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CONSTRUCTIONS OF HYPOTRACEABLE DIGRAPHS

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ABSTRACT

A hypotraceable digraph is a digraph D which is not traceable, i.e., does not contain a (directed) hamiltonian path, but has the property that for every node v in D the node-deleted digraph D-v is traceable. It was proved recently that hypotraceable digraphs of order n exit for all $n \geq 7$. We present here three different constructions to obtain further classes of hypotraceable digraphs which are either strongly connected or have a source and/or a sink.

1. Introduction and Notation

It was shown recently that the intractability of the symmetric travelling salesman problem is closely related with the difficulty of characterizing hypohamiltonian and hypotraceable (undirected) graphs. Namely, it was proved (c.f. [1]) that many of them induce facets of the monotone symmetric travelling salesman polytope.

We study hypohamiltonian and hypotraceable digraphs in order to check whether analogous results also hold with respect to the asymmetric travelling salesman problem. Indeed, it was shown in [2] and [3] that many of the hypohamiltonian digraphs obtained in [4], and some of the hypotraceable digraphs we present here, induce facets of the monotone asymmetric travelling salesman polytope.

Most of the known results on hypotraceable (undirected) graphs are due to Thomassen. In [7] and [8] Thomassen constructed hypotraceable graphs of order 34, 37, 39 and higher. A tight lower bound on the order of such graphs is, however, unknown.

The smallest known cubic hypotraceable graph was constructed by J. D. Horton and has order 40 [6]. In [9] the existence of an infinite family of cubic planar hypotraceable graphs is shown.

The question of the existence of hypotraceable digraphs was recently settled [5]. It was proved that such digraphs of order n exist iff $n \geq 7$, and that for each $k \geq 3$ there are infinitely many hypotraceable oriented graphs (digraphs with no 2-cycle) with a source and a sink and precisely k strong components. It was also shown in [5] that there are strongly connected hypotraceable oriented graphs and that there are hypotraceable digraphs with precisely two strong components one of which is a source. Finally, it was shown that hypotraceable (or hypohamiltonian) digraphs may contain large complete subdigraphs and large tournaments.

We denote (undirected) graphs by $G = [V, E]$ and edges $e \in E$ by $e = \{u,v\}$. A graph G is called hypotraceable if it is not traceable (i.e. does not contain a hamiltonian chain) but $G - v$ is traceable for all v in G.

A digraph $G = (V,E)$ consists of a finite node set V ($|V|$ is called the order of G) and an arc set of ordered pairs $e = (u,v)$, $u \neq v$, of nodes; multiple arcs are not allowed.

$$N^+(v) := \{ u : (v,u) \in E \},$$

$$N^-(v) := \{ u : (u,v) \in E \}.$$

$$N(v) := N^+(v) \cup N^-(v)$$

is the set of neighbours of v.

$$d^+(v) := |N^+(v)|$$

is called outdegree of v,

$$d^-(v) := |N^-(v)|$$

is called indegree of v,

$$d(v) := d^+(v) + d^-(v)$$

is called degree of v. A non-empty set of arcs

$$P = \{ (v_1,v_2);(v_2,v_3);\dots;(v_{k-1},v_k) \} \subset E$$

is called a path of length k-1 if $v_i \neq v_j$ for $i \neq j$ and is denoted by $[v_1,v_2,\dots,v_k]$. If $(v_k,v_1) \in E$ then

$$C := P \cup \{(v_k, v_1)\}$$

is called a circuit of length k, denoted by $(v_1,v_2,\dots,v_k)$. A path (circuit) of length $|V| - 1$ ($|V|$) is called hamiltonian. $G - v$ is the digraph with node set $V - \{v\}$ and the set of all arcs in E which do not contain v. $G = (V,E)$ is called traceable (hamiltonian) if G contains a hamiltonian path (circuit). G = (V,E) is called hypotraceable (hypohamiltonian) if G is not traceable (hamiltonian) but G - v is traceable (hamiltonian) for all $v \in V$. For $u,v \in V$ we define G - (u,v) to be the digraph (V, E - { (u,v) }), G + (u,v) is the digraph (V, E ∪ {(u,v)}).

2. Trivial Properties, Directing Hypotraceable Graphs

As the following lemma shows digraphs of the class considered cannot have all possible node degrees.

Lemma 2.1.

Let G = (V,E) be a hypotraceable digraph of order n.

Then for all $v \in V$

a) $d^+(v) \geq 1$ implies $d^+(v) \geq 2$,

b) $d^-(v) \geq 1$ implies $d^-(v) \geq 2$,

c) $d(v) \leq n - 2$.

Proof:

Suppose that $d^+(v) = 1$ and let $(v,w) \in E$. Every hamiltonian path P in G - w has v as its endpoint because the outdegree of v in G - w is 0. But then P ∪ {(v,w)} is a hamiltonian path in G. Contradiction! b) follows similarly.
c) Let v be any node in V, and let $P=[v_1,\dots v_{n-1}]$ be a hamiltonian path in G - v. Obviously (v,v_1) and (v_{n-1},v) cannot be in E and $(v_i,v) \in E$ implies $(v,v_{i+1}) \notin E$.

Suppose there is a path $Q = [v_p,\dots v_q] \subset P$ such that all nodes v_j, $p \leq j \leq q$, are neighbours of v and that v_p, v_q are doubly linked to v (i.e. (v_p,v), (v,v_p), (v,v_q), $(v_q,v) \in E$). As $(v_p,v) \in E$ implies $(v,v_{p+1}) \notin E$, i.e. $(v_{p+1},v) \in E$, hence $(v,v_{p+2}) \notin E$, etc. We obtain that $(v,v_q) \notin E$, a contradiction.
It follows that between every pair v_p,v_q of nodes that are doubly linked to v there is a node v_j, $p < j < q$, which is not a neighbour of v.

If we consider the sequence of nodes $w_1, w_2,\dots,w_k$ such that w_i is either doubly linked to v or not a neighbour of v where the indexing is comparable to the ordering in P, then no doubly linked nodes are consecutive, hence in the worst case this sequence is alternating. Similar arguments as above show that w_1 and w_k cannot be doubly linked to v. Therefore

$$d(v) \leq (n-1) - k + 2\,(k-1)\,/\,2$$
$$= n - 2$$

□

A node $v \in V$ with $d^-(v) = 0$ will be called a source, a node v with $d^+(v) = 0$ a sink. A source (sink) v with $d^+(v) = 2$ ($d^-(v) = 2$) will be called minimal. The third type of "minimal" nodes will be called distinguished. Such a node satisfies $d^+(v) = d^-(v) = 2$ and $|N(v)|=3$. Examples in the sequel will show that there are hypotraceable digraphs with distinguished nodes, minimal sources and sinks and nodes with degree n - 2.

Clearly, hypotraceable digraphs have to be connected. Suppose there would be an articulation node $w \in V$. Then G - w would be disconnected and could not contain a hamiltonian path. Therefore, hypotraceable digraphs are 2-connected. Later examples show that there are hypotraceable digraphs which are not 3-connected; therefore the minimum degree of connectivity of hypotraceable digraphs is lower than the one of hypohamiltonian digraphs, c.f.[4].

A digraph is strongly connected, or strong, if for every two nodes u and v there is a path from u to v (and a path from v to u).

Lemma 2.2.

The smallest nontrivial strong component of a hypotraceable digraph has at least five nodes.

Proof:

Let $H = (W,F)$ be a nontrivial strong component of a hypotraceable digraph $G = (V,E)$ such that $|W|$ is as small as possible.

It can be easily seen that H cannot have order 2 or 3. Suppose H has order 4. If H contains no circuit of length 4 then a contradiction is easily obtained. Let us consider then, the case that H contains a circuit of length 4 (this implies $|V| \geq 5$). The set V can obviously be partitioned into different node sets X^+, W, X^- (X^+ or X^- might be empty) such that:

1) any node in $W \cup X^+$ can be reached from any node in $X^- \cup W$ by a path

and that

2) there is no path from any node in $W \cup X^+$ to any node in X^-.

A node in X^- (resp. X^+) which is adjacent to a node v of H will be denoted by v^- (resp. v^+). Let $W = \{a, b, c, d\}$ and let P be a hamiltonian path in G - a. Then there are two possibilities:

Case 1:

There exists a circuit C in H, of length 4, such that P contains exactly 2 arcs of C.

Case 2:

There is no circuit in H, of length 4, such that P contains 2 arcs of the circuit.

In the first case consider w.l.o.g. $C = (a, b, c, d)$, $(b,c) \in P$, $(c,d) \in P$.

Therefore, P is of the form

$$P = P^- \cup [b^-, b, c, d, d^+] \cup P^+$$

where P^- (P^+) is a hamiltonian path in the subdigraph induced by X^- (X^+). Let Q be a path in G - b. Q can neither contain an arc (a^-,a) nor (a,a^+), otherwise G would be traceable. Furthermore, Q cannot contain the arc (a,c). This implies that Q cannot contain an arc (d^-,d). Therefore, Q has to contain an arc (c^-,c) and necessarily it contains the arcs (c,a) and (a,d). But then

$$P^- \cup [b^-, b, c, a, d, d^+] \cup P^+$$

is a hamiltonian path in G. Contradiction! Similar constructions also lead to a contradiction in the second case. □

In section 4 we will construct an infinite class of hypotraceable digraphs which have the property that all strong components contain five nodes. This shows that Lemma 2.2 is best possible and that hypotraceable digraphs can be highly non-strongly connected.

A simple way to get hypotraceable digraphs is to take hypotraceable graphs and direct them appropriately. Let $G = [V, E]$ be a graph, we substitute each edge $e = \{u,v\} \in E$ by the two arcs (u,v), (v,u) and denote the digraph obtained by

$$\vec{G} = (V, \vec{E})$$

and call $\vec{G}$ a trivially directed graph.

Theorem 2.3.

Let $G = [V,E]$ be a hypotraceable graph, then the following holds:

a) $\vec{G} = (V,\vec{E})$ is a hypotraceable digraph

b) $\vec{G} - e$ is a hypotraceable digraph for all $e \in \vec{E}$.

Proof:

Obvious. □

Thus using the results in [7] and [8] we can obtain hypotraceable digraphs of order 34, 37, 39, and higher.

Case b) of Theorem 2.3 is not the only way to modify trivially directed hypotraceable graphs. Depending on the underlying hypotraceable graph there are in general many possibilities to add or remove arcs from $\vec{G}$ and still maintaining the hypotraceable properties.

Another way to get possibly non-isomorphic hypotraceable digraphs of the same order is to reverse the direction of every arc. Clearly, every path $[v_1,\dots,v_k]$ in G will be a path $[v_k,\dots,v_1]$ in the reverse of G; hence, the defining properties of hypotraceable digraphs are kept.

3. Some Basic Hypotraceable Digraphs

In order to obtain hypotraceable digraphs of higher order we use the "basic" hypotraceable digraphs shown in Fig. 3.1 to 3.5.

The digraph T_7 is the only one which can be obtained by means of the construction presented in [5].

It is not difficult to check that the seven digraphs in Fig. 3.1. to Fig. 3.5. are hypotraceable; because of its length a proof is omitted (a short recursive computer program could do this job). In each of these digraphs an arc from the source to the sink can be added which obviously does not create a hamiltonian path. Hence there exist nonisomorphic hypotraceable digraphs of order n = 7,8,..., 13.

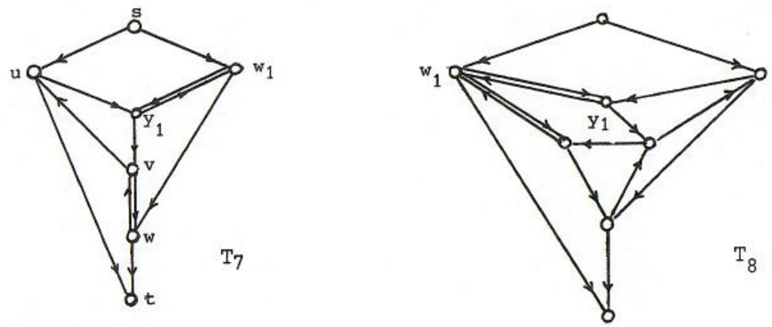

FIGURE 3.1

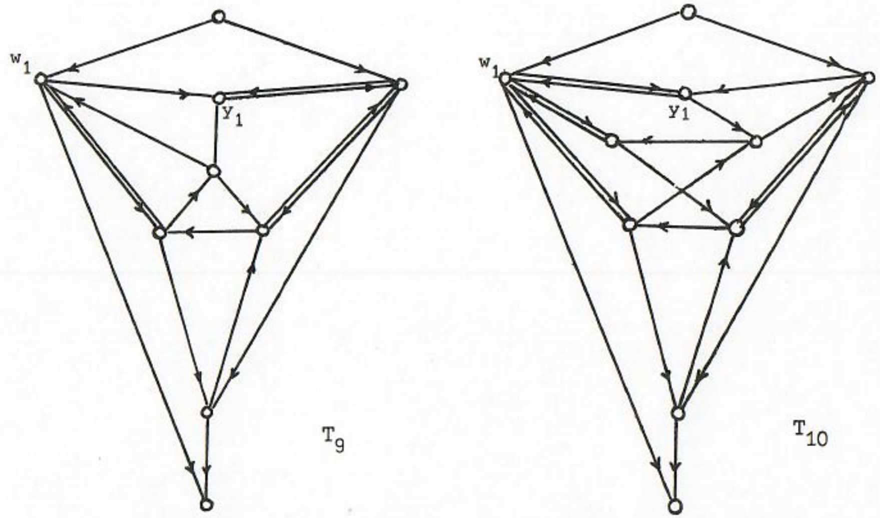

FIGURE 3.2

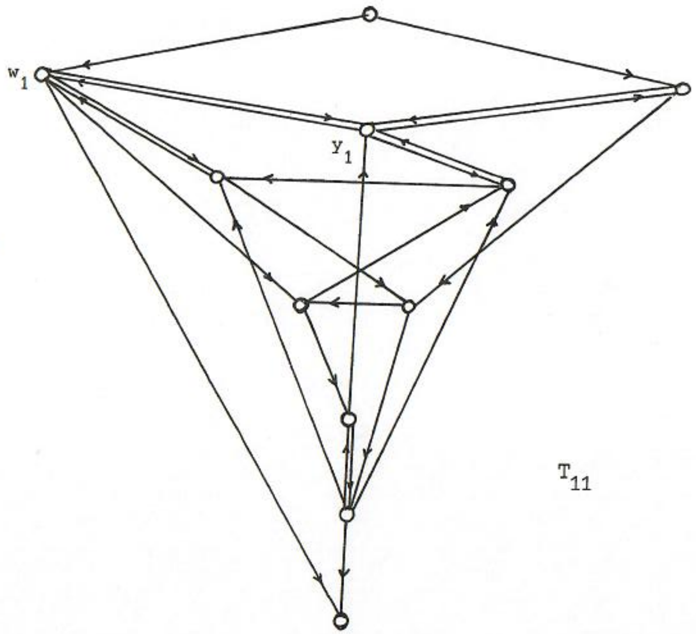

FIGURE 3.3

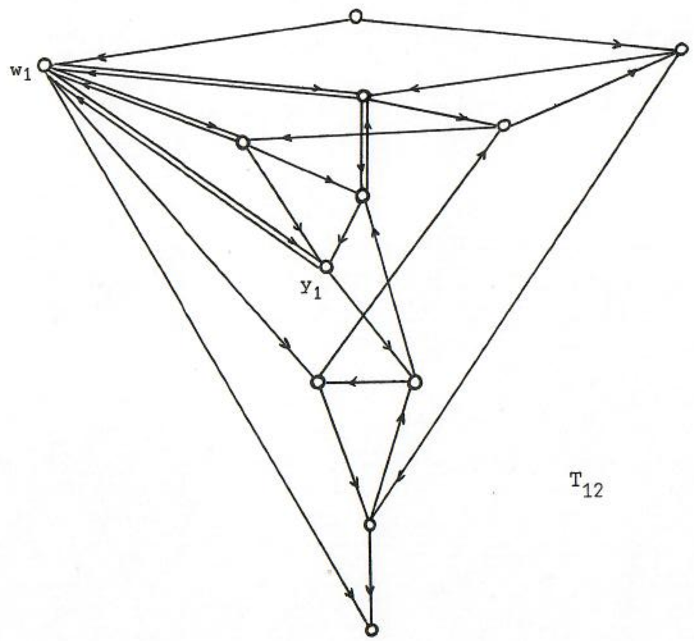

FIGURE 3.4

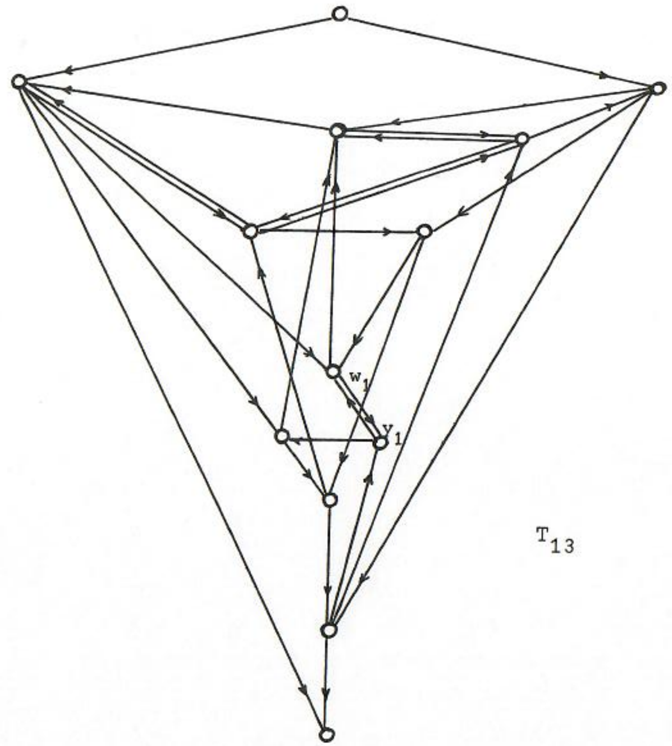

FIGURE 3.5

Even when the arc (s,t) from the source to the sink is added this new hypotraceable digraph is in general not maximal yet, i.e. we may add more arcs without getting a traceable digraph. Let T_7' be the digraph which contains all arcs of T_7 and furthermore the arcs (s,t), (u,v), and (w,w_1). It is easily seen that T_7' is also hypotraceable and that the addition of any further arc would generate a hamiltonian path. Thus T_7' is maximal hypotraceable and contains four nodes namely u,v,w,w_1 which have the maximal possible degree $n - 2 = 5$. Clearly there are many other ways of creating hypotraceable digraphs that contain T_7 or similary $T_8,\dots,T_{13}$. However, because of Lemma 2.1. no arc containing a node of maximal degree can be added without producing a hamiltonian path, e.g. no arc containing the node w_1 can be added to T_8 or T_{10}.

4. Construction of Hypotraceable Digraphs

In the following we will demonstrate three ways to construct hypotraceable digraphs from other digraphs. In the first method two "supertraceable" digraphs are linked to obtain one hypotraceable digraph. This construction produces hypotraceable digraphs of order $n \geq 12$ with source and sink. The second method combines two hypotraceable digraphs with source and sink to get a hypotraceable digraph with source but not sink. For the third method four hypohamiltonian digraphs are needed to obtain one strongly connected hypotraceable digraph, this method is a directed version of Thomassen's construction for hypotraceable graphs, c.f. [7].

Definition 4.1.

A digraph $G = (V,E)$, $|V| \geq 3$, is called <u>supertraceable</u> if it contains two non-empty special node sets $S, T \subset V$ with the following properties:

a) Each node $s \in S$ is the initial node of a hamiltonian path in G.

b) For every node $v \in V$ there exists a node $t \in T$ and a hamiltonian path P in $G - v$ with initial node t.

c) No node $t \in T$ is the initial node of a hamiltonian path in G.

If we have a supertraceable digraph $G = (V,E)$ with special node sets S,T then the reverse digraph $G^r = (V, E^r)$, where $E^r = \{(u,v) : (v,u) \in E\}$, has the above properties a), b), c) with respect to S and T where "initial" is replaced by "terminal" in the definition.

Supertraceable digraphs give hypotraceable digraphs in the following way:

Construction HT1:

Let $G_1 = (V_1,E_1)$ and $G_2' = (V_2,E_2')$ be two disjoint supertraceable digraphs with special node sets S_1, T_1 and S_2, T_2 respectively; let $G_2 = (V_2,E_2)$ be the reverse of G_2'. Let $A := \{(s,t) : s \in S_2, t \in T_1\}$, $B := \{(t,s) : t \in T_2, s \in S_1\}$, and define $G = (W,F)$ to be the digraph with

$$W := V_1 \cup V_2, \quad F := E_1 \cup E_2 \cup A \cup B.$$

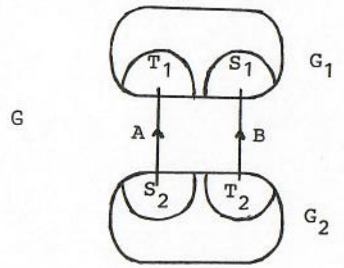

FIGURE 4.1

Theorem 4.2.

The digraph $G = (W,F)$ as defined in Construction HT1 is hypotraceable.

Proof:

a) Suppose G contains a hamiltonian path $P = [v_1, \ldots, v_n]$. P necessarily contains an arc $(v_i, v_{i+1}) \in A \cup B$. If $(v_i,v_{i+1}) \in A$ then the partial path $[v_{i+1}, \ldots, v_n] \subset P$ is a hamiltonian path in G_1; as $v_{i+1} \in T_1$ this contradicts c) of 4.1.
Similarly, $(v_i,v_{i+1}) \in B$ contradicts $v_i \in T_2$.

b) W.l.o.g. let $v \in V_1$. $G_1 - v$ contains a hamiltonian path P_1 with initial node $t \in T_1$, G_2 contains a hamiltonian path P_2 with terminal node $s \in S_2$, hence $(s,t) \in A$. Therefore $P_2 \cup \{(s,t)\} \cup P_1$ is a hamiltonian path in G - v.

This shows that G is hypotraceable. □

To prove the usefulness of Construction HT1 it remains to show that the class of supertraceable digraphs is not empty.

Remark 4.3.

For every hypotraceable digraph $G = (V,E)$ with source s the digraph G - s is supertraceable.

Proof:

G - s contains hamiltonian paths. Let S be the set of initial nodes of these paths. For every $v \in V - \{s\}$ there are hamiltonian paths in G - v which all necessarily start in s. If we remove the first arc of these paths we obtain hamiltonian paths in (G-s) - v. Let T be the set of initial nodes of these paths. By definition, S and T satisfy a), b) of Definition 4.1. Suppose, $t \in T$ is the initial node of a hamiltonian path P in G - s. By construction, $(s,t) \in E$ for all $t \in T$, hence $\{(s,t)\} \cup P$ is a hamiltonian path in G, a contradiction. Therefore, c) of 4.1. is also satisfied. □

Remark 4.4.

For every hypotraceable digraph G = (V,E) with sink t the digraph G - t is the reverse of a supertraceable digraph.

Proof:

Revert G and apply 4.3 □

A converse of Remark 4.3. which is another (trivial) construction of hypotraceable digraphs is obviously also true: Given a supertraceable digraph G = (V,E) with special sets S,T, then the digraph G' consisting of the node set V plus a new node s (source) and the arcs E plus all arcs (s,t), t ∈ T , is hypotraceable.

As the seven digraphs $T_7, T_8, \ldots, T_{13}$ shown in fig. 3.1 to 3.5 each have both a source and a sink we can construct several different supertraceable digraphs by removing the source or the sink. All new hypotraceable digraphs obtained by using two of these supertraceable digraphs in Construction HT1 will also have a source and a sink. By iteratively applying construction HT1 we get hypotraceable digraphs of all orders $n \geq 12$ and hence an infinite class of these digraphs. The number of hypotraceable digraphs of order n constructable in this way is not small at all. Consider e.g. the digraph T_7' defined in section 3; $T_7' - s$ is supertraceable where the special sets are $T = \{u, w_1\}$, and S any non empty subset of $\{v, w, y_1\}$, hence there are seven possible choices of S. Combining these supertraceable digraphs in various ways we get quite a number of nonisomorphic hypotraceable digraphs of order 12.

As the building blocks $T_7, \ldots, T_{13}$ resp. have one strongly connected component of size 5,...,11 resp. only, all newly constructed digraphs have strong components of these sizes only. By linking source-or sink-deleted digraphs T_7 repeatedly using construction HT1 we obtain an infinite number of hypotraceable digraphs with maximal (and minimal see 2.2) strong components having five nodes. Furthermore, all these digraphs are 2-connected but not 3-connected.

Summing up the foregoing remarks we have shown:

Theorem 4.5.

a) There are non-isomorphic hypotraceable digraphs of order n for every $n \geq 7$.

b) For all $n \geq 7$ there are hypotraceable digraphs of order n which are not strongly connected and have a minimal source and a minimal sink.

c) For every $n \geq 12$ there exists a hypotraceable digraph of order n the maximal strong component of which contains 5,6 or 7 nodes only.

Construction HT1 using the source-or-sink-deleted digraphs T_7 yields hypotraceable digraphs of order 5k + 2 , $k \geq 2$ with maximal strong components of size five and a source and a sink. Clearly, by the Lemma 2.2 there can be no hypotraceable digraphs of order 5k + 3 and 5k + 4, $k \geq 1$, the maximal strong component of which has order 5. However, there could exist hypotraceable digraphs of this kind having order 5k or 5k + 1, $k \geq 2$. These digraphs would either have neither source nor sink or a source or a sink but not both. Suppose there is such a digraph G = (V,E) without a sink. This digraph necessarily has a "last" strongly connected component (W,F), i.e. a component such that $(w,v) \notin E$ for all $w \in W$ and all $v \in V-W$. Thus, (W,F) is a strongly connected supertraceable digraph of order 5. By simple (but lengthy) enumeration we have shown that there are no supertraceable digraphs of order 5 at all. This proves that such a digraph G = (V,E) does not exist, hence, there are hypotraceable digraphs of order n, the maximal strongly connected component of which has five nodes only, if and only if n = 5k + 2, $k \geq 1$, i.e. using digraphs T_7 (and modifications of these) in Construction HT1 is the only way to get such hypotraceable digraphs.

The following construction was designed to obtain hypotraceable digraps which have a source but no sink or vice-versa (by reversion).

CONSTRUCTION HT2:

Let G_1 and G_2 be two disjoint hypotraceable digraphs with source and sink. Let u_1, v_1 be the source resp. sink of G_1 and y_1 the terminal node of a hamiltonian path in $G_1 - v_1$.

Let u_2, v_2 be the source resp. sink of G_2 and x_2 the initial node of a hamiltonian path in $G_2 - u_2$. Furthermore, we assume that G_1 has a node w_1, $w_1 \neq u_1, w_1 \neq v_1$, $w_1 \neq y_1$, such that the following conditions are satisfied:

C1) G_1 does not contain two node-disjoint paths $Q = [w_1, \ldots, y_1]$ and $Q' = [u_1, \ldots, v_1]$ which contain all nodes of G_1, and

C2) G_1 does not contain two node-disjoint paths $R = [u_1, \ldots, y_1]$ and $R' = [w_1, \ldots, v_1]$ which contain all nodes of G_1.

Let G be the digraph obtained by adding the digraphs G_1 and G_2 identifying the nodes v_1 and x_2 into a node z, and by adding the arcs $A = \{(v_2,u_2), (y_1,u_2), (v_2,w_1)\}$ (c.f. Fig.4.2).

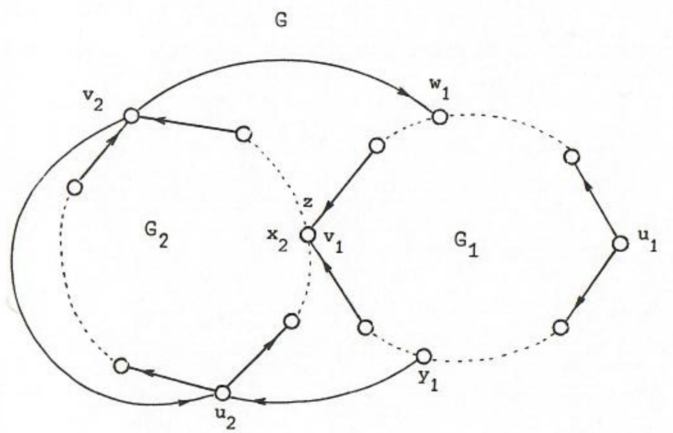

FIGURE 4.2

Theorem 4.6.

Given two hypotraceable digraphs G_1 and G_2 with source and sink, where G_1 has the mentioned properties, then the digraph G obtained by Construction HT2 is hypotraceable.

Proof:

a) Suppose G contains a hamiltonian path P. Then P has to contain at least one of the arcs of $A = \{(v_2,u_2), (v_2,w_1), (y_1,u_2)\}$

a_1) $P \cap A = \{(v_2,u_2)\}$. Then, P has to contain a hamiltonian path in G_1 (from u_1 to v_1), which is impossible.

a_2) $P \cap A = \{(v_2,w_1)\}$. In this case, P has to contain a hamiltonian path in G_2 (from x_2 to v_2), what cannot happen.

a_3) $P \cap A = \{(y_1,u_2)\}$. Then, P has to contain either a hamiltonian path in G_2 (from u_2 to v_2) or a hamiltonian path in G_1 (from u_1 to y_1), which is impossible.

a_4) $P \cap A = \{(v_2,w_1), (y_1,u_2)\}$. In this case, there are two possibilities:

a_{41}) $P = [u_1, \ldots, v_2, w_1, \ldots, y_1, u_2, \ldots]$ or

a_{42}) $P = [u_1, \ldots, y_1, u_2, \ldots, v_2, w_1, \ldots]$.

In the case a_{41}), in order to be a hamiltonian path in G, P has to contain two node-disjoint paths $Q = [w_1, \ldots, y_1]$ and $Q' = [u_1, \ldots, v_1]$ in G_1 which contain all nodes of G_1. By condition C1) this cannot happen.

In the case a_{42}) , P has to contain two node-disjoint paths $R = [u_1, .., y_1]$ and $R' = [w_1, \ldots, v_1]$ in G_1 which contain all nodes of G_1, which contradicts condition C2).

b) We show that G - t is traceable for any node t in G.

b_1) $t = u_2$. Let P_1 be a hamiltonian path in $G_1 - w_1$ (from u_1 to $v_1 = z$) and P_2 a hamiltonian path in $G_2 - u_2$ (from $x_2 = z$ to v_2). Then, $P_1 \cup P_2 \cup \{(v_2,w_1)\}$ is a hamiltonian path in $G - t$.

b_2) t in G_1, $t \neq v_1$. Let Q_1 be a hamiltonian path in $G_1 - t$ (with terminal node $v_1 = z$) and Q_2 a hamiltonian path in $G_2 - u_2$ (with initial node $x_2 = z$). Then $Q_1 \cup Q_2 \cup \{(v_2,u_2)\}$ is a hamiltonian path in $G - t$.

b_3) t in G_2, $t \neq u_2$. Let R_1 be a hamiltonian path in $G_1 - v_1$ (from u_1 to y_1) and R_2 a hamiltonian path in $G_2 - t$. Then, $R_1 \cup \{(y_1, u_2)\} \cup R_2$ is a hamiltonian path in $G - t$. □

Remark 4.7.

The hypotraceable digraphs T_7, T_8, T_9, T_{10}, T_{11}, T_{12} and T_{13} shown in figures 3.1. to 3.5. satisfy the conditions required (for the digraph G_1) in Construction HT2, where the nodes w_1 and y_1 are those indicated in the respective figures. Therefore, they can be used (as G_1) with any hypotraceable digraph with source and sink (as G_2) to obtain new hypotraceable digraphs. □

Remark 4.7. together with Theorem 4.5.b) implies:

Theorem 4.8.

a) For every $n \geq 13$ there exist hypotraceable digraphs of order n which contain a source or a sink s but not both.

b) For each of these hypotraceable digraphs $G - s$ is strongly connected. □

The following construction is a directed version of Thomassen's construction, c.f. [7], and produces strongly connected hypotraceable digraphs.

Definition 4.9.

If x is a distinguished node of a digraph G then the neighbours of x will be numbered as indicated in Fig 4.3. □

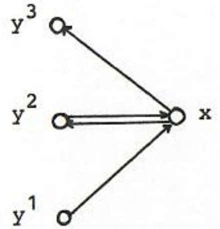

FIGURE 4.3

Definition 4.10. (c.f. [4])

Let x be a distinguished node of a hypohamiltonian digraph G and let y^1, y^2, y^3 be the neighbours of x, numbered as in definition 4.9. (G,x) is said to have property β if there are no hamiltonian paths from y^1 to y^2, from y^1 to y^3 and from y^2 to y^3 in $G - x$. □

Construction HT3:

Let G_1, G_2, G_3, G_4 be four disjoint hypohamiltonian digraphs with distinguished nodes x_1, x_2, x_3, x_4 resp. whose neighbours y^i_j $(j = 1,\dots,4)$ are numbered as in Definition 4.9. We assume that all pairs (G_1,x_1), (G_2,x_2), (G_3,x_3), (G_4,x_4) have property ß. Let $H_i = G_i - x_i$, $i = 1,\dots,4$. We add the digraphs H_i identifying the nodes y_1^1, y_3^3 into a node z_1, the nodes y_4^3, y_2^1 into a node z_2 and add the arcs

$$(y_1^3 , y_2^3), (y_2^3 , y_1^3), (y_1^2 , y_2^2), (y_2^2 , y_1^2),$$

$$(y_3^2 , y_4^2), (y_4^2 , y_3^2), (y_3^1 , y_4^1), (y_4^1 , y_3^1)$$

calling the resulting digraph G (c.f. Fig. 4.4). F_1 is the union of H_1 and H_2 together with the arcs linking H_1 and H_2; F_2 is the union of H_3 and H_4 together with the arcs linking H_3 and H_4.

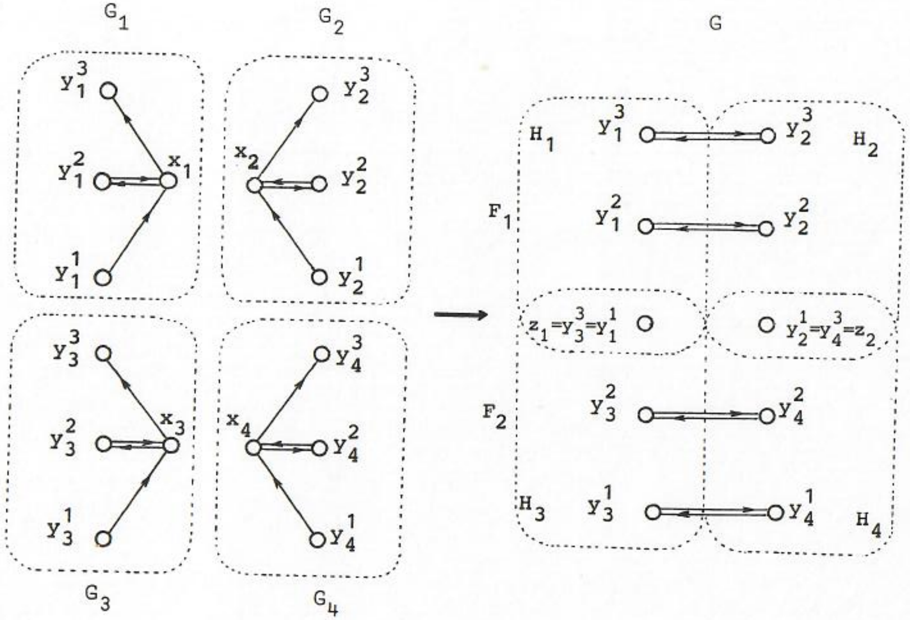

FIGURE 4.4

THEOREM 4.11.

Given four hypohamiltonian digraphs G_1, G_2, G_3, G_4, with the above properties then the digraph G obtained by Construction HT3 is hypotraceable.

Proof:

a) We have to show that $G - v$ is traceable for all nodes v. Let v be a node of H_1. $G_1 - v$ is hamiltonian, so $H_1 - v$ has a hamiltonian path P_1 from y_1^3 to y_1^2, or from y_1^3 to y_1^1 or from y_1^2 to y_1^1.

a_1) P_1 goes from y_1^3 to y_1^2. $G_2 - y_2^3$ contains a hamiltonian circuit containing $[y_2^1, x_2, y_2^2]$, we drop this path and obtain a hamiltonian path P_2 in $H_2 - y_2^3$. $G_4 - y_4^1$ contains a hamiltonian circuit which contains $[y_4^2, x_4, y_4^3]$, dropping this path we obtain a hamiltonian path P_4 in $H_4 - y_4^1$. $G_3 - y_3^3$ contains a hamiltonian circuit containing $[y_3^1, x_3, y_3^2]$, we drop this path and obtain a hamiltonian path P_3 in $H_3 - y_3^3$. Now

$$\{(y_2^3 , y_1^3)\} \cup P_1 \cup \{(y_1^2 , y_2^2)\} \cup P_2 \cup$$
$$\cup P_4 \cup \{(y_4^2 , y_3^2)\} \cup P_3 \cup \{(y_3^1 , y_4^1)\}$$

is a hamiltonian path in $G - v$.

a_2) P_1 goes from y_1^3 to y_1^1. H_2 contains a hamiltonian circuit C_2 containing an arc (y_2^3 , u). We drop this arc obtaining a hamiltonian path P_2 in H_2. $G_3 - y_3^1$ contains a hamiltonian circuit containing $[y_3^2, x_3, y_3^3]$, dropping this path we obtain a hamiltonian path P_3 in $H_3 - y_3^1$. $G_4 - y_4^3$ contains a hamiltonian circuit containing $[y_4^1, x_4, y_4^2]$, we drop this path obtaining a path P_4 in $H_4 - y_4^3$. The path

$$P_2 \cup \{(y_2^3, y_1^3)\} \cup P_1 \cup P_3 \cup \{(y_3^2, y_4^2)\} \cup P_4 \cup \{(y_4^1, y_3^1)\}$$

is hamiltonian in $G - v$

a_3) P_1 goes from y_1^2 to y_1^1. This case is done as a_2), instead of dropping (y_2^3, u) in C_2 we drop the arc (y_2^2, u) and link P_2 and P_1 through (y_2^2, y_1^2).

As H_1 and H_2 are symmetric the same result follows if v is in H_2. Using similar constructions as above we can obtain a hamiltonian path analogously, if v is in F_2.

b) Suppose G contains a hamiltonian path P from w_1 to w_2. P includes the nodes z_1 and z_2, therefore it is composed either of paths

b_1) P_1 from w_1 to z_1, P_2 from z_1 to z_2, P_3 from z_2 to w_2

or

b_2) P_1 from w_1 to z_2, P_2 from z_2 to z_1, P_3 from z_1 to w_2.

The paths P_1, P_3 may have length zero. P_2 is entirely contained in F_1 or F_2, and at least one of the paths P_1, P_3 is contained in F_2, F_1 resp. Therefore we have to consider several cases.

b_{11}) P_2 in F_1, P_2 contains exactly one of the arcs (y_1^3, y_2^3), (y_1^2, y_2^2).

1. P_1 in F_1, P_3 in F_2. $P_1 \cup P_2$ is a hamiltonian path in F_1.

1.1. If w_1 is in H_1, then $P_1 \cup P_2$ contains a hamiltonian path in H_2 from either y_2^2 or y_2^3 to y_2^1. Adding to this path we obtain either $[y_2^1, x_2, y_2^2]$ or $[y_2^1, x_2, y_2^3]$ we obtain a hamiltonian circuit in G_2. Contradiction.

1.2. If w_1 is in H_2 then $P_1 \cup P_2$ contains a hamiltonian path in H_1. This path goes either from y_1^3 to y_1^2 or from y_1^2 to y_1^3. In the first case we add $[y_1^2, x_1, y_1^3]$ obtaining a hamiltonian circuit in G_1, a contradiction, the second case is impossible because (G_1, x_1) has property β.

2. P_1 in F_2, P_3 in F_1. $P_2 \cup P_3$ is a hamiltonian path in F_1.

2.1. If w_2 is in H_2 then $P_2 \cup P_3$ contains a hamiltonian path in H_1 from y_1^1 to either y_1^2 or y_1^3, since (G_1, x_1) has property β this is impossible.

2.2. If w_2 is in H_1 then $P_2 \cup P_3$ contains a hamiltonian path in H_2 going from y_2^3 to y_2^2 or from y_2^2 to y_2^3. In the first case we add $[y_2^2, x_2, y_2^3]$ obtaining a hamiltonian circuit in G_2, a contradiction, the second case cannot occur as (G_2, x_2) has property β.

3. P_1 in F_2, P_3 in F_2. Then P_2 contains a hamiltonian path in H_2. We obtain a contradiction as in 1.1.

b_{12}) P_2 in F_2.

1. P_1 in F_2, P_3 in F_1. $P_1 \cup P_2$ is a hamiltonian path in F_2.

1.1. If w_1 is in H_3 then P_2 contains a hamiltonian path in H_4 from either y_4^1 or y_4^2 to y_4^3, a contradiction since (G_4, x_4) has property β .

1.2. If w_1 is in H_4 then $P_1 \cup P_2$ contains a hamiltonian path in H_3 going from either y_3^2 to y_3^1 or from y_3^1 to y_3^2. In the first case we add $[y_3^1, x_3, y_3^2]$ obtaining a hamiltonian circuit in G_3, contradiction; as (G_3, x_3) has property β the second case is impossible.

2. P_1 in F_1, P_3 in F_2. Then $P_2 \cup P_3$ is a hamiltonian path in F_2.

2.1. If w_2 is in H_4, then P_2 contains a hamiltonian path in H_3, going from y_3^3 to either y_3^1 or y_3^2, we add either $[y_3^1, x_3, y_3^3]$ or $[y_3^2, x_3, y_3^3]$ obtaining a hamiltonian circuit in G_3, contradiction.

2.2. If w_2 is in H_3, then $P_2 \cup P_3$ contains a hamiltonian path in H_4 going from either y_4^2 to y_4^1 or from y_4^1 to y_4^2. In the first case we add $[y_4^1, x_4, y_4^2]$ getting a hamiltonian circuit in G_4, the second case is impossible since (G_4, x_4) has property β .

3. P_1 in F_1, P_3 in F_1. P_2 contains a hamiltonian path in H_3 going from y_3^3 to either y_3^1 or y_3^2, like case 2.1.

Case b_2) follows from b_1) because of symmetry.

In order to be able to use construction HT3 we need hypohamiltonian digraphs G_i $(i = 1,2,3,4)$ with distinguished nodes x_i such that the pairs (G_i, x_i) have property β . Let $p \geq 3$ and odd , $V := \{a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p\}$,

$$E := \{(a_i, a_{i+1}), (b_i, b_{i+1}) : i = 1, \ldots, p-1\} \cup$$

$$\{(a_p, a_1), (b_p, b_1)\} \cup \{(a_i, b_i), (b_i, a_i) : i = 1, \ldots, p\}$$

then the digraphs $M_p = (V,E)$ are called <u>odd Marguerites</u>. All nodes $x \in V$ are obviously distinguished . It was shown in [4] that odd Marguerites are hypohamiltonian digraphs and that for every $p \geq 5$, odd, and every $x \in V$ the pair (M_p, x) has property β

Furthermore, the hypohamiltonian digraphs Y_8 and Y_9 shown in Fig. 3.2 in [4] have property β with respect to the distinguished nodes marked x in Fig. 3.2 in [4] . This can be checked easily by enumeration.

By using four of the digraphs Y_8 mentioned above Construction HT3 yields the hypotraceable digraph of order 26 shown in Fig. 4.5.

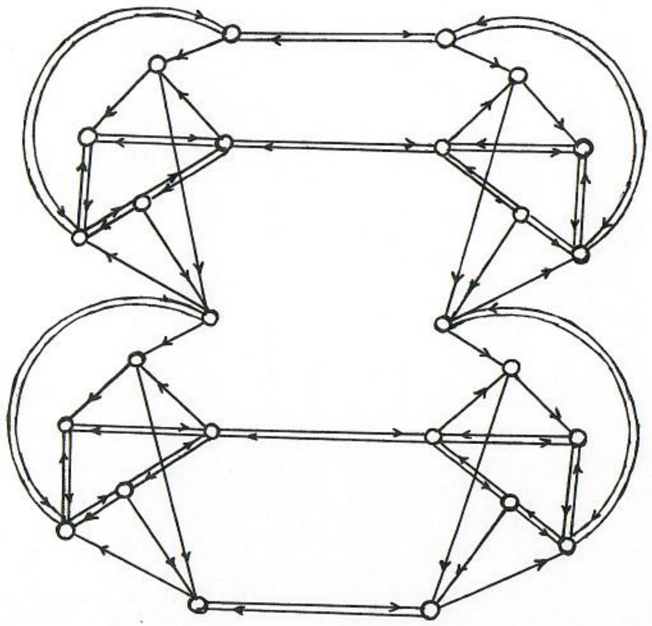

FIGURE 4.5

All hypotraceable digraphs obtained by Construction HT3 are obviously strongly connected. By combining the digraphs Y_8, Y_9 and M_p, $p \geq 5$ and odd, appropriately, we get

Theorem 4.12.

Construction HT3 gives strongly connected hypotraceable digraphs of order n for all $n \geq 26$.

Remark. It is shown in [5] that there exist strongly connected hypotraceable digraphs of order n for all $n \geq 12$.

REFERENCES

[1] M.Grötschel, On the monotone symmetric travelling salesman problem: hypohamiltonian/hypotraceable graphs and facets. Math. Operations Res.5 (1980) 285-292.

[2] M.Grötschel and Y.Wakabayashi, On the structure of the monotone asymmetric travelling salesman polytope I: hypohamiltonian facets. Discrete Math. 34 (1981) 43-59.

[3] M. Grötschel and Y. Wakabayashi, On the structure of the monotone asymmetric travelling salesman polytope II: hypotraceable facets. Math. Programming Studies 14 (1981) 77-97.

[4] M.Grötschel and Y.Wakabayashi, Hypohamiltonian digraphs. Methods of Operations Res.36 (1980) 99-119.

[5] M.Grötschel, C.Thomassen and Y.Wakabayashi,Hypotraceable digraphs. J.Graph Theory, Vol.4 (1980), 377-381 .

[6] R.R.Guy, Monthly research problems, 1969-1973. Amer.Math. Monthly 80 (1973) 1120-1128.

[7] C.Thomassen, Hypohamiltonian and Hypotraceable graphs. Discrete Math. 9 (1974) 91-96.

[8] C.Thomassen, Planar and infinite hypohamiltonian and hypotraceable graphs. Discrete Math. 14 (1963) 337-389.

[9] C.Thomassen, Planar cubic hypohamiltonian and hypotraceable graphs. J. Comb. Theory (B), 30 (1981) 36-44.