3$.*

(Topologically k-connected means that there are no holes of dimension $\leq k$. For (simply) connected complexes this is equivalent to the fact that all i-homological groups vanish for $i = 0,1,\ldots,k$.)

Lovász finally proves a theorem on the connectivity of neighborhood complexes of graphs from which he can infer that the neighborhood complex of a Kneser graph $N(KG(n,k))$ is topologically $(n-2k-1)$-connected. This establishes that the Kneser graph $KG(n,k)$ has chromatic number $n-2k+2$.

This connection (and the whole proof) immediately led to intensive research. Other proofs of this theorem were found (among them "book proofs" of Barany [10] and Green [63]), but all lower bounds for the chromatic number of Kneser graphs use or at least imitate Lovász's topological proof. Matoušek's book [128] surveys in detail various implications and modifications of the proof techniques. For example, it has been shown in [71] that the k-times generalized Mycielski construction has chromatic number $k+2$, and again, topological arguments are the basis of the only known proof of this fact. The paper [71] contains the following interesting construction of graphs G_k.

Put $[k] = \{1,2,\dots,k\}$. The vertices of G_k are all pairs (i,A) where $i \notin A$ and A is a nonepty subset of $[k]$. (i,A) and (j,B) form an edge in G_k if $i \in B, j \in A$ and A and B are disjoint. This "Kneser-like" graph G_k has remarkable properties: Its chromatic number is k, it is critical (i.e., every proper subgraph has a smaller chromatic number) and every strongly k-colorable graph has a homomorphism into it; it is the unique graph with this property. (A *strong coloring* of a graphs is a coloring where the neighborhood of any color class forms a stable set. Such a graph obviously has no triangles.)

The only known proof of these properties is an adaptation of Lovász's topological proof.

These examples of graphs were instrumental in the recent disproof of the Hedetniemi conjecture (that intended to establish a connection between the direct product of two graphs and their chromatic number which we mentioned in Section 2; see also [150]) and also in the study of *gap problems* for constraint satisfaction problems. Related questions in this area are called *promised problems*. A typical question here is: How difficult is to 5-color graphs or hypergraphs under the assumption that we know they are 3-colourable, see [13, 35], and [160].

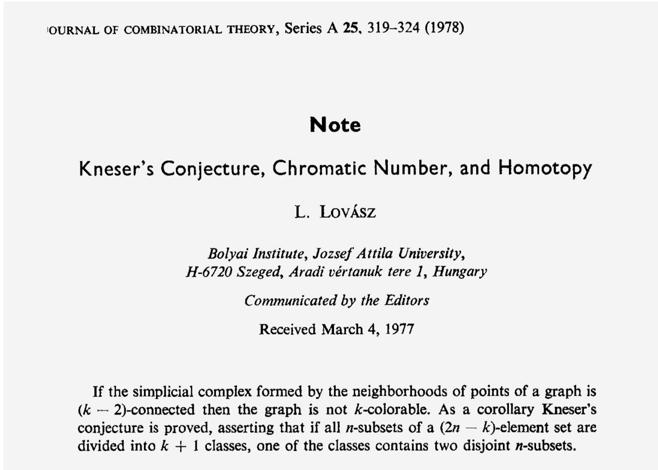
Lovász's paper opened a whole area whose fruits are still continuing to appear. Matoušek in the preface to [128] rightly wrote that Lovász's proof of the Kneser conjecture is a masterpiece of imagination.

Yet, in typical Lovász style, it was published just as a note (see Fig. 6).

Lovász's solution of the Kneser problem did not exhaust his topological imagination nor the potential of topological methods in combinatorics. He returned to this approach frequently during his career, often in collaboration with Lex Schrijver. We mention just one of the highlights of their cooperation.

Motivated by estimating the maximum multiplicity of the second eigenvalue of Schrödinger operators, Colin de Verdière introduced a new invariant for graphs G, denoted $\mu(G)$, based on spectral properties of matrices associated with G. He proved that $\mu(G) \le 1$ if and only if G is a disjoint union of paths, that $\mu(G) \le 2$ if and only if G is outerplanar, and that $\mu(G) \le 3$ if and only if G is planar.

Robertson, Seymour, and Thomas showed that a graph G is linklessly embeddable if and only if G does not have any of the seven graphs in the Petersen family as a mi-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Note

Kneser's Conjecture, Chromatic Number, and Homotopy

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If the simplicial complex formed by the neighborhoods of points of a graph is $(k-2)$-connected then the graph is not k-colorable. As a corollary Kneser's conjecture is proved, asserting that if all n-subsets of a $(2n-k)$-element set are divided into $k+1$ classes, one of the classes contains two disjoint n-subsets.

Fig. 6 The beginning of the article [105] starting topological combinatorics

nor. Their combinatorial result implies that $\mu(G) \leq 4$ if G is linklessly embeddable, and they conjectured that $\mu(G) \leq 4$ if and only if G is linklessly embeddable. Lovász and Schrijver, see [117], proved the only if part of this topological characterization. The key ingredient of their proof is a new Borsuk-type theorem on the existence of antipodal links, which is an extension of a polyhedral version of Borsuk's theorem due to Bajmóczy and Bárány. The combination of all these results provides a fascinating characterization of graphs G satisfying $\mu(G) \leq 4$ by means of spectral, combinatorial, and topological properties. Topological methods seem to keep on flourishing in combinatorics and graph theory.

6 Geometric Graphs and Exterior Algebra

> L. Lovász. Flats in matroids and geometric graphs. In *Combinatorial Surveys*. Proc. 6 British Comb. Conf. Academic Press, pages 45–86, 1977.

Many of Lovász's proofs deal with graphs (and hypergraphs) and make use of some additional structures. The Shannon Capacity paper, see Section 8, involved a geometric structure which was added (orthogonal representation) so that the problem could be solved. To solve the Kneser problem, discussed in Section 5, Lovász employed results from topology. To recognize that methodology from other mathematical fields can be utilized, needs of course mathematical maturity, skill, and imagination. We want to highlight that this is a different strategy than merely studying embeddings of graphs (e.g., graphs on surfaces): the special embeddings are being incorporated in proofs as tools in order to solve a (different) problem.

A very special example of this is the Lovász-article [104] which is a remarkable paper for multiple reasons.

The paper was published as an invited lecture in the proceedings of 6th British Combinatorial conference. These proceedings volumes usually contain surveys of recent developments. In contrast, the Lovász paper – full of new ideas – solved an important problem and unleashed research in two different areas: First, it started research in graphs where vertices are forming a matroid; Lovász uses here the term *geometric (or pregeometric) graphs*, and this generalization is essential for solving the problem. Secondly, the paper started the application of exterior algebra in combinatorics. Particularly, Lovász defined exterior calculus in matroids and Grassman graded matroids.

Why is Lovász introducing this general machinery? Well, he is explicit about that in the introduction:

> *This paper was intended to deal with the covering problems in graphs. It has turned out, however, that their study becomes much simpler if a more general structure, which we shall call geometric graph, is considered.*

Lovász later on used the term geometric graph in a broader sense, and he recently wrote the book [113] treating the whole area in detail.

What were the "covering problems" of [104]?

The starting point was an old problem due to Tibor Gallai related to τ-critical graphs: The *covering number* of graph $G = (V, E)$, usually denoted by $\tau(G)$, is the minimum cardinality of a set $A \subseteq V$ such that every edge of G meets A. (Such a set A is also called *hitting set*.)

$\tau(G)$ is a "hard" combinatorial parameter (ultimately related to the stability number $\alpha(G)$ and the chromatic number $\chi(G)$).

One approach to gain information about the covering number is to consider graphs that are critical with respect to this parameter. A graph $G = (V, E)$ is *τ-critical* if $\tau(G) > \tau(G - e)$ for every edge $e \in E$. Gallai proved in 1961 that every τ-critical graph G satisfies $|V| \leq 2\tau(G)$. So given τ there are only finitely many τ-critical graphs and this implies a "finite basis theorem".

However, a much stronger statement holds. Let us denote the gap in the above inequality by $\delta(G) := 2\tau(G) - |V(G)|$. Then one can observe that, given a τ-critical graph G, the graph G' obtained from G by subdividing an edge of G by an even number of vertices is also τ-critical, and obviously $\delta(G) = \delta(G')$. Gallai conjectured that this is the only operation that does not destroy τ-criticality and that the number of τ-critical graphs with a given value δ is (essentially) finite. And this was the motivation of Lovász for his paper [104] in which he proved this conjecture.

Theorem. *The number of connected τ-critical graphs G with gap $\delta(G) = 2\tau(G) - |V(G)| = \delta$ and all vertex degrees ≥ 3 is at most $2^{5\delta^2}$.*

The proof of this result is complex. In fact, Lovász develops several new tools. The whole paper makes effective use of geometric graphs (where vertices form a matroid). This allows Lovász to carry on a subtle refinement of induction procedures. He makes magnificent use of his vast experience with matchings and generalized factors (this was the subject of his doctoral thesis supervised by T. Gallai) which found its way into his early book on matching theory [115] with M. Plummer. The proof also implicitly contains the "skew Bollobás theorem" (in a matroid setting) about an extremal problem for set intersections of pairs of sets and many other inspiring ideas, in particular, the surprising utilization of exterior algebra. This aspect of the paper [104] also generated a whole new theory.

We shall illustrate the use of exterior algebra by the simpler example of the (Prague) dimension of graphs (treated in another Lovász paper [114]).

It is easy to prove that every graph is an (induced) subgraph of the direct product of complete graphs (the product we introduced in Section 2). The smallest number of such a set of complete graphs is called *dimension* $\dim(G)$ of the graph G.

Thus, $\dim(K_n) = 1$ and $\dim(K_n \times K_n \times \cdots \times K_n) \le t$ (direct product of t copies of K_n).

It is very nice that we have equality here. The proof in [114] is one of the first applications of exterior algebra in combinatorics which was initiated in [104].

Theorem. $\dim(K_n^t) = t$ *for every* $t \ge 1,\ n \ge 2$.

K_2^2 is isomorphic to $K_2 + K_2$ and K_2^t is isomorphic to a perfect matching (i.e., disjoint edges) of size 2^{t-1}.

It suffices to prove $\dim(K_2^t) \ge t$. Given a representation $f : K_2^t \to K_N^d$, we put explicitly:

$f(i) = a_i = (a_i^1, \ldots, a_i^d)$ and $f(i') = b_i = (b_i^1, \ldots, b_i^d)$ (we think of matchings having edges $\{i,\ i'\}$ $i = 1, \ldots, 2^{t-1}$). Clearly all these 2^t vectors are distinct.

The condition that f is an embedding can be then captured by $\prod_{k=1}^d (a_i^k - b_j^k) \ne 0$ if and only if $i = j$, $\prod_{k=1}^d (a_i^k - a_j^k) = 0$, and $\prod_{k=1}^d (b_i^k - b_j^k) = 0$ for all i, j.

But these expressions can be written even more concisely by means of scalar products of vectors in the exterior algebra, i.e., the same technique which we mentioned above in connection with τ-critical graphs. Towards this end, for a vector $x = (x^1, \ldots, x^d)$, we define 2^d-dimensional vectors

$$x^* = \big(x^*(K) \mid K \subseteq \{1, \ldots, d\}\big),\ x^\# = \big(x^\#(K) \mid K \subseteq \{1, \ldots, d\}\big)$$
$$\text{by} \quad x^*(K) = \prod_{i \in K} x^i \quad \text{and} \quad x^\#(K) = \prod_{i \notin K} -x^i.$$

The above expressions can be then written as

$$\prod_{k=1}^{d} (a_i^k - b_j^k) = \sum \Big(\prod_{k \in K} a_i^k \cdot \prod_{k \notin K} -b_j^k \Big| K \subseteq \{1, \ldots, d\} \Big) = \sum_K a_i^*(K) \cdot b_j^\#(K) = a_i^* \cdot b_j^\#.$$

Thus $a_i^* \cdot b_j^\# \ne 0$ iff $i = j$. Similarly we have $b_i^* \cdot a_j^\# \ne 0$ iff $i = j$ while $a_i^* \cdot\ a_j^\# = b_i^* \cdot b_j^\# = 0$ for all $i,\ j$.

It follows then that the set of 2^t vectors $a_i^*, b_j^*,\ 1 \le i, j \le 2^t$ is linearly independent in the vector space of dimension 2^d and thus $t \le d$.

Again, no other (say combinatorial) proof is known.

7 Perfect Graphs and Computational Complexity

L. Lovász. A characterization of perfect graphs. *J. Comb. Theory* 13:95–98, 1972.

L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* 2:253–267, 1972.

This section addresses a particular class of graphs that is tightly connected with four important parameters. For a graph $G = (V, E)$ with vertex set V and edge set E, a *stable set* (also called independent set) is a set of vertices such that no two vertices are adjacent. The largest size of a stable set of vertices is denoted by $\alpha(G)$ and called *stability number*. Similarly, the largest size of a *clique* (mutually adjacent vertices) is denoted by $\omega(G)$ and called *clique number*, the *chromatic number* $\chi(G)$ is the smallest number of stable sets (each stable set is a color class) covering all vertices of G, and the *clique covering number* $\bar{\chi}(G)$ is the smallest number of cliques covering all vertices of G.

If the vertices of a graph are colored so that no two adjacent vertices have the same color then, obviously, the smallest number $\chi(G)$ of colors of such a coloring must be at least as large as the largest number $\omega(G)$ of mutually adjacent vertices, i.e., $\omega(G) \leq \chi(G)$. And similarly, the stability number $\alpha(G)$ cannot be larger than the smallest number $\bar{\chi}(G)$ of cliques covering all vertices of a graph G, i.e., $\alpha(G) \leq \bar{\chi}(G)$.

In the beginning of the 1960s Claude Berge, see [15, 16], called a graph G *perfect* if $\omega(H) = \chi(H)$ holds for all induced subgraphs H of G. In the *complement* $\bar{G}$ of G, two vertices are connected by an edge if and only if they are not connected in G, and thus, $\alpha(G) = \omega(\bar{G})$ and $\chi(G) = \bar{\chi}(\bar{G})$. Berge conjectured:

A graph G is perfect if and only if its complement $\bar{G}$ is perfect.

This conjecture (called *weak perfect graph conjecture*) started a massive search for classes of perfect graphs. Examples are, for instance, bipartite graphs and their line graphs, interval graphs, parity graphs, and comparability graphs; Schrijver [144] describes many of these graphs in detail in Chapter 66, Hougardy [80] gives a survey of these graphs and provides a list of 120 classes. More importantly, intensive attempts to solve the conjecture began. Fulkerson introduced pluperfect graphs in [56] and, developing in [57] the antiblocking theory for this purpose, he came very close to its solution – as he outlines in [58]. Just a lemma (later called *replication lemma*) was missing. Lovász [100] solved the conjecture by proving the replication lemma, pointing out, though, that the more difficult step was done first by Fulkerson. In a subsequent paper, Lovász [101] provided a new characterization of perfect graphs as follows:

Theorem. *A graph $G = (V, E)$ is perfect if the following holds: $\omega(H)\alpha(H) \geq |V(H)|$ for all induced subgraphs $H = (V(H), E(H))$ of G.*

This Theorem immediately implies the weak perfect graph conjecture since the condition given in it is invariant under taking graph complementation. The perfect graph theorem is also a generalization of the well-known theorems of König on bipartite matching and Dilworth on partially ordered sets. It generated particular interest in the characterization of conditions under which the Duality Theorem of linear programming holds in integer variables and initiated related investigations in polyhedral combinatorics.

Due to its importance and elegance, the Lovász's article [100] was reprinted in the collection *Classic Papers in Combinatorics*, edited by I. Gessel and G. C. Rota [60].

The beginning of the 1970s was a particularly productive time period for László Lovász. He was solving one open problem after the other. These years firmly established his international position as the world foremost researcher in graph theory and combinatorics.

As in many other cases, Lovász was not just looking for a proof of the weak perfect graph conjecture, he looked for a more general mathematical setting for which it is possible to prove farther reaching results that imply the conjecture. In [101] Lovász considered a hypergraph approach. We sketch the construction.

Recall that a hypergraph H is a non-empty finite collection of finite sets called edges; the elements of the edges are the vertices of H. The *chromatic index* of a hypergraph H is the least number of colors with which the edges can be colored so that edges with the same color are disjoint. The number of edges containing a given vertex is called the *degree* of the vertex. The largest degree of a vertex of H is called the *degree of* H.

Clearly, the degree of H is a lower bound on the chromatic index of H. Lovász called a hypergraph H *normal* if the degree and the chromatic index are the same for every partial hypergraph of H. Let us call a set T of vertices a *transversal* (or hitting set) if T meets every edge of H and denote its minimum cardinality by $\tau(H)$. (We just point out that $\tau(H)$ is the hypergraph generalization of $\tau(G)$ for graphs discussed in Section 6.) If we denote by $\nu(H)$ the maximum number of edges of H that are pairwise disjoint, then we obviously have $\nu(H) \leq \tau(H)$. Lovász called a hypergraph H *τ-normal* if this inequality holds with equality for all partial hypergraphs of H. He also introduced procedures to associate with every hypergraph H its edge graph $G(H)$ and with every graph G a hypergraph $H(G)$ and proved the following:

Theorem. *A hypergraph H is normal if and only if its edge graph $G(H)$ is perfect; G is perfect if and only if $H(G)$ is normal; H is τ-normal if and only if $\bar{G}(\bar{H})$ is perfect; $\bar{G}$ is perfect if and only if $H(G)$ is τ-normal.*

Corollary. *A hypergraph is normal if and only if it is τ-normal.*

This hypergraph generalization immediately implies the weak perfect graph conjecture.

A side remark: In Section 4 we mentioned the Erdős–Faber–Lovász conjecture. This appears in this context in the following two equivalent forms: (1) The chromatic index of hypergraphs consisting of n edges such that each edge contains n vertices and any two edges have exactly one vertex in common is n. (2) For graphs G consisting of n cliques of size n so that two of these cliques have one vertex in common, $\omega(G)$ equals $\chi(G)$. As indicated before the conjecture is true for large n, see [84].

Berge [16] also conjectured – later called *strong perfect graph conjecture* – that a graph is perfect if and only if it does neither contain an odd cycle nor the complement of an odd cycle as an induced subgraph. After a long sequence of contributions of many researchers, this conjecture was finally solved in 2006 by Chudnovsky, Robertson, Seymour, and Thomas [26].

During the early 1970s computational complexity theory took off, see Wigderson's book [158] for an up-to-date survey. The classes of decision problems that can be solved in polynomial time, denoted by $\mathcal{P}$, and those solvable in nondeterministic polynomial time, denoted by $\mathcal{NP}$, were introduced. S. Cook [29] and L. A. Levin [96] independently showed the existence of $\mathcal{NP}$-*complete* problems, which are decision problems in $\mathcal{NP}$ with the property that, if they can be solved with a polynomial time algorithm, then $\mathcal{P} = \mathcal{NP}$. Whether $\mathcal{P}$ is equal to $\mathcal{NP}$ is one of the great open problems in mathematics and computer science.

Optimization problems can be phrased as decision problems by asking whether, for a given value t, there exists a feasible solution with value at least (or at most) t. If the decision problem associated this way to an optimization problem is $\mathcal{NP}$-complete, the optimization problem is called $\mathcal{NP}$-*hard*. For example, if a graph $G = (V, E)$ with rational weights w_v for every vertex $v \in V$, is given and one wants to find a stable set S in V such that the sum of the weights of the vertices in S is as large as possible, we have a typical combinatorial optimization problem. The associated decision problem asks if there is a stable set whose value is at least t. If this decision problem can be solved in polynomial time, the stable set problem can also be solved in polynomial time by binary search. And vice versa, a polynomial time algorithm for the (weighted) stable set problem would prove that $\mathcal{P} = \mathcal{NP}$.

Karp [86] showed that many graph-theoretical problems, such as computing the value of the four parameters $\alpha(G)$, $\omega(G)$, $\chi(G)$, and $\bar{\chi}(G)$, introduced above, are $\mathcal{NP}$-hard for general graphs G. The immediate question came up: Is that also true for perfect graphs, or can their special structure be exploited to design polynomial time algorithms? This challenge triggered significant developments that we outline later.

Another side remark: Lovász was one of many contributors to one of the most astonishing results in complexity theory, the PCP Theorem. This theorem is the highlight of a long sequence of research on interactive proofs and probabilistically checkable proofs. It states that every decision problem in $\mathcal{NP}$ has probabilistically checkable proofs of constant query complexity using only a logarithmic number of random bits. Nine persons (including Lovász) received the Gödel Prize 2002 "*for the PCP theorem and its applications to hardness of approximation*". A consequence of the PCP Theorem is, for instance, that many well-known optimization problems, including the stable set problem mentioned above and the shortest vector problem for lattices to be introduced subsequently, cannot be approximated efficiently unless $\mathcal{P} = \mathcal{NP}$.

8 The Shannon Capacity of a Graph and Orthogonal Representations

L. Lovász. On the Shannon capacity of graphs. *IEEE Trans. Inform. Theory* 25:1–7, 1979.

L. Lovász. Graphs and geometry. *Amer. Math. Soc.* 2019.

Suppose the vertices of a graph G represent letters of an alphabet and the edges uv of G indicate that the two letters of the alphabet represented by u and v can be confused, e.g., when transmitted over a noisy communication channel. It is obvious that the largest number of one-letter messages that can be sent without danger of confusion is the largest number of vertices mutually not adjacent, i.e., the stability number $\alpha(G)$. Two k-letter words are confusable if their i-th letters, $1 \leq i \leq k$, are confusable or equal.

Let G^k denote the k-th Cartesian product of G. Words with k-letters can be transmitted without danger of confusion if they are unequal and inconfusable in at least one letter. This implies that $\alpha(G^k)$ is the maximum number of inconfusable k-letter words. Forming k-letter words from a stable set of size $\alpha(G)$, one can easily construct $\alpha(G)^k$ inconfusable words. This proves that $\alpha(G)^k \leq \alpha(G^k)$.

Shannon [146] introduced the number

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)},$$

where the second equation follows from $\alpha(G^{k+l}) \geq \alpha(G^k)\alpha(G^l)$. $\Theta(G)$, today called the *Shannon capacity* of G, is a measure of the information that can be transmitted across a noisy communication channel. Shannon proved that $\Theta(G) = \alpha(G)$ for graphs which can be covered by $\alpha(G)$ cliques. Perfect graphs have this property and thus belong to this class. How can one determine $\Theta(G)$ in other cases? Lovász, see [106], invented an ingenious upper bound on the Shannon capacity as follows:

Let $G = (V, E)$ be a graph. An *orthonormal representation* of G is a sequence $(u_i \mid i \in V)$ of $|V|$ vectors $u_i \in \mathbb{R}^N$, where N is some positive integer, such that $\|u_i\| = 1$ for all $i \in V$ and $u_i^T u_j = 0$ for all pairs i, j of nonadjacent vertices. Trivially, every graph has an orthonormal representation (just take all the vectors u_i mutually orthogonal in $\mathbb{R}^V$). Figure 7 shows a less trivial orthonormal representation of the pentagon C_5 in $\mathbb{R}^3$. It is constructed as follows. Consider an umbrella with five ribs of unit length (representing the nodes of C_5) and open it in such a way that nonadjacent ribs are orthogonal. Clearly, this can be achieved in $\mathbb{R}^3$ and gives an orthonormal representation of the pentagon. The central handle (of unit length) is also shown.

Where $(u_i \mid i \in V), u_i \in \mathbb{R}^N$, ranges over all orthonormal representations of G and $c \in \mathbb{R}^N$ over all vectors of unit length, let

$$\vartheta(G, w) := \min_{\{c,(u_i)\}} \max_{i\in V} \frac{w_i}{(c^T u_i)^2}.$$

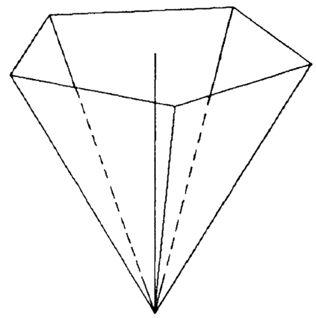


Fig. 7 Orthonormal representation of the 5-cycle in $\mathbb{R}^3$

The quotient has to be interpreted as follows. If $w_i = 0$ then we take $w_i/(c^T u_i)^2 = 0$ even if $c^T u_i = 0$. If $w_i > 0$ but $c^T u_i = 0$ then we take $w_i/(c^T u_i)^2 = +\infty$.

Lovász proved that, if the vertex weights w_i above are all equal to 1 and G is the pentagon graph C_5, i.e., the 5-cycle, then the value of $\vartheta(G,w)$ is $\sqrt{5}$ and equal to the Shannon capacity $\Theta(C_5)$ of C_5.

This looks like a tiny achievement, but at present, this is the only known Shannon capacity of a non-perfect graph. In fact, the complexity of determining the Shannon capacity of a general graph is today still open. Much more important, Lovász provided several different characterizations of the function ϑ (called the *Lovász ϑ-function*) that became, as we show later, important ingredients for proving that the four graph parameters $\alpha(G)$, $\omega(G)$, $\chi(G)$, and $\bar{\chi}(G)$ can be computed in polynomial time for perfect graphs G.

In his recent book [113], Lovász investigated the representation of graphs as geometric objects in great depth. His main message is that such representations are not merely a way to visualize graphs, but important mathematical tools. The range of applications is wide. We mention three examples: rigidity of frameworks and mobility of mechanisms in engineering, learning theory in computer science, the Ising and Fortuin–Kasteleyn model, and conformal invariance in statistical physics. Orthogonal representations of graphs are treated in Chapters 10 to 12. Lovász shows that orthogonal representations are, in addition to the stability and chromatic number, related to several fundamental properties of graphs such as connectivity and tree-width. Among many other aspects, he also discusses a quantum version of the Shannon capacity problem, as well as two further interesting applications of orthogonal representations to the theory of hidden variables and in the construction of strangely entangled states. These are exciting topics in quantum physics that we cannot cover here.

9 The Ellipsoid Method

P. Gács, L. Lovász. Khachiyan's algorithm for linear programming. *Math. Prog. Study* 14: 61–68, 1981.

One of the major open complexity problems in the 1970s was the question whether linear programs (LPs) can be solved in polynomial time. The simplex algorithm did (and still does) work well in practice, but for all known variants of this algorithm, there exist sequences of LP-instances for which the running time is exponential. In 1979 Khachiyan indicated in [87] how the ellipsoid method, an algorithm devised for nonlinear nondifferentiable optimization based on work of Shor and Yudin and Nemirovskiĭ, can be modified to check the feasibility of a system of linear inequalities in polynomial time. Employing binary search or a sliding objective function technique, this implies that linear programs are solvable in polynomial time. Linear programs arise almost everywhere in industry, and their fast solution is of economic importance. Thus, Khachiyan's achievement received significant attention in the nonscientific media; it even made it on the front page of the New York Times on November 7, 1979. Most of these statements, though, were exaggerations or misinterpretations.

We sketch the method. Let P be polyhedron defined by a system of linear inequalities $Ax \leq b$. We assume that P is full-dimensional or empty; and for simplifying the exposition, we also assume that P is bounded, i.e., a polytope. The ellipsoid method utilizes the following facts. Given $Ax \leq b$ with rational coefficients, then numbers r and R can be computed in time polynomial in the encoding length of A and b with the following properties. If P is nonempty, the ball B of radius R around the origin contains P, and P contains a ball S of radius r.

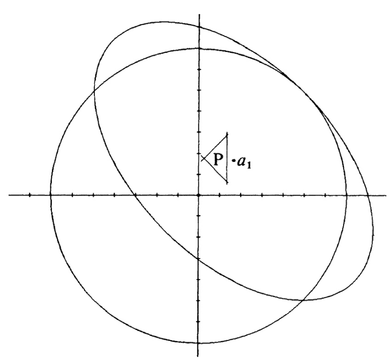


Fig. 8 The first step of the ellipsoid method

The basic ellipsoid method begins with the ball B and center $a_0 = 0$ as initial ellipsoid E_0. In a general step it checks whether the center a_k of the current ellipsoid E_k, $0 \leq k$, is contained in P. If this is the case, a point in P is found and $Ax \leq b$ is feasible. If not, there must be an inequality in the system $Ax \leq b$ that is violated by a_k. Using this inequality, a new ellipsoid E_{k+1} is computed that contains P and has a volume that is – by a constant shrinking rate – smaller than the volume of the previous ellipsoid E_k (cf. Fig. 8). This way a sequence of points a_k and shrinking ellipsoids E_k is created. Using variants of the formulas for determining the Löwner–John-ellipsoid of a convex body, one can prove that the volume shrinking rate satisfies $\mathrm{vol}(E_{k+1})/\mathrm{vol}(E_k) < e^{-1/(2n)} < 1$ and that the ellipsoid method either discovers a point in P or, after a number N of steps that is polynomial in the encoding length of A and b, the ellipsoid E_N has a volume that is smaller than that of the small ball S. This can only happen in case P is empty. All computations carried out can be made with rational numbers of polynomial size in such a way that nonemptiness of P is certified by a finding a feasible solution or the emptiness of P is guaranteed by the mentioned volume argument, see [69] for details.

This method was a total surprise for the linear programming community. A polynomial time termination proof employing shrinking volumes, the combination of geometric and number theoretic "tricks" (e.g., making a low-dimensional polyhedron full-dimensional, reduction to the bounded case, careful rounding of the real numbers that appear in the update-formulas, and various necessary estimation processes) puzzled the LP-specialists. The brief article by Khachiyan (four pages), written in Russian, needed interpretation. One of the first papers explaining the approach and adding missing details was a preprint by Gács and Lovász [59]. It appeared in the fall of 1979 (and was published in 1981). This paper made Khachiyan's important contribution accessible to a wide audience and had a significant bearing on the boom of follow-up research on the ellipsoid method.

The ellipsoid method, though provably a polynomial time algorithm, performs poorly in practice. Its appearance, however, sparked successful research efforts that led to new LP-algorithms, based on various ideas from nonlinear programming, often also influenced by differential and other types of geometry, that are theoretically and practically fast. They run under the names interior point or barrier methods. New implementations of the simplex algorithm improved its performance significantly as well. The ellipsoid method, on the other hand, turned out to have fundamental theoretical power as an elegant and versatile tool to prove the polynomial time solvability of many geometric and combinatorial optimization problems. The next chapter has details.

10 Oracle-Polynomial Time Algorithms and Convex Bodies

M. Grötschel, L. Lovász, A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* 1:169–197, 1981.

M. Grötschel, L. Lovász, A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, Springer, 1988.

In a general step of the ellipsoid method, one has to verify that the center of the current ellipsoid is in the polyhedron $P = \{x \in \mathbb{R}_n \mid Ax \leq b\}$. This is usually done by substituting the center into the given inequality system $Ax \leq b$. A reasonable idea is to replace this substitution by an algorithm that checks feasibility and provides a violated inequality in case the center is not in P. Two cases, relevant in real-world applications, where this generalization might be helpful come immediately into mind.

The first case is the traditional transformation of combinatorial optimization problems into linear programs. The idea is, for a given combinatorial optimization problem, to define the convex hull of all incidence vectors of feasible solutions and to try to find a linear system describing this polytope, at least partially. The number of facets of such polytopes is often exponentially large in the encoding length of the combinatorial problem. This holds for $\mathcal{NP}$-hard problems and even for some problems solvable in polynomial time. One such instance is the matching problem. This is implied by the result of Rothvoss [142] that the matching problem has "exponential extension complexity". Substituting the ellipsoid center into a linear system of exponential size makes the running time of ellipsoid algorithm exponential. Can one replace the substitution by a polynomial time algorithm?

The second case are convex sets and is even more demanding. Convex sets are intersections of potentially infinitely many halfspaces. Can one optimize over exponentially many linear inequalities in polynomial time?

The roots of this research program were laid by Grötschel, Lovász, and Schrijver in [66] and were fully worked out in [69]. The results were the starting point of what

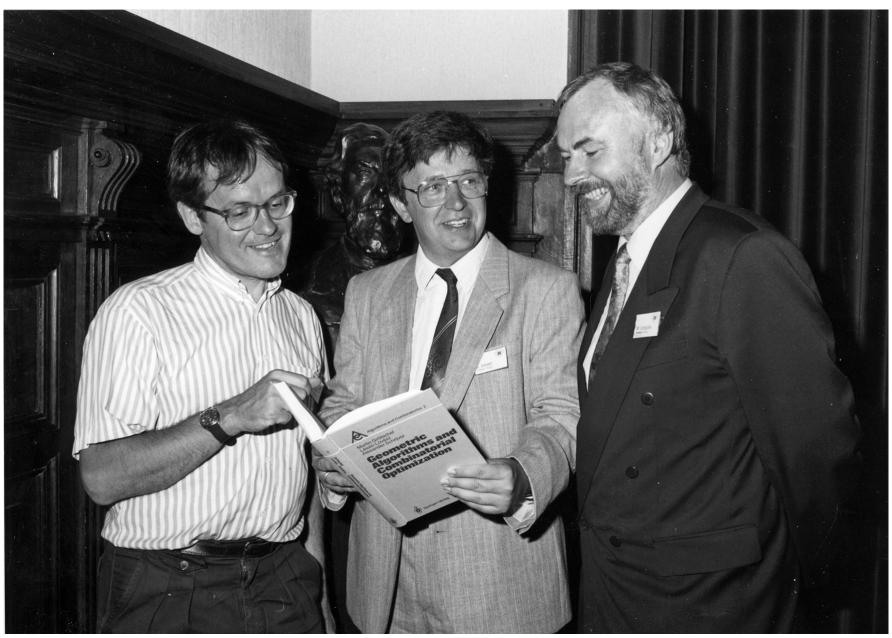


Fig. 9 A. Schrijver, L. Lovász, M. Grötschel at the International Symposium on Mathematical Programming in Amsterdam, 1991 (Photo: Nationaal Foto-Persbureau B. V.)

Gritzmann and Klee [64] called an algorithmic theory of convex bodies, or briefly, *computational convexity*. We outline important steps of this approach.

Suppose now that we have some convex set $K \subseteq \mathbb{R}_n$ and we want to obtain information about properties of K. Let us formulate three questions that are typical in this context:

The Strong Optimization Problem (SOPT). *Given a vector $c \in \mathbb{R}^n$, find a vector $y \in K$ that maximizes $c^T x$ on K, or assert that K is empty.*

The Strong Separation Problem (SSEP). *Given a vector $y \in \mathbb{R}^n$, decide whether $y \in K$, and if not, find a hyperplane that separates y from K; more exactly, find a vector $c \in \mathbb{R}^n$ such that $c^T y > \max\{c^T x \mid x \in K\}$.*

The Strong Membership Problem (SMEM). *Given a vector $y \in \mathbb{R}^n$, decide whether $y \in K$.*

It is clear that the strong membership problem can be solved if either the strong optimization or the strong separation problem can be solved. What about the other way around? And what do we have to assume about K, what is the input length of K, and how do we estimate running times? Before addressing these issues, we observe that, if we allow arbitrary convex sets K, the unique solution of an optimization problem over K may have irrational coordinates. To deal with such issues we have to allow margins and to accept approximate solutions. Let us define, for the Euclidean norm and a rational number $\epsilon > 0$,

$$S(K,\epsilon) := \left\{x \in \mathbb{R}^n \mid \|x-y\| \leq \epsilon \text{ for some } y \in K\right\},\ S(K,-\epsilon) := \left\{x \in K \mid S(x,\epsilon) \subseteq K\right\}.$$

Points in $S(K,\epsilon)$ can be viewed as "almost in K", while points in $S(K,-\epsilon)$ as "deep in K". The exactness requirements of the strong problems above can be softened as follows:

The Weak Optimization Problem (WOPT). *Given a vector $c \in \mathbb{Q}^n$ and a rational number $\epsilon > 0$, either*

(i) *find a vector $y \in \mathbb{Q}^n$ such that $y \in S(K,\epsilon)$ and $c^T x \leq c^T y + \epsilon$ for all $x \in S(K,-\epsilon)$ (i.e., y is almost in K and almost maximizes $c^T x$ over the points deep in K), or*

(ii) *assert that $S(K,-\epsilon)$ is empty.*

The Weak Separation Problem (WSEP). *Given a vector $y \in \mathbb{Q}^n$ and a rational number $\delta > 0$, either*

(i) *assert that $y \in S(K,\delta)$, or*

(ii) *find a vector $c \in \mathbb{Q}^n$ with $\|c\|_\infty = 1$ such that $c^T x \leq c^T y + \delta$ for every $x \in S(K,-\delta)$ (i.e., find an almost separating hyperplane).*

The Weak Membership Problem (WMEM). *Given a vector $y \in \mathbb{Q}^n$ and a rational number $\delta > 0$, either*

(i) *assert that $y \in S(K,\delta)$, or*

(ii) *assert that $y \notin S(K,-\delta)$.*

We are interested in the algorithmic relations between these problems. To do this we make use of the *oracle algorithm concept*. An *oracle* is a device that solves a certain problem for us. Its typical use is as follows. We feed some input string to the oracle, and the oracle returns another string specifying the solution (of which we hope that it helps solving our original problem). We make no assumption on the way the oracle finds its solution. An oracle algorithm is an algorithm in the usual sense whose power is enlarged by allowing querying an oracle and using the oracle answer for determining its next computational steps.

If a query to and an answer of the oracle are counted as one step each, we can determine the running time of an oracle algorithm in the usual way. The output of the oracle may, however, be huge so that reading it may take exponential time. Since our aim is to design polynomial time algorithms, we require that for every oracle we have a polynomial q, such that for every query of encoding length at most l, the answer of the oracle has length at most $q(l)$. Under this assumption we say that an oracle algorithm has *oracle-polynomial running time* if its usual running time plus the running time of the interaction with the oracle is bounded by a polynomial in the input length of the original problem. A consequence of this set-up is that, if an oracle can be realized by a polynomial time algorithm on a real computational device, an oracle-polynomial algorithm is in fact a polynomial time algorithm in the usual sense.

For ease of exposition, we restrict ourselves to considering convex bodies K only. A convex set $K \subseteq \mathbb{R}^n$ that is compact and has dimension n is called *convex body*. To perform computations, we have to assume that the convex body K is given by a mathematical description. Let us briefly call it $\mathrm{Name}(K)$. Then the encoding length of K is defined as the dimension n plus the encoding length of $\mathrm{Name}(K)$. To determine the algorithmic relations between the problems above, we assume that a convex body is given by an oracle for the solution of one of the problems and we investigate whether any of the other problems can be solved employing the oracle. The running times are measured as usual in the size of the input. This is, in the cases described here, the encoding length of K (as defined above) to which we have to add, if they appear in the problem statement, the following: the encoding lengths of the parameters ϵ and δ, the encoding lengths of the objective function c and the vector y, and moreover the encoding lengths of the additional data (the radii r and R, and the center a_0 of a ball) appearing in the statements of the theorems. The following was proved in [69]:

Theorem. (a) *There exists an-oracle polynomial time algorithm that solves the weak membership problem for every convex body K in $\mathbb{R}^n$ given by a weak optimization or a weak separation oracle.*

(b) *There exists an oracle-polynomial time algorithm that solves the weak separation problem for every convex body K in $\mathbb{R}^n$ given by a weak optimization oracle.*

(c) *There exists an oracle-polynomial time algorithm that solves the weak optimization problem for every convex body K in $\mathbb{R}^n$ given by a weak separation algorithm, provided a radius $R > 0$ of a ball around the origin containing K is given as well.*

(d) *There exists an oracle-polynomial time algorithm that solves the weak optimization problem for every convex body K in $\mathbb{R}^n$ given by a weak membership algorithm, provided the following data are given as well: a vector a_0 and a radius $r > 0$ such that $S(a_0, r) \subseteq K$, and a radius $R > 0$ with $K \subseteq S(0, R)$.*

This theorem establishes the oracle-polynomial time equivalence of WOPT, WSEP, and WMEM under mild additional assumptions. Moreover, the oracle-polynomial time equivalence of the strong versions SOPT, SSEP, and SMEM of these problems can be derived from the results above (assuming, of course, that K is given such that exact answers are possible). One can prove on the other hand that, if we would drop one of the additional requirements in the theorem such as the knowledge of radii r or R or the vector a_0, it is impossible to derive oracle-polynomial time algorithms.

A consequence of the last result, see [66] and [69], is the polynomial time solvability of convex function minimization – in the following weak sense:

Theorem. *There exists an oracle-polynomial time algorithm that solves the following problem:*
Input: *A convex body K given by a weak membership oracle, a rational number $\epsilon > 0$, radii r, $R > 0$, a vector a_0 such that $S(a_0, r) \subseteq K \subseteq S(0, R)$, and a convex function $f : \mathbb{R}^n \to \mathbb{R}$ given by an oracle that, for every $x \in \mathbb{Q}^n$ and $\delta > 0$, returns a rational number t such that $|f(x) - t| < \delta$.*
Output: *A vector $y \in S(K, \epsilon)$ such that $f(y) < f(x) + \epsilon$ for all $x \in S(K, -\epsilon)$.*

This is the first polynomial time solvability result for convex minimization.

11 Polyhedra, Low Dimensionality, and the LLL Algorithm

M. Grötschel, L. Lovász, A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring Polynomials with rational coefficients. *Mathematische Annalen* 261(4):515–534, 1982.

The Abel prize citation states (correctly, of course): "*The LLL algorithm is only one among many of Lovász's visionary contributions*". It may be surprising to learn that its invention was triggered by a technical problem arising in the analysis of the ellipsoid method. We explain its origin and usefulness in this context.

Since square roots appear in the update formulas defining the ellipsoid method, computing with irrational numbers is unavoidable. Careful rounding is necessary to reach the desired approximation of an optimal value or solution. In various applications exact solutions can in fact be obtained by appropriate rounding. In integer programming, e.g., the solution vectors are required to have integral entries, and if the objective function is integral, the optimal value v^* is integral as well. If one can tune the ellipsoid method so that it guarantees to find an approximation v of the optimal value v^* such that $|v - v^*| < 1/2$, then one can simply round v to the next integer to find the true optimum value. Such considerations are the key to pass from "weak solutions" to "strong solutions", i.e., derive exact from approximate results. This straightforward rounding unfortunately is often not sufficient.

We sketch the case of optimizing a linear objective function over a polytope $P \subseteq \mathbb{R}^n$. We say that P has *facet-complexity at most* φ if there exists a system of inequalities with rational coefficients that has solution set P and such that the encoding length of each inequality of the system is at most φ. No assumption about the number of inequalities is made. Let us define the encoding length of P to be $n+\varphi$, call such a polyhedron *well-described,* and denote it by $(P;\ n,\varphi)$. One can prove that the encoding length of each vertex of $(P;\ n,\varphi)$ is at most $4n^2\varphi$ and that, if P is full-dimensional, P contains a ball B_P with radius $2^{-7n^3\varphi}$.

To illustrate the annoying "technical problem" that triggered the invention of the LLL algorithm, let us consider a well-described polytope $P \subseteq \mathbb{R}^n$ that is not full-dimensional; for ease of exposition, say P has dimension $n-1$. The ellipsoid method would not work in this case. To get around this problem, one needs to carefully blow P up to a polytope P' that contains P and is full-dimensional such that running the ellipsoid method on P' approximately delivers the desired result for P. This can be done but is technically tedious and requires ugly pre- and post-processing.

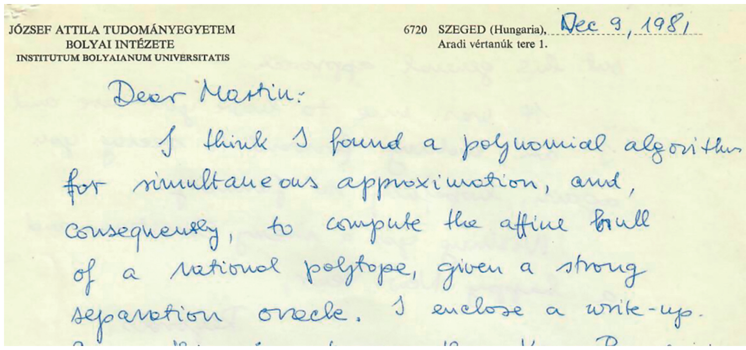
Let us instead make a bold step and run the ellipsoid method on P directly. We suppose P is given by a separation oracle. Since P is low-dimensional it is highly unlikely that the ellipsoid method finds a feasible solution in one of its iterations. After a number N of iterations that is polynomial in $n+\varphi$, the N-th ellipsoid E_N contains P and has a volume that is smaller than the volume of B_P, the ball P would contain if P were full-dimensional. This is contradictory. The basic ellipsoid method, assuming a full-dimensional polytope P is given, would conclude now that P is empty. But E_N contains information that one may be able to employ.

Let $H = \{x \in \mathbb{R}^n \mid a^T x = \alpha\}$ be the unique hyperplane containing P. Then $a^T x = \alpha$ is the (up to scaling) unique equation defining H. The last ellipsoid E_N, having such a small volume, must obviously be very "flat" in the direction perpendicular to H. In other words, the symmetry hyperplane F belonging to the shortest axis of E_N must be very close to H. Is it possible to find $a^T x = \alpha$ by rounding the coefficients of the linear equation defining this symmetry hyperplane F? A positive answer would be an elegant way to avoid the blow-up mentioned and the numerical problems associated with it.

The authors of [66] and [69] were at this point in the fall of 1981 and realized that such a rounding can be done – in principle – using the following classical theorem of Dirichlet [36] on the existence of a solution of a simultaneous Diophantine approximation problem.

Theorem. *Given any real numbers* $\alpha_1,\ldots,\alpha_n$ *and* $0 < \epsilon < 1$*, there exist integers* $p_1,\ldots,p_n$*, and* q *such that* $1 < q < \epsilon^{-n}$ *and* $|\alpha_i - p_i/q| < \epsilon/q$ *for* $i = 1,\ldots,n$.

No polynomial algorithm is known to compute such integers. And at the end of their writing session, no progress was achieved. About three months later the following letter from L. Lovász arrived:


JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZETE
INSTITUTUM BOLYAIANUM UNIVERSITATIS

6720 SZEGED (Hungaria), Dec 9, 1981
Aradi vértanúk tere 1.

Dear Martin:

I think I found a polynomial algorithm for simultaneous approximation, and, consequently, to compute the affine hull of a rational polytope, given a strong separation oracle. I enclose a write-up.

Fig. 10 Beginning of a letter from L. Lovász

Lovász approached the approximation problem via the consideration of (integral) lattices. If $\{b_1, \dots, b_n\}$ is a basis of $\mathbb{R}^n$, then the set $L = L(b_1, \dots, b_n)$ that is generated by taking all integral linear combinations of the vectors b_i is called a *lattice* with basis $\{b_1, \dots, b_n\}$. Integral lattices have been studied in number theory for a very long time (with contributors such as Gauss, Minkowski, Landau, and many others). Clearly, a lattice may have different bases, and it may be interesting to find a "minimal basis" $\{a_1, \dots, a_n\}$ of L, i.e., a basis such that the product of the norms of the a_i is as small as possible. However, this problem is $\mathcal{NP}$-hard. Lovász introduced the quite technical notion of a *reduced basis*, which we do not explain here, that is a weak form of a minimal basis and proved:

Theorem. *There is a polynomial time algorithm that, for any given linearly independent vectors $\{b_1, \dots, b_n\}$ in $\mathbb{Q}^n$, finds a reduced basis of the lattice $L(b_1, \dots, b_n)$.*

The algorithm, called LLL algorithm, to achieve this starts with the Gram–Schmidt orthogonalization and then performs carefully designed exchange operations. Proving polynomiality requires not only controlling the number of steps, but in particular, the estimation of the encoding lengths of all numbers appearing in the course of the algorithm. A consequence of this algorithm is the following weak form of Dirichlet's theorem.

Theorem. *There exists a polynomial time algorithm that, given rational numbers $\alpha_1, \dots, \alpha_n$ and $0 < \epsilon < 1$, computes integers $p_1, \dots, p_n$, and q such that and $1 \le q \le 2^{n(n+1)/4}\epsilon^{-n}$ and $|\alpha_i q - p_i| < \epsilon$ for $i = 1, \dots, n$.*

This algorithm, based on computing a reduced basis, made it possible to compute via simultaneous Diophantine approximation, the coefficients of the equation $a^T x = \alpha$ defining the hyperplane H containing the well-described polytope $(P;\ n, \varphi)$ as indicated above. By iterating this process, the affine hull of any lower dimensional polytope can be determined in oracle-polynomial time.

For well-described polyhedra $(P;\ n, \varphi)$, the restriction to the bounded case can also be dropped, and one can show the following:

Theorem. *Any of the following three problems:*
– *strong separation*
– *strong violation*
– *strong optimization*
can be solved in oracle-polynomial time for any well-described polyhedron $(P;\, n, \varphi)$ *given by an oracle for any of the other two problems.*

For a linear program given by a system of rational linear inequalities, the strong separation problem can be trivially solved by substituting a given rational vector y into the inequalities, i.e., linear programs can be solved in polynomial time.

Employing the LLL algorithm and results of András Frank and Éva Tardos [53] one can, in fact, derive a general result about optimization problems for polyhedra and their dual problems in strongly polynomial time. *Strongly polynomial* means that the number of elementary arithmetic operations to solve an optimization problem over a well-described polyhedron and to solve its dual problem does not depend on the encoding length of the objective function. More precisely, the following can be shown:

Theorem. *There exist algorithms that, for any well-described polyhedron* $(P;\, n, \varphi)$ *specified by a strong separation oracle, and for any given vector* $c \in \mathbb{Q}^n$*,*
(a) *solve the strong optimization problem* $\max\{c^T x \mid x \in P\}$*, and*
(b) *find an optimum vertex solution of* $\max\{c^T x \mid x \in P\}$ *if one exists, and*
(c) *find a basic optimum standard dual solution if one exists.*
The number of calls on the separation oracle, and the number of elementary arithmetic operations executed by the algorithms are bounded by a polynomial in φ*. All arithmetic operations are performed on numbers whose encoding length is bounded by a polynomial in* φ *and the encoding length of the objective function vector* c*.*

An important application of this theorem is that one can turn many polynomial time combinatorial optimization algorithms into strongly polynomial algorithms.

Summarizing: The search for an elegant proof that avoids tedious numerical estimates was the driving force for the invention of the LLL algorithm.

12 The LLL Algorithm and its Consequences

A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen* 261(4):515–534, 1982.

The basis reduction algorithm by L. Lovász to solve a problem, that initially looked like a technicality, had a significant impact on the book [69] as outlined in Section 11. Its deep impact on other fields came really unexpected, even for Lovász himself as can be inferred from his letter, see Fig. 11.

We consider this as one of the occasional miracles in mathematics where a result that was prompted by the desire to find an elegant solution for a technical detail has consequences that are simply beyond imagination.

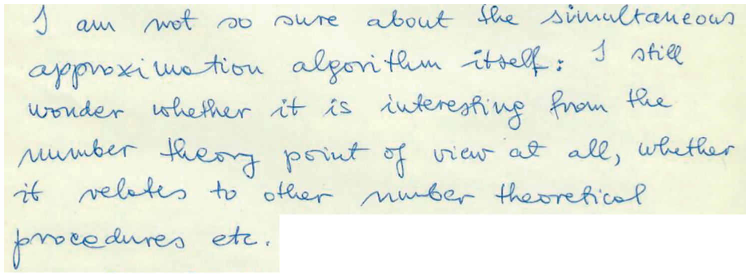

I am not so sure about the simultaneous approximation algorithm itself: I still wonder whether it is interesting from the number theory point of view at all, whether it relates to other number theoretical procedures etc.

Fig. 11 Cutout from a Lovász letter

Lovász informed not only the coauthors Grötschel and Schrijver of his book [69] about his achievement, but also Hendrik Lenstra. Employing tools from the geometry of numbers, Hendrik had (briefly before) made the substantial discovery that integer programs (IPs) can be solved in polynomial time when the dimension is fixed. Concerning this, he was in discussion with Lovász who pointed out that some of the steps of Hendrik's IP-algorithm could be improved, see [95].

Hendrik got excited about the news because his brother Arjen was (together with two fellow students) about to implement a method to factor univariate polynomials over algebraic number fields. Zassenhaus had suggested to use the Berlekamp–Hensel approach for this which, however, could be "*very, very much exponential*" according to Arjen. A few days after Lovász's letter had arrived, Hendrik became convinced that the basis reduction algorithm implies that there is a polynomial time algorithm for factorization in the ring $\mathbb{Q}(X)$ of univariate polynomials over the rational numbers. At that time this looked inconceivable as one did not (and still does not) know a polynomial time algorithm for finding the factors of an integer. After working out the details, Hendrik's observation turned out to be true. The two Lenstra brothers and Lovász combined their contributions and wrote the joint paper [94]. Believing that polynomial time factoring of polynomials over the rational numbers (an unexpected result) is the most important contribution of their work, they agreed to mention only this aspect in the paper title. The full story of this cooperation is nicely described in the article of I. Smeets [151].

It turned out that basis reduction has applications that reach much further than linear programming or polynomial factorization. It is beyond the scope of this article to highlight here the wide range of applications of the basis reduction algorithm which – in contrast to the ellipsoid method – is usable in practice. We mention two concrete examples.

Odlyzko and te Riele [137] used the basis reduction algorithm to disprove the Mertens conjecture, a conjecture standing in number theory since 1897, which – if true – would have implied the Riemann hypothesis. This disproof was surprising as there was extensive computational evidence that the Mertens conjecture is true.

Lagarias and Odlyzko [92] employed the lattice basis reduction algorithm to launch a polynomial time attack on knapsack-based public-key cryptosystems which made these cryptosystems unsafe.

The LLL algorithm, in fact, created a revolution in cryptography. It is known that the widely used public-key schemes such as the RSA or elliptic-curve cryptosystems can be defeated if Shor's quantum polynomial time factoring algorithm can be implemented on a quantum computer. Many cryptographers are convinced that certain lattice problems cannot be solved efficiently. Based on this, some lattice-based constructions appear to be resistant to attack by both classical and quantum computers. For surveys see Regev [141] or Micciancio and Goldwasser [131]. The National Institute of Standards and Technology (NIST) and other institutions are currently preparing cryptography standards for the post-quantum era. The first Quantum-Resistant Cryptographic Algorithms were announced by NIST in July 2022. Lattices play a major role here, and lattice basis reduction algorithms have become standard tools to test the security of cryptosystems.

Instead of attempting to comprehensively document the impact of Lovász's work on basis reduction, we point to the book by Nguyen and Vallée [136] entitled *The LLL Algorithm: Survey and Applications* which consists of a collection of broad overviews of fields where the LLL algorithm is employed. Chapters, written by specialists in the respective fields, cover, for instance, applications in number theory, Diophantine approximation, integer programming, cryptography, geometry of provable security, inapproximability, and improvements of the LLL algorithm. A reviewer of this book wrote:

> The LLL algorithm embodies the power of lattice reduction on a wide range of problems in pure and applied fields [...] [and] the success of LLL attests to the triumph of theory in computer science.

Finally, the algorithm Lovász designed to find a reduced lattice basis is usually called LLL algorithm, because it appeared in a paper written by three authors whose last names starts with L. Of course, the Lenstra brothers do not claim that it is their invention, they also attribute it to L. Lovász. But LLL algorithm has become the usually employed name of the algorithm.

13 Cutting Planes and the Solution of Practical Applications

> M. Grötschel, L. Lovász, A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, Berlin, 1988.

László Lovász has, in addition to inventing beautiful theory, designed many algorithms, concentrating particularly on polynomial time algorithms. The theory and the algorithms Lovász developed had significant impact on computational practice. Chapter 8 of [69] "Combinatorial Optimization: A Tour d'Horizon" is a highly condensed overview of the applicational potential that arises from combinations of the many insights provided by the ellipsoid method, the LLL algorithm, and further ideas. These have contributed to the astonishing computational success stories that evolved in the last thirty to forty years in combinatorial optimization. We sketch some of these aspects.

In combinatorial optimization, a typical approach is, as indicated before, to attack a problem by transforming it into a linear programming problem with integer variables.

Take the traveling salesman problem, for instance. Given a complete graph $G = (V, E)$ on n vertices and a distance c_e for every edge $e \in E$, we look for a Hamiltonian cycle (briefly: tour) of minimum length. If H is a tour, let $x^H \in \mathbb{R}^E$ be its *incidence vector*, i.e., the e-th component x_e^H of x^H is equal to 1 if $e \in H$, otherwise it is 0. The traveling salesman polytope $\mathrm{TSP}(G)$ of G is the convex hull of all incidence vectors of tours in G. $\mathrm{TSP}(G)$ is a polytope in $\mathbb{R}^{n(n-1)/2}$. To apply the linear programming approach, we now have to find a linear inequality system, so that the integral solutions of the linear program are exactly the incidence vectors of tours. Such linear programs are called *LP-relaxations*. Let $\delta(W)$ denote the set of edges in E with one endvertex of e in W and the other in $V \setminus W$, and let $x(\delta(W))$ denote the sum over all variables x_e with $e \in \delta(W)$. It is well known that the following linear program:

$$
\begin{aligned}
0 \le x_e &\le 1 \quad \text{for all } e \in E \\
x\big(\delta(\{w\})\big) &= 2 \quad \text{for all } w \in V \\
x\big(\delta(W)\big) &\ge 2 \quad \text{for all } W \subseteq V \text{ with } 2 \le |W| \le |V| - 2
\end{aligned}
$$

is an LP-relaxation of the TSP. The third type of inequalities is called *subtour elimination constraints*.

Let us call the polytope defined by the linear system above $\mathrm{TSPLP}(G)$. All vertices of the traveling salesman polytope $\mathrm{TSP}(G)$ are vertices of $\mathrm{TSPLP}(G)$. But $\mathrm{TSPLP}(G)$ has many nonintegral vertices as well. About 2^n inequalities define $\mathrm{TSPLP}(G)$. This renders the straightforward LP-solution approach hopeless. The facet complexity φ of $\mathrm{TSPLP}(G)$, however, is small since the entries of every inequality or equation are only 0 or 1 and the right-hand sides are 0, 1, or 2. Thus the facet complexity of $\mathrm{TSPLP}(G)$ is linear in the number of variables $|E| = n(n-1)/2$. Due to the oracle-polynomial time equivalence of strong separation and strong optimization, linear programs over $\mathrm{TSGLP}(G)$ can be solved in polynomial time – provided, given a vector $y \in \mathbb{Q}^E$, one can find a fast separation algorithm for the subtour elimination constraints.

This can in fact be done, as was observed by Hong [77]. One assigns the value y_e to every edge $e \in E$ as a capacity and computes (this can be done quickly) a minimum nonempty cut $\delta(W^*)$ in this capacitated graph $G = (V, E)$. If $y(\delta(W^*)) < 2$, a violated inequality is found, otherwise y satisfies all subtour elimination constraints. This is an example of a linear program appearing in many practical applications with an exponential number of inequalities that, nevertheless, can be solved in polynomial time. An optimal solution of a linear program over $\mathrm{TSPLP}(G)$ is usually nonintegral but provides a very good lower bound on the optimum TSP-value in practice. Finding a provably optimal solution needs additional effort, though.

In 1954 Dantzig, Fulkerson, and Johnson [33] proposed in a seminal paper to solve combinatorial optimization problems such as the traveling salesman problem by starting with some LP-relaxation, checking whether the optimum solution y is the incidence vector of a tour (in this case the problem is solved), and if not searching for inequalities valid for TSP(G) that are violated by y, adding these to the current LP as *cutting planes*, and to continue. This was one of the first proposals to solve linear and integer programs using cutting planes in an iterative process. The cutting plane search in this case was done manually, the LPs were solved by the simplex method. Four years later Gomory [61] invented an automatic cutting plane generation scheme (called *Gomory cuts*) for which he could prove finite termination. This looked like a promising approach to solve integer programs.

However, the computer implementations of this and related approaches in the 1960s and 1970s were not successful in practice. Moreover, theoretical results of Chvátal [27] and others showed that there are series of examples for which the number of cutting plane additions cannot be effectively bounded. Hoping for unimportance of these negative aspects in real-world applications, the idea came up in the 1970s to study combinatorial optimization problems of practical relevance and to look for cutting planes that define facets of the investigated polytopes. These are cuts that cut as deep as possible. The first implementations employing a combination of manual and heuristic searches for facet defining cutting planes at the end of the 1970s indicated practical success. Soon after, the ellipsoid method theory with the principle of polynomial time equivalence of optimization and separation was developed and demonstrated that this approach is a viable idea, and that linear optimization over exponentially large systems of linear inequalities is possible in polynomial time – at least theoretically.

Despite serious attempts, no implementation of the ellipsoid method has shown satisfactory numerical performance in computational practice. By replacing it with new implementations of the dual simplex algorithm, the theoretical polynomial time termination is lost, but astonishing computational results were achieved by many researchers in combinatorial optimization. Of course, lots of additional features (such as presolve techniques, heuristic primal and dual searches, branch and bound, robust numerics, etc.) were implemented as well. The new insights gave a significant push to the theoretical and applied part of combinatorial optimization. Problems with many industrial applications such as linear ordering; set partitioning and packing; knapsack; clustering; various types of matching; connectivity; path, flow and other network problems; max cut; unconstrained Boolean quadratic programming; stable sets; several variations of coloring; and vehicle and passenger routing could be solved for instances of practically relevant sizes. The discovery of new classes of facets and fast separation procedures (exact and heuristic) has been an important ingredient of this solution methodology. To indicate at least one example of practically useful separation algorithms we mention the paper [138] of Padberg and Rao that describes sophisticated and fast separation algorithms for various ramifications of the matching polytope. A large number of separation algorithms are, of course, described in the book [69].

This research activity goes on and brings application relevant instances of many $\mathcal{NP}$-hard combinatorial optimization problems to the realm of practical solvability. For the traveling salesman problem, for example, the "solvability world record" was 42 cities in 1954, it went to 120 in 1977, 2392 in 1987, and in 2017 a TSP with 109,399 cities could be solved to optimality, see the Webpage of Bill Cook [31], his

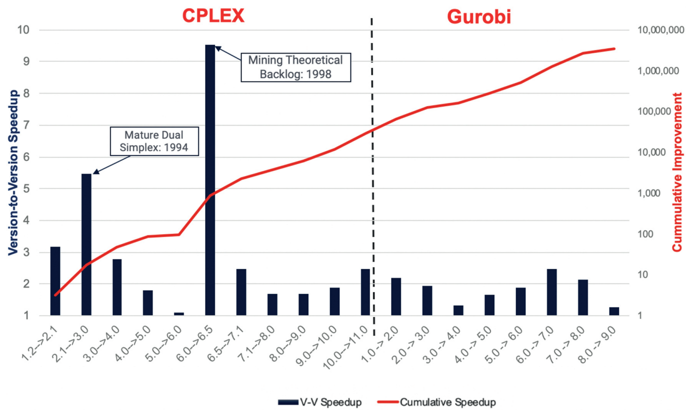


Fig. 12 MIP-code performance 1990–2019 (courtesy Robert E. Bixby)

book [30], and the book [8] by Applegate, Bixby, Chvátal, and Cook for comprehensive information. The solution process includes linear programming technology (its theory and implementation) that is able to prove, for example, that a vector in dimension 10^{10} satisfies more than $2^{100,000}$ constraints and is optimal for this system. This is really breathtaking.

The success stories indicated above, and the theoretical and practical lessons learned from these began to be harvested and improved by the developers of commercial optimization software in the 1990s. One reason for this is that many mixed-integer optimization problems (MIPs) occurring in industry contain subproblems that are combinatorial optimization problems for which large classes of facet-defining inequalities have been discovered. Efficient separation algorithms for these inequalities were successfully added to the existing MIP-codes. The graphic in Fig. 12, presented with the permission of Bob Bixby, shows the development of the commercial mixed integer programming codes CPLEX and Gurobi in the 30 years from 1990 to 2019. The large bar (pointed at by "Mining Theoretical Backlog") shows an almost tenfold speedup that is obtained from one version of the code to the next in which cutting plane technology (including a fresh implementation of Gomory cuts) was introduced together with various supporting features. The overall message is that the MIP technology in 2019 runs 3.5 million times faster than the codes of 1990. That speedup is due to mathematical and implementation improvements and is independent of the hardware speedup during this period. This is real progress indeed. Cutting plane technology contributed to it significantly.

14 Computing Optimal Stable Sets and Colorings in Perfect Graphs

M. Grötschel, L. Lovász, A. Schrijver. Polynomial algorithms for perfect graphs. *Annals of Discrete Math*. 21:325–256, 1984.

M. Grötschel, L. Lovász, A. Schrijver. Relaxations of vertex packing. *J. Combin. Theory* B 40:330–343, 1986.

The extension of the ellipsoid method to convex bodies outlined in Section 10 was driven by the hope that one could solve the stable set and the coloring problem in perfect graphs in polynomial time with this methodology, see Section 7. The successful attempt is presented in the articles [66, 67], and [68]. We describe the stable set case.

For a graph $G = (V, E)$ and a stable set $S \subseteq V$, one can define the incidence vector x^S in $\mathbb{R}^V$ as follows: the i-th component x_i^S of x^S is equal to 1 if the vertex $i \in V$ is an element of S, and it is 0 otherwise. The stable set polytope of G is the convex hull of all incidence vectors of stable sets S of G, i.e.,

$$\mathrm{STAB}(G) := \mathrm{conv}\left\{x^S \in \mathbb{R}^V \mid S \subseteq V \text{ stable set}\right\}.$$

Let $w : V \to \mathbb{Q}$ be any weighting of the vertices of G (we may assume that all weights are positive) and denote the largest weight of a stable set in G by $\alpha(G, w)$. Then $\alpha(G, w)$ is the maximum value of the linear function $w^T x$ for $x \in \mathrm{STAB}(G)$, in other words, $\alpha(G, w)$ can be computed by solving a linear program over $\mathrm{STAB}(G)$. For this observation to be of any use, we have to find inequalities defining $\mathrm{STAB}(G)$. Consider the polytope defined by

$$\begin{aligned}\mathrm{QSTAB}(G) := \Big\{x \in \mathbb{R}^V \,|x_i \geq 0 \quad \forall i \in V,\ x_i + x_j \leq 1 \quad \forall ij \in E,\\ x(Q) \leq 1 \quad \forall Q \subseteq V \text{ clique}\Big\},\end{aligned}$$

where $x(Q)$ denotes the sum of all $x_i, i \in Q$. The corresponding inequality is called *clique constraint*. Since the intersection of a clique and a stable set contains at most one vertex, all clique constraints are satisfied by all incidence vectors of stable sets. This implies $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$ and optimizing over $\mathrm{QSTAB}(G)$ is an LP-relaxation of the stable set problem.

The stable set problem is $\mathcal{NP}$-hard. Therefore, solving linear programs over $\mathrm{STAB}(G)$ is $\mathcal{NP}$-hard as well. For some combinatorial optimization problems, their natural LP-relaxation is solvable in polynomial time. A sobering observation is that, for general graphs, solving linear programs over $\mathrm{QSTAB}(G)$ is also $\mathcal{NP}$-hard. So, in general, nothing is gained algorithmically. For perfect graphs, though, this approach combined with a tighter relaxation delivers the desired result.

Lovász's Shannon capacity article [106] suggests studying a different relaxation of the stable set problem.

Let $(u_i \mid i \in V), u_i \in \mathbb{R}^N$, be any orthonormal representation of G and let $c \in \mathbb{R}^N$ with $\|c\| = 1$. Then for any stable set $S \subseteq V$, the vectors $u_i, i \in S$, are mutually orthogonal and hence,

$$\sum_{i \in S} (c^T u_i)^2 \le 1.$$

Since $\sum_{i \in V} (c^T u_i)^2 x_i^S = \sum_{i \in S} (c^T u_i)^2$, we see that the inequality

$$\sum_{i \in V} (c^T u_i)^2 x_i \le 1 \tag{ORC}$$

holds for the incidence vector $x^S \in \mathbb{R}^V$ of any stable S set of nodes of G. Thus, (ORC) is a valid inequality for $\mathrm{STAB}(G)$ for any orthonormal representation $(u_i \mid i \in V)$ of G, where $u_i \in \mathbb{R}^N$, and any unit vector $c \in \mathbb{R}^N$. We shall call (ORC) the *orthonormal representation constraints* for $\mathrm{STAB}(G)$.

Utilizing these inequalities, the following set was introduced in [68]. For any graph $G = (V, E)$ let

$$\begin{aligned} \mathrm{TH}(G) := \big\{ x \in \mathbb{R}^V \mid x_i \ge 0 \quad \forall i \in V, \\ \text{and } x \text{ satisfies all orthonormal representation constraints} \big\}. \end{aligned}$$

$\mathrm{TH}(G)$ is the solution set of infinitely many linear inequalities and thus a convex set. Since for every clique Q, its clique constraint appears as an orthonormal representation constraint (given a clique $Q \subseteq V$, let $\{u_i \mid i \in V \setminus Q\} \cup \{c\}$ be mutually orthogonal unit vectors and set $u_j = c$ for $j \in Q$) and every incidence vector of a stable set satisfies all such inequalities, we obtain:

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G).$$

An important fact is, that the Lovász theta function $\vartheta(G, w)$ introduced in Section 8 can also be characterized as follows:

$$\vartheta(G, w) = \max\big\{ w^T x \mid x \in \mathrm{TH}(G) \big\}.$$

$\mathrm{TH}(G)$ is contained in the unit ball, and it is easy to find the center of a ball contained in the interior of $\mathrm{TH}(G)$. Thus, $\mathrm{TH}(G)$ is a convex body satisfying the assumptions required for the oracle-polynomial time equivalence of weak optimization, separation, and membership. The desired result is, of course, the following:

Theorem. *The weak optimization problem for* $\mathrm{TH}(G)$ *is solvable in polynomial time for any graph* $G = (V, E)$.

Lovász, see [106], established several characterizations for his ϑ-function. They can be used in various ways to prove this theorem. One proof, worked out in detail in [66] and [69], is based on the following characterization:

$$\begin{aligned} &\vartheta(G, w) = \max\{ \bar{w}^T B \bar{w} \mid B \in \mathcal{K} \}, \\ &\text{where } \mathcal{K} := \{ B \in \mathbb{R}^{V \times V} \mid B \in \mathcal{D} \cap \mathcal{M} \text{ and } \operatorname{tr}(B) = 1 \}. \end{aligned}$$

Above, $\mathcal{D}$ is the set of positive semidefinite matrices, $\mathcal{M}$ the set of symmetric matrices B that satisfy $b_{ij} = 0$ whenever ij is an edge in G, and $\bar{w}$ denotes the vector whose entries are the square roots of the values $w_i, i \in V$. The main part of the proof consists in showing that the weak membership problem for $\mathcal{K}$ can be solved in polynomial time, and the core of this proof is established by showing whether a symmetric matrix is positive definite.

A by-product of the proof is the first polynomial time algorithm for optimization problems containing positive semidefinite constraints, a major result that led to considerable follow-up research such as the design of polynomial time interior point (and other) algorithms for semidefinite programming.

Another way to establish the above theorem is by utilizing the following fact:

$$\vartheta(G, w) = \min\left\{\Lambda(A + W) \mid A \in \mathcal{M}^{\perp}\right\},$$

where Λ denotes the largest eigenvalue, $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$, and W the symmetric $V \times V$-matrix whose entries are the square roots of $w_i w_j$. $\Lambda(A+W)$ is a convex function that ranges over a linear space, and thus, we can obtain $\vartheta(G, w)$ via an unconstrained convex function optimization problem in polynomial time.

A third way to prove the theorem was demonstrated in [116], and this approach turned out to be one of the starting points for a generalization of this technique. Lovász and Schrijver developed in this article a general lift-and-project method that constructs higher-dimensional polyhedra (or, in some cases, convex sets) whose projection approximates the convex hull of 0-1 valued solutions of a system of linear inequalities. An important feature of these approximations is that one can optimize any linear objective function over them in polynomial time. Lift-and-project methods have been extended in many directions and are still an area of intensive research. The recent (not even exhaustive) survey by Fawzi, Gouveia, Parrilo, Saunderson, and Thomas [51] discusses the contributions of almost one hundred articles and illustrates the richness of this topic by presenting examples from many different areas of mathematics and its applications.

We refrain from describing the technically challenging details of this lift-and-project technique and return to stable sets in perfect graphs.

A combination results of Fulkerson [57] and Chvátal [28] yields:

Theorem. $\mathrm{STAB}(G) = \mathrm{QSTAG}(G)$ *if and only if G is perfect.*

And since we already know that $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$ holds, we obtain:

Corollary. $\mathrm{STAB}(G) = \mathrm{TH}(G) = \mathrm{QSTAG}(G)$ *if and only if G is perfect.*

Since the weak optimization problem for $\mathrm{TH}(G)$ can be solved in polynomial time and since, in case G is perfect, $\mathrm{TH}(G)$ is a well-described polyhedron, the strong optimization problem for $\mathrm{TH}(G)$ can be solved in polynomial time. This yields the desired result:

Theorem. *The stable set problem can be solved in polynomial time for perfect graphs.*

We can now employ the fact that, if a linear program can be solved in polynomial time, the dual linear program can also be solved in polynomial time, see Section 11. By proving that, in this case, an optimum basic solution of the dual program can be transformed in polynomial time into an integral optimum basic solution one can find an optimum solution of the weighted clique covering problem. Since the cliques of a graph G are the stable sets of the complementary graph $\bar{G}$ of G and the colorings of G are the clique covering of $\bar{G}$, we can conclude:

Theorem. *For perfect graphs, the stable set, the clique, the coloring, and the clique covering problem can be solved in polynomial time. This also holds for the weighted versions of these problems.*

15 Submodular Functions

L. Lovász. Submodular functions and convexity. In *Mathematical Programming: The State of the Art* (eds. A. Bachem, M. Grötschel, B. Korte), Springer, pages 235–257, 1983.

Let E be a finite set. A function $f : 2^E \to \mathbb{R}$ is called *submodular* on 2^E (the power set of E) if

$$f(S \cap T) + f(S \cup T) \leq f(S) + f(T) \text{ for all } S, T \subseteq E.$$

Submodular functions play an important role in lattice theory, geometry, graph theory, and particularly, in matroid theory and matroidal optimization problems. The rank function of a matroid, for example, is submodular as well as the capacity function of the cuts in directed and undirected graphs.

Two polyhedra can be associated with a submodular function $f : 2^E \to \mathbb{R}$ in a natural way

$$P_f := \{x \in \mathbb{R}^E \mid x(F) \leq f(F) \text{ for all } F \subseteq E,\ x \geq 0\},$$
$$EP_f := \{x \in \mathbb{R}^E \mid x(F) \leq f(F) \text{ for all } F \subseteq E\}.$$

P_f is called the *polymatroid* associated with the submodular function f, EP_f the *extended polymatroid* associated with f. A deep theorem of Edmonds [43] states that if f and g are two integer valued submodular functions then all vertices of $P_f \cap P_g$ as well as all vertices of $EP_f \cap EP_g$ are integral. This theorem contains a large number of integrality results in polyhedral combinatorics; it particularly generalizes the matroid intersection theorem.

To address algorithmic questions concerning the structures introduced above, we assume that a submodular function f is given by an oracle that returns the value $f(S)$ for every query $S \subseteq E$. We also assume that we know an upper bound β on the encoding length of the output of the oracle. With these assumptions we define the encoding length of the submodular function as $|E| + \beta$.

It is well-known that, for any nonnegative linear objective function, the greedy algorithm finds an optimum vertex of EP_f in oracle-polynomial time, and that this vertex is integral provided the submodular function f is integer valued. Optimizing over polymatroids or the intersections of two polymatroids or the intersections of two extended polymatroids and finding integral optima is more complicated and needs careful analysis. The most important algorithmic problem in this context is:

Submodular Function Minimization. *Given a submodular function $f : 2^E \to \mathbb{Q}$, find a set $S \subseteq E$ minimizing f.*

Lovász has built in [110] a bridge between submodularity and convexity by showing that submodular functions are discrete analogues of convex functions and has thus provided the key to the algorithmic solution of the submodular function minimization problem. The link is established as follows.

Let $f : 2^E \to \mathbb{R}$ be any set function. For every subset $T \subseteq E$, let x^T be its incidence vector and set

$$\hat{f}(x^T) := f(T).$$

This way $\hat{f}$ is defined on all 0/1-vectors. Note that every nonzero nonnegative vector $y \in \mathbb{R}^E$ can be expressed uniquely as

$$y = \lambda_1 x^{T_1} + \lambda_2 x^{T_2} + \ldots + \lambda_k x^{T_k},$$
$$\text{such that} \quad \lambda_i > 0, i = 1, \ldots, k \quad \text{and} \quad \emptyset \neq T_1 \subset T_2 \subset \ldots \subset T_k \subseteq E.$$

Then

$$\hat{f}(y) := \lambda_1 f(T_1) + \lambda_2 f(T_2) + \ldots + \lambda_k f(T_k)$$

is a well-defined extension of the set function f (called *Lovász extension of f*) to the nonnegative orthant. Lovász proved in [110]:

Theorem. *Let $f : 2^E \to \mathbb{R}$ be any set function and $\hat{f}$ its extension to nonnegative vectors. Then $\hat{f}$ is convex if and only if f is submodular.*

Lemma. *Let $f : 2^E \to \mathbb{R}$ be set function with $f(\emptyset) = 0$. Then*

$$\min\{f(S) \mid S \subseteq E\} = \min\{\hat{f}(x) \mid x \in [0,1]^E\}.$$

Thus, instead of minimizing a set function f over E, it suffices to minimize its Lovász extension $\hat{f}$ over the unit hypercube. We observe that $\hat{f}(x)$ can be evaluated in oracle-polynomial time using the oracle defining f and that, if f is submodular, then $\hat{f}$ is convex. We know already from Section 10 that convex functions can be minimized in oracle-polynomial time. (The assumption $f(\emptyset) = 0$ is irrelevant, if necessary, we can replace f by the function $f - f(\emptyset)$.) This yields:

Theorem. *Let $f : 2^E \to \mathbb{Q}$ be a submodular function. Then a subset S of E minimizing f can be found in oracle polynomial time.*

This theorem implies the polynomial time solvability of many combinatorial optimization problems, including the computation of a minimum capacity cut in a graph. It has various ramifications such as solvability in strongly polynomial time, as outlined in [110] and [69].

The running time of the polynomial time algorithm sketched above makes it, however, infeasible for practical use. New and better polynomial time algorithms, not employing the ellipsoid method, have been devised by Schrijver [143] and Iwata, Fleischer, and Fujishige [81].

16 Volume Computation

L. Lovász. How to compute the volume? *Jber. d. Dt. Math.-Vereinigung*, Jubiläumstagung 1990, B. G. Teubner, Stuttgart, pages 138–151, 1992.

Since the convergence of all versions of the ellipsoid method depends on sequentially shrinking the volume of an ellipsoid containing the given convex body K, it is tempting to ask whether the algorithm can be tuned to provide a reasonable estimate of the volume of K. The key idea in this context is, of course, to come up with an algorithmic version of the Löwner–John theorem, that states, that, for a convex body K in $\mathbb{R}^n$, there exists a unique ellipsoid E of minimal volume containing K; moreover, K contains the ellipsoid obtained from E by shrinking it from its center by a factor of n. In formulas, let $E(A,a) := \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \le 1\}$ denote the ellipsoid defined by a positive definite matrix A with center $a \in \mathbb{R}^n$ then the Löwner–John theorem states

$$E(n^{-2}A,a) \subseteq K \subseteq E(A,a),$$

if $E(A,a)$ is the Löwner–John ellipsoid E of K. Algorithmically, the following could be achieved in the Grötschel–Lovász–Schrijver book [69].

Theorem. *There exists an oracle-polynomial time algorithm that finds, for any convex body K given by the space dimension n, a weak separation oracle and two real numbers r and R with the property that K is contained in the ball of radius R around the origin and contains a ball of radius r, an ellipsoid $E(A,a)$ such that*

$$E\Big(\frac{1}{n(n+1)^2}A,a\Big) \subseteq K \subseteq E(A,a).$$

With more effort and making additional assumptions such as central symmetry or requiring that a system of defining linear inequalities is explicitly given (in the polytopal case), the factor $1/(n(n+1)^2$ in front of the matrix A above can be slightly improved, but not fundamentally. If one declares the volume of the interior ellipsoid as an approximation of the volume of K, the relative error turns out to be $2^n n^{3n/2}$, which appears to be outrageously bad.

Surprisingly, the error is not as bad as it looks since subsequently Elekes [45] and others proved that no oracle-polynomial time algorithm can compute, for a convex body K as given above, the volume of K with a much better relative error. We quote a result of Bárány and Füredi [11].

Theorem. *Consider a polynomial time algorithm which assigns to every convex body K given by a membership oracle an upper bound $w(K)$ on its volume* $\mathrm{vol}(K)$. *Then there is a constant $c > 0$ such that in every dimension n there exists a convex body K for which $w(K) > n^{cn}\,\mathrm{vol}(K)$.*

Following up, various authors proved more negative results on the deterministic approximation of the volume, width, diameter and other convexity parameters.

These negative results fueled the investigation of stochastic approaches to estimate the volume of a convex body. Instead of giving a deterministic guarantee, one could try to calculate a number that is close to the true value of the volume with high probability employing a randomized algorithm.

A side remark: Khachiyan [88] and Lawrence [93] proved that, for every dimension n, one can construct systems of rational inequalities defining polytopes P so that the encoding length of the rational number p/q representing the true volume of P requires a number of digits that is exponential in the encoding length of the inequality system. Hence, exact volumes of convex bodies cannot be computed in polynomial time since specifying the exact volume requires exponential space.

A fundamental breakthrough was achieved in Dyer, Frieze, and Kannan [40] who provided a randomized polynomial time approximation scheme for the volume approximation problem where K is given by a membership oracle. The ingredients of their algorithm are a multiphase Monte-Carlo algorithm (using the so-called product estimator) to reduce volume computation to sampling, the utilization of Markov chain techniques for sampling, and the use of the conductance bound on the mixing time, due to Jerrum and Sinclair [82]. The running time of the algorithm is roughly $O(n^{23})$ which is truly prohibitive. The exponent 23 of n was subsequently reduced considerably by adding further techniques and improved estimates to the toolbox of randomized algorithms, including rapid mixing, harmonic functions, connection to the heat kernel, isoperimetric inequalities, discrete forms of Cheeger inequality, and many more.

Lovász played an important role in the exponent shrinking race. For example, the exponent went down to 16 (Lovász and Simonovits [120]), to 10 (Lovász [111]), to 8 (Dyer and Frieze [39]), to 7 (Lovász and Simonovits [121]), to 5 (Kannan, Lovász, and Simonovits [85]), and to 4 (Lovász and Vempala [124]). A nice survey of the many tricky issues in designing randomized algorithms for volume computation and their analysis is the article by Simonovits [149].

The race for better algorithms has not stopped. On September 3, 2022, the new record was published on arXiv by Jia, Laddha, Lee, and Vempala [83]. The authors show that the volume of a convex body in $\mathbb{R}^n$ defined by a membership oracle can be computed to within relative error ϵ using $\tilde{O}(n^3\psi^2 + n^3/\epsilon^2)$ oracle queries, where ψ is the KLS constant. With the current bound of $\psi = \tilde{O}(1)$, this gives an $\tilde{O}(n^3/\epsilon^2)$ algorithm, improving on the Lovász–Vempala $\tilde{O}(n^4/\epsilon^2)$ algorithm.

17 Analysis, Algebra, and Graph Limits

L. Lovász, *Large Networks and Graph Limits*. American Mathematical Society, 2012.

We have already indicated that many of the results mentioned in our article seem to be of permanent importance and are used again and again: the Lovász Local Lemma, algorithmic consequences of the ellipsoid method, topological combinatorics, and the LLL algorithm, to name just few. Very recently Lovász's mathematics culminated in a topic that somehow combines this into an all-in-one subject: like a late symphony of a grand composer displaying the experience of the master and an echo of his/her life. We believe that this happened with the subject of graph limits founded and developed by Lovász with co-authors and students in the last 15 years. Here is a brief sketch of this fascinating development.

We have seen in Section 2 that the homomorphism function $\hom(F,G)$ and the Lovász vector $L(G)$ are determining every graph G up to an isomorphism. With a proper scaling this leads to the notion of *homomorphism density* $t(F,G)$, which is the probability that a random mapping between sets of vertices of F and G is a homomorphism: $t(F,G) = \frac{\hom(F,G)}{v(G)^{v(F)}}$ where $v(G)$ denotes the number of vertices of graph G.

This definition is close to the sampling density and one motivation for introducing it. One can observe that homomorphism densities do not determine a graph up to an isomorphism but up to a "blowing up of vertices". (This is a procedure by which vertices are replaced by a certain number of twin copies.) It is perhaps more important that one can then define *convergence of a sequence* of finite graphs $G_1, G_2, \ldots, G_n, \ldots$ as the convergence of homomorphism densities $t(F,G_n)$ for every graph F. This convergence concept (and various other notions of convergence) were introduced and investigated in the article [22] of C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi.

Hence, a sequence of graphs converges if, for every F, all homomorphism densities (or F-sampling densities) converge. Does this convergence have a real (geometrical) meaning? Are there limit graphs or, perhaps, other limit objects?

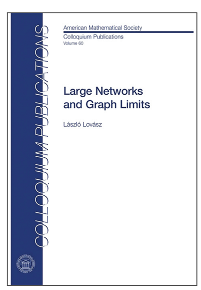


Fig. 13 Lovász's Graph Limits book

It appears that these questions have non-trivial yet positive answers and these were the starting point of a very rich and interesting area. In fact, they generated a whole new theory. Here is a sample of some of the results.

L. Lovász and B. Szegedy proved the following in [122]:

Theorem. *A sequence of graphs (with unbounded size) is converging if and only if it converges to a symmetric measurable function* $W : [0,1]^2 \to [0,1]$. *Moreover, up to a measurable bijection, such a function W is uniquely determined.*

Explicitly, this means that for every graph $F = (V,E)$ the homomorphism densities $t(F,G_n)$ are converging to:

$$t(F,W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i,x_j) \prod_{i \in V} dx_i$$

Such functions W are called *graphons*. Graphon is a very intuitive notion and the convergence of a graph sequence to a graphon looks like a movie. It leads to “pixel” pictures like those on samples shown in Figure 14 (taken from Lovász‘s book [112]).

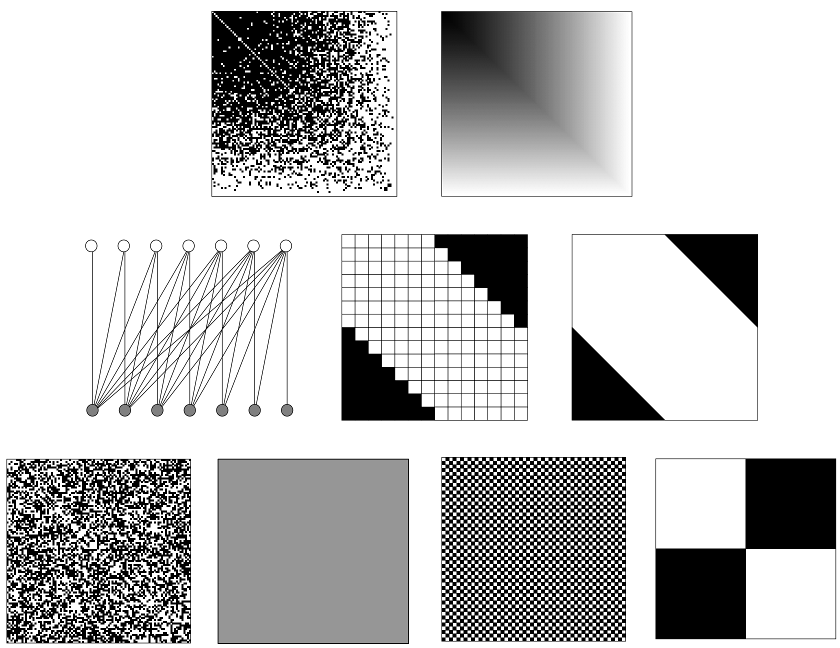


Fig. 14 Samples of graphons

The first row of Fig. 14 shows on the left a randomly grown uniform attachment graph with 100 nodes, and on the right a (continuous) function approximating it. The picture on the right side is a grayscale image of the function $U(x,y) =$

$1-\max(x,y)$. The second row of Fig. 14 indicates the construction of the graphon for the "halfgraph" (the graph on the left side). The bottom part indicates the influence of ordering and the regularity Lemma in its simplest form. Note that the sequence of random graphs is converging to a graphon W that is a constant function. It is important that the same is true for "quasirandom graphs".

Graphon is not just an intuitive notion, it has mathematical relevance. This setting extends work of Aldous [2] and Hoover [79] in probability theory on exchangeable random graphs (see, e.g., [9]). Graphons are also not just a generalization. They present a convenient and useful way to study extremal problems for graphs (such as to find maximum number of edges of a graph satisfying given local properties).

These problems then often take the form of linear inequalities. Lovász introduced graph algebras (of "quantum graphs") with nice "pictorial" proofs, see [112]), and independently Alexander Razborov developed "flag algebras" [140] which proved to be a very efficient tool in various extremal problems, see, e.g., [73] and [70].

The graph algebra of Lovász and Razborov was motivated by early examples provided by the Caccetta–Häggkvist conjecture, see [20], the Sidorenko conjecture [148], and the early paper [50] of Erdős, Lovász, and Spencer on topological properties of the graphcopy function.

A typical extremal problem may be expressed as a fact that a certain linear inequality built from homomorphism densities of graphs is nonnegative. This in turn led Lovász to a question whether any such inequality can be deduced from a sum of squares of "quantum graphs". A related question was formulated by Razborov [140] whether the validity of any such inequality can be solved by "Cauchy–Schwarz Calculus". However, Hamed Hatami and Serguei Norin [74] showed that both these questions have a negative answer in general as the related problems are algorithmically undecidable. So, extremal problems may be more difficult as originally thought. This was further supported by the universality results of Cooper, Grzesik, Král, Martins, and L. M. Lovász, see [32] and [70], claiming particularly that every graphon may be extended to a "finitely forcible" graphon.

This approach also provides an understanding of the celebrated Szemerédi regularity lemma. The Szemerédi regularity lemma in this interpretation means an approximation of every graph (and every graphon) by means of a "small" pixel image where almost all entries are constant (but may be different for different pixels).

The key of the approach of [22] is to characterize convergence using the *cut metric* $d_\square(G,H)$ (based on the cut norm introduced by R. Frieze and R. Kannan in [55]). If the homomorphism density is defined by scaled subgraph density, then the cut metric is, somewhat dually, characterized by means of a scaled density of partitions.

The cut metric $d_\square(G,H)$ for finite graphs G,H on the same vertex set V is defined as

$$\max_{S,T\subseteq V}\frac{|e_G(S,T)|-|e_H(S,T)|}{|V\times V|}$$

i.e., as the scaled difference of the-sizes of cuts in G and H; above $e_G(S,T)$ is the number of edges of G between sets S and T. (This definition can be extended to graphs on different vertex sets. This is technical and it takes three full pages in [112]). Interestingly, the cut distance for a graphon W is more easily defined than in the finite case: it is induced by the norm:

$$\|W\| = \sup_{S,T \subseteq [0,1]} \int_{S \times T} W(x,y)\,dx\,dy$$

The cut norm is also very natural and fitting from an algorithmic point of view; and it is bounded by the Grothendieck norm up to a multiplicative constant (as shown by Alon and Naor [5]).

As a culmination of several auxiliary results, one obtains that the convergence is indeed induced by a distance. This is the key fact in many applications and was proved by Lovász and Szegedy in [122]:

Theorem. *If (G_n) is a sequence of graphs of unbounded size, then (G_n) is a converging sequence if and only if (G_n) is a Cauchy sequence with respect to cut distance $d_{\square}(G_i, G_j)$.*

The following result was proved by Lovász and Szegedy in [123]. Lovász considers it as one of the basic results treated in his book [112].

Theorem. *The space of all graphons W with cut distance is compact.*

This compactness theorem may be viewed as the roof result for the Szemerédi regularity Lemma and its various extensions. It also displays the usefulness of the limit language and of the much more general setting. This area was studied extensively, for instance by Borgs, Chayes, Elek, Lovász, Sós, Szegedy, Vesztergombi, and Tao in [23, 44, 123, 155].

The mathematical richness of this area is best illustrated by the Appendix A of [112] which contains the following sections: Möbius functions; the Tutte polynomial; some background in probability and measure theory; moments and the moment problem; ultraproduct and ultralimit; Vapnik–Chervonenkis dimension; nonnegative polynomials; categories. Obviously, it is impossible to present here more than a glimpse of what the book [112] covers.

Note that the above results are interesting for dense graphs. For sparse graphs (for example for graphs with constant degrees) one has to devise a different approach. Limit objects are now called graphings and modelings. For them results similar to above three theorems are not known. This is treated, e.g., by Benjamini and Schramm [17] and by Nešetřil and Ossona de Mendez [134]; see again [112].

It is amazing that the area of graphs and their limits can be traced back to Lovász's very early algebraic results (mentioned in Section 2). Some forty years later it blossomed in the inspiring climate of the Microsoft Research Theory Group at Redmond in an atmosphere of concentrated research and quality, with persons such as Michael Freedman, Oded Schramm and many other great visitors and with László Lovász as a driving force.

18 Final Remarks

Let us finish the fireworks of beautiful theorems ranging over many parts of mathematics and theoretical computer science by adding a few general remarks.

It happens very rarely that a well-known and long-standing open problem is solved by a novel technique that immediately influences not just that area, but other parts of mathematics as well. Lovász not only accomplished this once. It is unbelievable that Lovász repeatedly offered to the world community exactly such solutions. Some of these proofs are really elegant and were included in the collections of other beautiful "book proofs", see [1] and [129].

In this article we concentrated on Lovász-results which had general influence, led to intensive research by many others, and sometimes spawned the emergence of whole new theories. Work in areas such as combinatorial optimization, applications of the ellipsoid method, algebraic graph theory, graph homomorphisms, topological graph theory, and graph limits is very difficult to imagine without the pioneering accomplishments of László Lovász.

In our Introduction we indicated that Lovász is both, a "problem solver" and a "theory builder", and pointed out that the trio *depth, elegance, and inspiration* is a particular signature of his work that makes his achievements unique. We do hope that the glimpse into his oeuvre and the scientific influence of his results, that we have offered here, provides at least a partial proof of our conviction.

Fig. 15 Several Lovász-books on a poster (by A. Goodall and J. N.) of the Charles University in Prague (Photo: Private)

To keep this article at a reasonable length we had to omit many topics on which Lovász left his marks. In particular, it was impossible to give adequate attention to the books he has written, see Fig. 15, and the influence they had and still have. To mend this omission, albeit very incompletely, we elucidate the contents and impact of four of his books – extremely briefly, though.

Lovász's third book *Combinatorial Problems and Exercises* [107] became – without any exaggeration – a bible for combinatorialists worldwide. This is a book organized in an unusual way. It has three parts: The first part consists of mostly easily formulated questions and problems, the second part contains hints for the solutions, and the third part thorough proofs with discussions. This of course, makes up the largest part.

Lovász convincingly claims in this book that discrete mathematics, at the time of publication, has grown out of an area with simple questions that are relatively easy to solve without much mathematical knowledge into a structured field with various branches consisting of central concepts and theorems forming a hierarchy and possessing a rich bouquet of proof techniques. Instead of presenting the theories analytically and deductively, Lovász designed his book with the purpose of helping interested readers to learn many of the existing techniques in combinatorics. And as he wrote in the introduction:

> *The most effective (but admittedly very-time consuming) way of learning such techniques is to solve (appropriately chosen) exercises and problems.*

We believe that this book significantly changed the level on which combinatorics (and graph theory in particular) was treated. It caught worldwide attention from the very start (see, e.g., the book review by Bollobás [19]) by combinatorialists, computer scientists, and mathematicians in general. It is remarkable that after more than 40 years of its existence the book, that mirrors the vast experience of the author, is still in print and in use.

A side remark: Combinatorics meetings usually have an open problems session where participants explain questions they are working on and have not solved yet. Lovász, with his wide knowledge of proof techniques, has always been outstanding in being able to solve many of the open problems on the spot.

Matching problems have played a considerable role in the development of graph theory. Well-known and important early results are, e.g., König's Matching Theorems, the Marriage Theorem, and Tutte's f-factor theorem. Matchings, b-matchings, T-joins, etc. have a rich structure theory. The Edmonds–Galai decomposition is one such example. Various matching problems and their ramifications appear in a large variety of applications of combinatorial optimization (e.g., the Chinese Postman Problem). Many of these are solvable with (highly nontrivial) polynomial time algorithms for which the pioneering work of J. Edmonds, see [41], laid the basis. Edmonds [42] achieved also a breakthrough in polyhedral combinatorics by providing a linear description of the matching polytope that does not simply follow from total unimodularity. Lovász [108] came up with a new and elegant proof of this result that was later often mimicked for the characterization of other polytopes arising in combinatorial optimization.

The book [115] *Matching Theory*, written by László Lovász and Mike Plummer, provides a broad view of this subject and covers the roughly 40 articles that Lovász has contributed to this field. We just want to highlight Chapters 10 and 11 of this book. Chapter 10 is devoted to the f-factor problem which asks whether, for a given graph $G = (V, E)$ and integers $f(v)$ for every vertex $v \in V$, there is a spanning subgraph H of G such that the degree of v in H is equal to $f(v)$. In a series of four papers that appeared 1970–1972, Lovász developed a generalization of the Edmonds–Galai Structure Theorem to the f-factor problem to provide an elegant answer of the f-factor problem. Chapter 11 introduces further generalizations such as the matroid and polymatroid matching problem which are interesting (and difficult) combinations of topics in graph and matroid theory. We refer to this Chapter of [115] and the article [109] for some of the results that can be shown in this context. Finally, this book contains in the preface a wonderful brief, yet in-depth survey of the historical development of matching theory.

Lovász's book *Large networks and graph limits* [112] is aiming in a different direction. It is the result of a stay of Lovász at the IAS in Princeton. We have dealt with parts of this book in Section 17. Graph limits became a very active field with contributions ranging from model theory, probability, functional analysis to theoretical computer science, network science and, of course, combinatorics. This theory fits very well with advanced combinatorics; for example, the role of Szemerédi's regularity lemma is highlighted and explained properly in this context. The basic theory of convergent graph sequences is derived in several settings; and multiple applications to parameter and property testing, extremal theory, and other applications are given. The book starts with an informal introduction into large graphs in a network science context, specifying the abundance of real applications, and questions to ask about them. This is followed by a lengthy chapter on the algebra of graph homomorphisms. This chapter can be read independently and is also of independent interest. But one of the main features of this book is to show how this algebra is connected to limit structures and limit distributions. It is amazing how much material was developed in this context in less than a decade. In the very nice preface, Lovász lists the branches of mathematics that come into play in his book and writes:

> *These connections with very different parts of mathematics made it quite difficult to write this book in a readable form* [...] [continuing that he found that] *the most exciting feature of this theory* [...] [is] *its rich connections with other parts of mathematics (classical and non-classical)* [...] [so that he] *decided to explain as many of these connections* [...] [as he] *could fit in the book*.

Summarizing, this book is a real tour de force.

The American Mathematical Society Colloquium Publications were established in 1905. So far 66 books were published in this AMS flagship book series "*offering the finest in scholarly mathematical publishing*". Vol. 60 is the book [112] *Large Networks and Graph Limits* discussed above, Vol. 65 is the book *Graphs and Geometry* [113], so far the last book written by Lovász.

Vol. 60 pictures the emergence and maturation of a new theory while Vol. 65 presents a wide spectrum of geometry related techniques (and tricks) to study graphs.

In twenty chapters (and three appendices) Lovász surveys many connections between graph theory and geometry concentrating on those which lie deeper. These are among others: rubber band representations, coin representations, orthogonal representation, and discrete analytic functions. Interestingly, this book is only about geometry, and thus topology is outside its scope. Nevertheless, the book contains some of the key discoveries of Lovász in a new context.

The Leitmotiv of the whole book [113] is described in the preface:

> *Graphs are usually represented as geometric objects drawn in the plane, consisting of vertices and curves connecting them. The main message of this book is that such a representation is not merely a way to visualize the graph, but an important mathematical tool. It is obvious that this geometry is crucial in engineering if you want to understand rigidity of frameworks and mobility of mechanisms. But even if there is no geometry directly connected to the graph-theoretic problem, a well-chosen geometric embedding has mathematical meaning and applications in proofs and algorithms. This thought emerged in the 1970s, and I found it quite fruitful*.

Lovász has been developing these thoughts for about forty years observing:

> *Many new results and new applications of the topic have also been emerging, even outside mathematics, like in statistical and quantum physics and computer science (learning theory). At some point I had to decide to round things up and publish this book*.

This finishes his preface. But he returns to these considerations in Chapter 20, “Concluding Thoughts”, on page 390 as follows:

> *I am certain that many new results of this nature will be obtained in the future (or are already in the literature, sometimes in a quite different disguise). Whether these will be collected and combined in another monograph, or integrated into science through some other platform provided by the fast changing technology of communication, I cannot predict. But the beauty of nontrivial connections between combinatorics, geometry, algebra and physics will remain here to inspire research*.

When reviewing the book [112] in the Bulletin of the American Mathematical Society, one of us quoted Michel Mendès France who once told him that envy is the right feeling when reading beautiful mathematics. Yes, this is the feeling one may have when reading Lovász’s books such as [112] and [113].

His exceptional research capabilities and his broad knowledge of mathematics are mirrored in Lovász’s public presentations and survey articles. He has the ability to explain difficult results in understandable language and, in particular, to display and illustrate connections between seemingly unrelated topics. Examples of that can, e.g., be found in the articles he contributed to the *Handbook of Combinatorics* [62], see also [91]. The titles of some of his survey and motivating articles contain phrases such as *One mathematics* or *Discrete and Continuous: Two sides of the same*. This reflects his philosophy that science is not a collection of independent topics but a tightly connected network to be discovered and understood. He contributed to this conviction also administratively by serving the scientific community in leading positions of the International Mathematical Union and the Hungarian Academy of Sciences.

The unity of mathematics and the role of mathematics in the world have been addressed again and again by László Lovász through many of his activities. Given the outstanding excellence in his own research and the huge experience as a professional in combination with admirable modesty the mathematical community can hardly think of a better representative.

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