A Polyhedral Study of the Asymmetric Traveling Salesman Problem with Time Windows

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The asymmetric traveling salesman problem with time windows (ATSP-TW) is a basic model for scheduling and routing applications. In this paper, we present a formulation of the problem involving only 0/1 variables associated with the arcs of the underlying digraph. This has the advantage of avoiding additional variables as well as the associated (typically very ineffective) linking constraints. In the formulation, time-window restrictions are modeled using "infeasible path elimination" constraints. We present the basic form of these constraints along with some possible strengthenings. Several other classes of valid inequalities derived from related asymmetric traveling salesman problems are also described, along with a lifting theorem. We also study the ATSP-TW polytope, P_{TW}, defined as the convex hull of the integer solutions of our model. We show that determining the dimension of P_{TW} is a strongly $\mathcal{NP}$-complete problem, even if only one time window is present. In this latter case, we provide a minimal equation system for P_{TW}. Computational experiments on the new formulation are reported in a companion paper, where we show that it outperforms alternative formulations on some classes of problem instances. © 2000 John Wiley & Sons, Inc.

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1. INTRODUCTION

The *asymmetric traveling salesman problem with time windows* (*ATSP-TW*), in the "open path" version considered in this paper, can be described as follows: Consider a complete digraph $G = (V, A)$ on $n := |V|$ nodes, having nonnegative arc *costs* c_{ij} and *setup times* t_{ij} associated with each arc $(i, j) \in A$. Nodes correspond to jobs to be processed in sequence and without preemption on a single machine. Arcs correspond to job transitions, the associated setup times t_{ij} giving the changeover time needed to process node j right after node i. A *processing time* $p_i \geq 0$, a *release date* $r_i \geq 0$, and a *deadline* $d_i \geq r_i$ are given for every node $i \in V$. The release date r_i denotes the earliest possible and the deadline d_i the latest possible starting time for processing node $i \in V$. The interval $[r_i, d_i]$ is called the *time window* for node i. The time window is called *active* if $r_i > 0$ or $d_i < +\infty$; a time window of the type $[0, +\infty)$ is called *relaxed*. We deal with the case where waiting times are allowed, that is, processing may reach a node $i \in V$ earlier than r_i and wait until the node is released. We assume that costs and setup times, as well as processing times, release dates, and deadlines are integer values and allow $d_i = +\infty$ for some nodes $i \in V$.

The *minimal time delay* for processing node j immediately after node i is given by

$$\vartheta_{ij} := p_i + t_{ij}.$$

In several applications, the triangle inequality on ϑ is satisfied, that is:

$$\vartheta_{ij} \leq \vartheta_{ik} + \vartheta_{kj}, \quad \text{for all } i, j, k \in V, |\{i, j, k\}| = 3. \tag{1.1}$$

The problem is to find a min-cost Hamiltonian path satisfying the time-window restrictions, that is, a node sequence with minimal total cost such that, for every node $i \in V$, the start time for processing (visiting) node $i \in V$ lies within the given time window $[r_i, d_i]$.

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By introducing an additional node (depot), the problem can easily be transformed into its "closed-tour" version in which one is interested in finding a min-cost Hamiltonian tour through all nodes, starting at the depot node at time 0 and satisfying the time-window restrictions for all other nodes.

Note that ATSP-TW reduces to the standard asymmetric TSP when the time windows are relaxed; hence, ATSP-TW with general time windows is an $\mathcal{NP}$-hard problem. In fact, Savelsbergh [33] showed that it is already strongly $\mathcal{NP}$-complete to find a feasible solution for the problem; this also follows from Garey and Johnson [26], who showed that it is strongly $\mathcal{NP}$-complete to find a feasible schedule for nonpreemptive single-machine scheduling with release times and deadlines. Furthermore, Tsitsiklis [35] showed that the problem with general time windows remains strongly $\mathcal{NP}$-complete, even if the nodes correspond to points on a line and all processing times equal 0.

The research in this paper was motivated by a joint project with industry that had the aim of minimizing the time needed for the unloaded moves of a stacker crane in an automated storage system (see Ascheuer [3] and Abdel-Hamid et al. [1] for details). As already mentioned, the ATSP-TW (in its "open-path" version) can be interpreted as a *one-machine scheduling problem* with time-windows and sequence-dependent setup times. We refer to Queyranne and Schulz [31] for a comprehensive survey on polyhedral approaches to scheduling problems.

Some different versions of the ATSP-TW were discussed in the literature on vehicle-routing problems. For example, ATSP-TW is a subproblem in the "*cluster-first, route-second approach,*" where the nodes that have to be visited are first clustered according to some heuristic criterion and then routed by solving an instance of ATSP-TW for each cluster. In other applications, soft time windows are considered, that is, a violation of the time windows is allowed but results in an additional penalty cost. For a survey on time-constrained routing and scheduling problems, see Desrochers et al. [19, 20], among others.

To our knowledge, only a few papers describe exact methods for solving to optimality the (symmetric) TSP with time windows. Christofides et al. [17] and Baker [7] described branch-and-bound algorithms, whereas Dumas et al. [21] solved the problem through dynamic programming. Polyhedral methods have turned out to be an appropriate tool to solve the unconstrained ATSP (Fischetti and Toth [25]) and the ATSP with precedence constraints (Ascheuer et al. [4, 6]). Hence, we started our attempt of extending these methods to take time windows into account.

In this paper, we address a polyhedral study of the ATSP-TW. The paper is organized as follows: Section 2 gives the notation used throughout and introduces the basic model that we use, in which the time-window restrictions are modeled using "infeasible path elimination" constraints. The polytope associated with the feasible solutions to this model is addressed in Section 3, where we investigate the polytope dimension in the very special case in which only one of the time windows is active. We show that, even in this oversimplified case, determining the polytope dimension is a strongly $\mathcal{NP}$-complete problem. Several classes of additional valid inequalities are introduced in Section 4. In particular, we describe different lifted forms of the basic infeasible path elimination constraints, new inequalities associated with concatenation of feasible paths, a lifting procedure, and a strengthened version of the (π, σ)-inequalities proposed by Balas et al. [12] for the ATSP with precedence constraints. Some conclusions are finally drawn in Section 5.

2. NOTATION AND MODELING

The ATSP-TW can be modeled in various ways: A "standard model" (cf. Desrochers and Laporte [18], among others) involves binary arc variables x_{ij} as well as node variables τ_i, indicating the time when node i is visited. The time-window restrictions are modeled with the help of the bound constraints $r_i \leq \tau_i \leq d_i$; linking between the x and τ variables is provided by a generalization of the Miller–Tucker–Zemlin inequalities, namely:

$$\tau_i + \vartheta_{ij} - (1 - x_{ij}) \cdot M \leq \tau_j \qquad \forall i, j \in V, i \neq j,$$

where M is a sufficiently large positive value. Note that these constraints involve a "big M" term that is known to cause computational problems.

Recently, Maffioli and Sciomachen [29] and van Eijl [36] proposed a different model involving $|A|$ additional variables y_{ij} which give the time when node i is left in direction to node j (with $y_{ij} = 0$ whenever $x_{ij} = 0$). The time-window restrictions are then modeled via the constraints $r_i x_{ij} \leq y_{ij} \leq d_i x_{ij}$, whereas the linking between x and y variables is established through the inequalities

$$\sum_{i=1}^{n} (y_{ij} + \vartheta_{ij} x_{ij}) \leq \sum_{k=1}^{n} y_{jk} \quad \forall j = 1, \dots, n.$$

The computational results reported in Bogatsch [14] and our companion paper [5] indicate that this model outperforms the standard one, mainly on instances where time windows are not too tight.

In this section, we provide a different way of modeling the ATSP-TW as an integer linear program, which avoids additional variables as well as "big M" coefficients. The time-window constraints are modeled implicitly by a class of *infeasible path constraints*. The infeasibility is not restricted to the case of time windows: Capacity constraints, for example, might be modeled as well in a similar vein.

We next introduce the main notation used in the sequel: Given a node set $W \subseteq V$, let

$$A(W) := \{(i, j) \in A | i, j \in W\}$$

denote the set of all arcs with tail and head in W. For any two node sets $U, W \subseteq V$ let,

$$(U : W) := \{(i,j) \in A | i \in U, j \in W\}$$

denote the set of arcs with tail in U and head in W. To simplify notation, we write $(W : j)$ and $(j : W)$ instead of $(W : \{j\})$ and $(\{j\} : W)$, respectively. If $U = \emptyset$ or $W = \emptyset$, then $(U : W) = \emptyset$. Given a node set $W \subset V, W \neq \emptyset$, we also define

$$\delta^-(W) := \{(i,j) \in A | i \in V \backslash W, j \in W\},$$

$$\delta^+(W) := \{(i,j) \in A | i \in W, j \in V \backslash W\},$$

$$\delta(W) := \delta^-(W) \cup \delta^+(W).$$

The arc set $\delta(W)$ is called a *cut*. To simplify notation, we write $\delta^-(v), \delta^+(v)$, and $\delta(v)$ instead of $\delta^-(\{v\}), \delta^+(\{v\})$, and $\delta(\{v\})$, respectively. The numbers $|\delta^-(v)|, |\delta^+(v)|$, and $|\delta(v)|$ are called the *indegree*, *outdegree*, and *degree* of node $v \in V$. A node with degree zero is called *isolated*.

For notational convenience, a path P consisting of the arc set $\{(v_i, v_{i+1}) | i = 1, \ldots, k-1\}$ is sometimes denoted by $P = (v_1, v_2, \ldots, v_k)$. If not stated differently, the path P is always intended to be open and *simple*, that is, $|P| = k - 1$ and $v_i \neq v_j$ for $i \neq j$. Moreover, we let

$$[P] := \{(v_i, v_j) \in A | 1 \leq i < j \leq k\}$$

denote the *transitive closure* of $P = (v_1, \ldots, v_k)$.

Given a path $P = (v_1, \ldots, v_k)$, the *earliest starting time* t_{v_i} at node v_i $(i = 1, \ldots, k)$ along P is computed as

$$t_{v_1} := r_{v_1}$$

$$t_{v_i} := \max\{t_{v_{i-1}} + \vartheta_{v_{i-1}v_i}, r_{v_i}\} \qquad \text{for } i = 2, \ldots, k.$$

Notice that the formula introduces "waiting times" $w_{v_i} := \max\{0, r_{v_{i-1}} - (t_{v_{i-1}} + \vartheta_{v_{i-1}v_i})\}$ which are positive whenever a node $v_i, i = 2, \ldots, k$, is reached before its release date. If $w_i = 0$ for all $i = 2, \ldots, k$, the path is called *minimal*. We denote by $\vartheta(P) := t_{v_k}$ the earliest starting time at the last node of P. For a minimal path, $\vartheta(P) = r_{v_1} + \sum_{i=1}^{k-1} \vartheta_{v_i v_{i+1}}$ holds. To simplify notation, we sometimes write $\vartheta(v_1, v_2, \ldots, v_k)$ instead of $\vartheta(P)$ for $P = (v_1, v_2, \ldots, v_k)$.

A Hamiltonian path $P = (v_1, \ldots, v_n)$ is called *feasible* if each node is visited within its time window, that is, $r_{v_i} \leq t_{v_i} \leq d_{v_i}$ for $i = 1, \ldots, n$. A path $P = (v_1, \ldots, v_k)$, where $2 \leq k \leq n$, is said to be *infeasible* if it does not occur as a subpath in any feasible Hamiltonian path. Deciding whether a given path P is feasible is clearly an $\mathcal{NP}$-complete problem, even when P contains only one node, as in this case it amounts to deciding whether a feasible Hamiltonian path exists. Easily checkable and obvious sufficient conditions for infeasibility are given in the following lemma:

Lemma (2.1). *A path $P = (v_1, \ldots, v_k)$ is infeasible if at least one of the following conditions holds*:

(*i*) *P violates the deadline for its last node v_k, that is, $\vartheta(P) > d_{v_k}$.*

(*ii*) *The triangle inequality* (1.1) *on ϑ is satisfied and there exists a node w not contained in P such that both paths $P_1 = (w, v_1, \ldots, v_k)$ and $P_2 = (v_1, \ldots, v_k, w)$ violate the given deadline on their last node, that is, $\vartheta(P_1) > d_{v_k}$ and $\vartheta(P_2) > d_w$.*

In case condition (*ii*) above is satisfied, we say that node w cannot be *covered by* (*an extension of*) *path P*. If the triangle inequality on ϑ is not satisfied, the condition can easily be generalized by considering the ϑ-shortest paths from w to v_1 and from v_k to w.

Time windows give rise to precedences among the nodes. For example, whenever the ϑ-shortest path from j to i is longer than $d_i - r_j$, we can conclude that i has to precede j in any feasible solution. Let $i \prec j$ denote the fact that i has to precede j in any feasible solution and let $G_P = (V, R)$ denote the *precedence digraph* where each arc $(i, j) \in R$ represents a precedence relationship $i \prec j$. Without loss of generality, we assume G_P to be acyclic and transitively closed. Moreover, let

$$\pi(v) := \{i \in V | (i, v) \in R\},$$

$$\sigma(v) := \{j \in V | (v, j) \in R\}$$

represent the set of the *predecessors* and *successors* of a node $v \in V$, respectively. Set, moreover, $\pi(X) := \cup_{v \in X} \pi(v)$ and $\sigma(X) := \cup_{v \in X} \sigma(v)$ for all $X \subseteq V$.

We next describe the basic model that we propose, written for the "open-path" version of ATSP-TW. A similar model can easily be defined for the "closed-tour" version of the problem. For each arc $(i, j) \in A$, we introduce a binary variable $x_{ij} \in \{0, 1\}$ with the interpretation

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \in A \text{ is chosen,} \\ 0, & \text{otherwise.} \end{cases}$$

For any $Q \subseteq A$, we write $x(Q)$ for $\sum_{(i,j) \in Q} x_{ij}$. To guarantee that the set of chosen arcs (represented by x) forms a feasible Hamiltonian path, we introduce the following model:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij}$$

$$\text{s.t.} \quad x(A) = n - 1 \tag{2.2}$$

$$x(\delta^-(i)) \leq 1 \qquad \forall\ i \in V \tag{2.3}$$

$$x(\delta^+(i)) \leq 1 \qquad \forall\ i \in V \tag{2.4}$$

$$x(A(W)) \leq |W| - 1 \qquad \forall\ W \subset V, |W| \geq 2 \tag{2.5}$$

$$x(P) \leq |P| - 1 \qquad \forall \text{ infeasible path } P \tag{2.6}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\ (i, j) \in A. \tag{2.7}$$

Constraints (2.5) forbid the occurrence of subtours; (2.2), …, (2.5), (2.7) force the solution to represent a Hamiltonian path, whereas the *infeasible path constraints* (2.6) forbid infeasible paths to be part of the solution. The formulation of the infeasible path constraints, as stated in (2.6), is very weak; in Section 4.1, we present possible strengthenings of these inequalities.

We want to remark here that objective functions such as minimizing the makespan or minimizing the waiting times cannot be incorporated easily in this model. On the other hand, note that no "big M" term is required in the model. Only arc variables are present, which implies that no linking constraints between arc and node variables are necessary. This suggests that (a strengthening of) this model can be superior to alternative ones, at least for some problem classes. The computational results presented in our companion paper [5] confirm this expectation.

3. POLYHEDRAL ANALYSIS

In this section, we aim at analyzing the dimension of the *ATSP-TW polytope*:

$$P_{TW} := conv\{x \in \mathbb{R}^A | x \text{ satisfies (2.2)–(2.7)}\},$$

defined as the convex hull of the characteristic vectors of all feasible Hamiltonian paths on the complete digraph $G = (V, A)$. Recall that it is a strongly $\mathcal{NP}$-complete problem to find a feasible ATSP-TW solution, that is, to decide (for the general case) whether P_{TW} is nonempty. We will therefore restrict ourselves to the study of "simpler" special cases, in particular, the one where only one time window is active. Rather unexpectedly, even in this oversimplified case, the polyhedral analysis is quite difficult, in that P_{TW} lies on a number of hyperplanes whose defining equations have no counterpart in the pure ATSP. Moreover, the solution of a min-cost Hamiltonian path problem is required to compute the dimension of P_{TW}, that is, determining this dimension is a strongly $\mathcal{NP}$-complete problem even when all time windows but one are relaxed.

In the remaining part of this section, we consider the case in which only the time window $[r_1, d_1]$ associated with node 1 (say) is active. All other time windows are relaxed to $[0, +\infty)$. Furthermore, we assume that the triangle inequality (1.1) on ϑ is satisfied.

Partition the node set as follows:

$$V = \{1\} \cup Q \cup W,$$

where

$$Q := \{j \in V \backslash \{1\} | \vartheta(j, 1) = r_j + \vartheta_{j1} > d_1\}$$

contains the nodes that cannot be sequenced before node 1 without violating the time window for node 1, and $W := V \backslash (\{1\} \cup Q)$. We define an undirected *feasibility graph* $G_F = (V \backslash \{1\}, E)$, where

$$E := \{ij | i, j \in V \backslash \{1\}, i \neq j, \min\{\vartheta(i, j, 1), \vartheta(j, i, 1)\} \leq d_1\}$$

contains all node pairs i, j such that either $(i, j, 1)$ or $(j, i, 1)$, or both, are feasible paths. Note that the nodes of Q (possibly among others) are isolated in G_F, that is, they are "incompatible" with all other nodes in $j \in V \backslash \{1\}$ (see Fig. 1 for an illustration).

Consider the following equations:

$$x_{j1} = 0, \qquad \forall j \in Q \tag{3.1}$$

$$x(\delta^+(1)) = 1, \qquad \text{if node 1 cannot be the final node of a feasible Hamiltonian path} \tag{3.2}$$

$$\sum_{j \in S_h} x(\delta^-(j)) = |S_h| - x(S_h : 1), h = 1, \ldots, m, \tag{3.3}$$

where $S_1, \ldots, S_m$ are the m (say) connected components of G_F. Note that adding up the m eqs. (3.3) leads to

$$\sum_{h=1}^{m} \left(\sum_{j \in S_h} x(\delta^-(j)) \right) = \sum_{h=1}^{m} |S_h| - \sum_{h=1}^{m} x(S_h : 1),$$

which can be rewritten as $x(A) = n - 1$, that is, eq. (2.2) is a linear combination of (3.3). Furthermore, note that checking the condition in (3.2) requires the solution of a min-cost Hamiltonian path problem.

Lemma (3.4). *Equations* (3.1)–(3.3) *are valid for* P_{TW}.

Proof. Validity is obvious for eqs. (3.1) and (3.2). As to eqs. (3.3), take any connected component S_h of G_F and note that no edge in G_F crosses the cut induced by S_h.

Consider the characteristic vector x of any feasible Hamiltonian path, and let k be its starting node, that is, $x(\delta^-(k)) = 0$. The left-hand side of (3.3) reads

$$\sum_{j \in S_h} x(\delta^-(j)) = \begin{cases} |S_h| - 1, & \text{if } k \in S_h, \\ |S_h|, & \text{otherwise.} \end{cases}$$

Hence, (3.3) is violated if and only if one of the following two cases occurs (recall that $1 \notin S_h$, as node 1 is not part of G_F and that we assume the triangle inequality to hold):

(i) $k \in S_h$ and $x(S_h : 1) = 0$ [see Fig. 2(a)]: Let j be such that $x_{j1} = 1$ (note that $k \in S_h$ implies $k \neq 1$, i.e., node 1 has indeed a predecessor j in the path); since $x(S_h : 1) = 0$, we must have $j \notin S_h$, which implies that k and j are "compatible," that is, $kj \in E$, a contradiction.

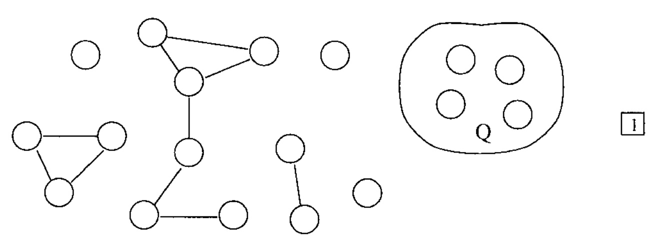

FIG. 1. The feasibility graph G_F (with node 1 added).

(ii) $k \notin S_h$ and $x(S_h : 1) = 1$ [see Fig. 2(b)]: Let $j \in S_h$ be such that $x_{j1} = 1$ and observe that $kj \in E$, again a contradiction.

∎

It is important to observe here that eqs. (3.1)–(3.3) are valid for the case of general time windows as well, node 1 being any node with an active time window.

Lemma (3.5). *Equations* (3.1)–(3.3) *are linearly independent.*

Proof. It is sufficient to exhibit for each equation $\alpha x = \alpha_0$ of the family (3.1)–(3.3) a point $x \in \mathbb{R}^A$ satisfying all the equations in the family except $\alpha x = \alpha_0$.

For each eq. (3.1), let x to be the characteristic vector of any path of the form $(j,\ 1,\ \Phi[S_1], \ldots, \Phi[S_{h_j-1}], \Phi[S_{h_j+1}], \ldots, \Phi[S_m])$, where $S_{h_j} = \{j\}$ is the singleton component of G_F containing node j and $\Phi[S_i]$ denotes any permutation of the nodes in S_i [see Fig. 3(a)].

For each eq. (3.3), let x to be the characteristic vector of the two paths $(1, \Phi[S_1], \ldots, \Phi[S_{h-1}], \Phi[S_{h+1}], \ldots, \Phi[S_m])$ and $(\Phi[S_h])$ [see Fig. 3(b)].

As to eq. (3.2), let x be the characteristic vector of any cycle on $n-1$ nodes of the form $(\Phi[S_1], \ldots, \Phi[S_m])$ [see Fig. 3(c)]. ∎

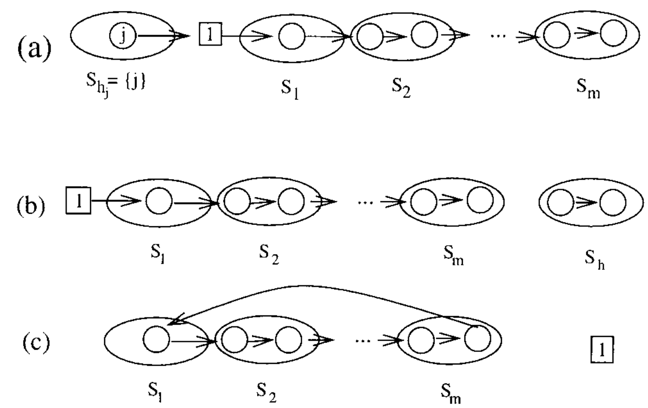

FIG. 3. Constructions for the proof of Lemma 3.5.

We now show that (3.1)–(3.3) define a minimal equation system for the polytope P_{TW}, that is, that no other linearly independent valid equation exists. Let

$$\mu = \begin{cases} 1, & \text{if the condition in (3.2) is satisfied, that is,} \\ & \text{node 1 cannot be the final node of a} \\ & \text{feasible Hamiltonian path,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (3.6). *Consider any ATSP-TW instance defined on a complete digraph $G = (V, A)$ with $n \geq 4$ nodes. If only the time window for one node is active, then*

$$dim(P_{TW}) = |A| - (|Q| + m + \mu).$$

Proof. As before, we assume w.l.o.g. that only the time window of node 1 is active. Lemmas (3.4) and (3.5) imply that $dim(P_{TW}) \leq |A| - (|Q| + m + \mu)$. We give a direct proof that this bound is tight, consisting of exhibiting $|A| - (|Q| + m + \mu) + 1$ affinely independent vertices of P_{TW}.

Consider first the face F of P_{TW} induced by $x(\delta^-(1)) \geq 0$, containing all feasible Hamiltonian paths starting with node 1. As we are assuming that $d_j = +\infty$ for all $j \neq 1$, every Hamiltonian path starting with node 1 is indeed feasible. It is then easy to see that there is a 1–1 correspondence between the vertices of F and the Hamiltonian tours of G; in particular, $dim(F)$ equals the dimension of the "closed-tour" ATSP polytope on G. Hence, $dim(F) = |A| - 2n + 1$, and a minimal equation system for F is given by

$$\begin{array}{lll} (i) & x_{j1} = 0 & \forall j \in V \backslash \{1\}, \\ (ii) & x(\delta^-(j)) = 1 - x_{j1} & \forall j \in V \backslash \{1\}, \\ (iii) & x(\delta^+(1)) = 1. & \end{array}$$

[Note that, because of (i), the right-hand side in (ii) is, in fact, 1.] Therefore, there exist $dim(F) + 1 = |A| - 2n + 2$ affinely independent vertices of this face. To prove our claim, we need $2n - 1 - |Q| - m - \mu$ additional affinely independent points, which we define in the following way:

A. For each $j \in (V \backslash \{1\}) \backslash Q$, we define the point $x^j \in P_{TW}$ associated with any feasible Hamiltonian path of the form $(j, 1, \ldots)$. The affine independence of each such point follows from eqs. (i), in that $x^j_{j1} = 1$, whereas $x_{j1} = 0$ for all other points defined previously. This construction produces $n - 1 - |Q|$ new points.
B. For each $h = 1, 2, \ldots, m$, in turn, consider the component S_h and let $T_h \subseteq E$ be any tree spanning S_h. Choose any root node $\rho \in S_h$ and give an orientation to the edges in T_h so as to obtain a directed tree (arborescence) rooted at ρ. Then, scan the nodes $v \in S_h \backslash \{\rho\}$ in any sequence, visiting each node after its father node in the arborescence, and let f_v be the father node of v in the arborescence. Since $f_v v \in E$, there exists a feasible Hamiltonian path of the form $(v, f_v, 1, \ldots)$ or $(f_v, v, 1, \ldots)$, whose characteristic vector y^v satisfies all the eqs. (ii) except those with $j = v$ and $j = f_v$. Because of the particular sequence in visiting the nodes, all the points constructed before the current y^v satisfy the eq. (ii) written for $j = v$;

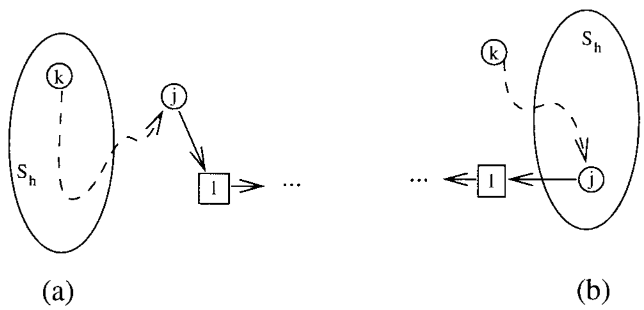

FIG. 2. Constructions for the proof of Lemma 3.4.

hence, each new point is affinely independent from the previous ones. The above construction produces $|S_h| - 1$ new points y^v for each component S_h, that is, $n - 1 - m$ new points in total.

C. If $\mu = 1$, we are done; otherwise, the last point to be constructed is associated with any feasible Hamiltonian path of the form $(\ldots, 1)$. The affine independence of this point follows from the fact that eq. (*iii*) is satisfied by all the previous points in which node 1 always appears in position 1, 2, or 3 (recall $n \geq 4$).

∎

As already mentioned, an implication of Theorem 3.6 is that determining the dimension of P_{TW} is a difficult problem even when all time windows except one are relaxed. To be more specific, consider the following decision problem:

PROBLEM "DIMENSION":

INSTANCE: Any ATSP-TW instance defined on a complete digraph $G = (V, A)$ with integer arc lengths $\vartheta_{ij} \geq 0$ satisfying the triangle inequality.

QUESTION: $dim(P_{TW}) = |A| - 1$?

Theorem (3.7). *Problem DIMENSION is strongly $\mathcal{NP}$-complete even if all time windows but one are relaxed.*

Proof. Because of the equation $x(A) = n - 1$, we know that $dim(P_{TW}) \leq |A| - 1$; hence, the answer "yes" can be certified concisely by exhibiting $|A|$ affinely independent vertices of P_{TW}. This shows that DIMENSION belongs to the class $\mathcal{NP}$. We next show that the problem is indeed strongly $\mathcal{NP}$-complete through a reduction from the following well-known strongly $\mathcal{NP}$-complete problem, HP:

> Given a complete digraph $\tilde{G} = (\tilde{V}, \tilde{A})$ with integer triangular arc lengths $\tilde{\vartheta}_{ij} \geq 0$ and an integer bound L, does $\tilde{G}$ contain a Hamiltonian path $\tilde{P}$ with $\sum_{(i,j)\in\tilde{P}} \tilde{\vartheta}_{ij} \leq L$?

Note that it is not restrictive to assume that $min\{\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ji}\} \leq L$ holds for all $i, j \in \tilde{V}, i \neq j$, since, otherwise, the answer to problem HP is trivially "no."

Given any instance of HP, one can define, in polynomial time, a new complete digraph $G = (V, A)$ with node set $V := \tilde{V} \cup \{1\}$ and set up the following ATSP-TW instance on G:

- $p_j := 0, r_j := 0, \quad \text{for all } j \in V$
- $d_j := \begin{cases} +\infty, & \text{for } j \in \tilde{V}, \\ L, & \text{for } j = 1 \end{cases}$
- $t_{ij} := \begin{cases} \tilde{\vartheta}_{ij}, & \text{for all } (i,j) \in \tilde{A}, \\ 0, & \text{for all } (i,j) \in \delta^-(1) \\ max\{\tilde{\vartheta}_{ij} : (i,j) \in \tilde{A}\}, & \text{for all } (i,j) \in \delta^+(1). \end{cases}$

This instance has just one active time window, namely, the one associated with node 1, and the values $\vartheta_{ij} := p_i + t_{ij} (= t_{ij})$ satisfy the triangle inequality, as required. In addition, the feasibility graph G_F is complete since $min\{\vartheta(i,j,1), \vartheta(j,i,1)\} = min\{\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ji}\} \leq L = d_1$ holds for all $i, j \in \tilde{V}, i \neq j$. It then follows from Theorem 3.6 that $dim(P_{TW}) = |A| - 1 - \mu$; hence, problem DIMENSION has an affirmative answer if and only if $\mu = 0$, a condition equivalent to the existence of a Hamiltonian path $\tilde{P}$ in $\tilde{G}$ with $\sum_{(i,j)\in\tilde{P}} \tilde{\vartheta}_{ij} \leq L$. This proves the claim. ∎

4. CLASSES OF VALID INEQUALITIES

In this section, we summarize known classes of inequalities valid for P_{TW}, present strengthenings of these inequalities, and state new classes of valid inequalities.

In the last section, we have seen that it is a difficult problem to establish the dimension of P_{TW} even for the simple case where only one time window is active. Another problem for the polyhedral study is that, even for fixed n and fixed time windows, little changes in the setup times t_{ij} may result in dramatical changes of the polytope structure, for example, increasing just one such coefficient may make many feasible paths or even the whole instance infeasible. Therefore, proving that certain classes of inequalities are facet-defining for P_{TW} appears a very difficult task; hence, we only prove the validity of the constraints that we propose. With the help of a computer program [16], we have verified for small instances with $n \leq 5$ that these inequalities are, indeed, facet-defining for most instances.

Not many classes of valid inequalities for the time-constrained ATSP can be found in the literature. If we ignore the setup times t_{ij}, ATSP-TW is related to *single-machine scheduling problems*, which require sequencing a set of jobs on a single machine with release times and deadlines. Balas [8] introduced inequalities for this class of problems, which can be incorporated in the "standard" ATSP-TW models in which node variables are used. Applegate and Cook [2] performed computational experiments with several of these classes for the job-shop scheduling problem in which, however, a different objective function (minimization of the makespan) is considered.

It can be seen easily that all classes of inequalities that are valid for the unconstrained ATSP polytope (the convex hull of Hamiltonian tours in G) have a version valid for P_{TW}, obtained by possibly modifying the inequality to take care of the "open-path" formulation that we consider. The ATSP polytope was studied extensively by, among others, Grötschel [27], Grötschel and Padberg [28], Balas [9], Fischetti [22–24], Balas and Fischetti [10, 11], Chopra and Rinaldi [15], and Queyranne and Wang [32]. Similarly, all classes of valid inequalities derived for the precedence-constrained ATSP (cf. Ascheuer [3], Balas et al. [12]) are valid for P_{TW}, by considering the precedence relationships implied by the time windows.

For some of these inequalities, we will present strengthened versions taking into account the time windows explicitely.

4.1. Infeasible Path-elimination Constraints

These constraints express the fact that certain paths are infeasible, that is, they violate a time-window constraint. For a given infeasible path $P = (v_1, \dots, v_k)$, the basic version of these inequalities is given by $x(P) \leq |P| - 1$. There exist, however, several possibilities to strengthen these inequalities. We next discuss in detail some of these lifted inequalities.

4.1.1. Tournament Constraints

Lemma (4.1). *For all infeasible simple paths $P = (v_1, \dots, v_k)$, the* tournament constraint

$$x([P]) := \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x_{v_i v_j} \leq k - 2 (= |P| - 1) \qquad (4.2)$$

is valid for P_{TW}; see Fig. 4 *for an illustration.*

Proof. Because of the degree inequalities $x(\delta^-(j)) \leq 1$, condition $\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x_{v_i v_j} > k - 2$ would imply that $\sum_{j=i+1}^{k} x_{v_i v_j} = 1$ for all $i = 1, \dots, k - 1$. This, in turn, implies that $x_{v_{k-1} v_k} = 1, x_{v_{k-2} v_{k-1}} + x_{v_{k-2} v_k} = 1$ (i.e., $x_{v_{k-2} v_{k-1}} = 1$), etc. But then one would have $x_{ij} = 1$ for all $(i, j) \in P$, impossible because of the infeasibility of P. ∎

Notice that the above proof also shows that, for any simple path $P, x([P]) = |P|$ iff $x_{ij} = 1$ for all $(i, j) \in P$.

The validity of a tournament constraint depends only on the infeasibility of a single path P. In the case when other infeasible paths through the nodes $v_1, \dots, v_k$ exist, the inequality can be further lifted in several ways: For example, in the tournament constraint represented in Figure 4, the coefficient of variable $x_{v_3 v_1}$ can be lifted to 1 if both paths (v_2, v_3, v_1, v_4) and (v_3, v_1, v_2, v_4) happen to be infeasible.

Given a node set $W \subseteq V$, recall that $\Phi[W]$ denotes a generic permutation of the nodes in W.

Theorem (4.3). *For each node set $Q = \{v_1, \dots, v_{k-1}\} \subset V$ and each node $v_k, \in V \backslash Q$ such that all paths of the form $(\Phi[Q], v_k)$ are infeasible, the inequality*

$$x(A(Q)) + x(Q : v_k) \leq k - 2 (= |Q| - 1) \qquad (4.4)$$

is valid for P_{TW}.

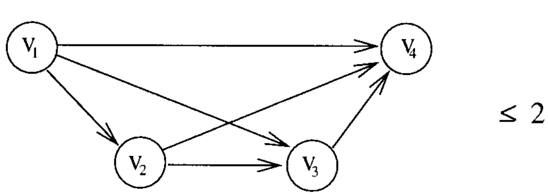

FIG. 4. The support graph of a tournament constraint on $k = 4$ nodes.

Proof. Let x be any vertex of P_{TW}, and let $G_x = (V, \{(i, j) \in A \mid x_{ij} = 1\})$ be its support graph. If vertex x violates (4.4), then $x(A(Q)) = |Q| - 1$ and $x(Q : v_k) = 1$; hence, G_x contains a path of the form $(\Phi[Q], v_k)$, impossible since any such path is infeasible by assumption. ∎

Theorem (4.5). *For each node set $S = \{v_2, \dots, v_{k-1}\} \subset V$ and any two nodes $v_1, v_k \in V \backslash S, v_1 \neq v_k$, such that all paths of the form $(v_1, \Phi[S], v_k)$ are infeasible, the inequality*

$$x(v_1 : S) + x(A(S)) + x(S : v_k) + x_{v_1 v_k} \leq k - 2 \qquad (4.6)$$

is valid for P_{TW}.

Proof. Analogous to that of the previous theorem: If x violates (4.6), then $x(v_1 : S) = x(S : v_k) = 1$ and $x(A(S)) = |S| - 1$; hence, G_x would contain an infeasible path of the form $(v_1, \Phi[S], v_k)$. ∎

Note that inequality (4.4) is a strengthening of the subtour elimination constraint $x(A(Q)) \leq |Q| - 1$. Moreover, both inequalities (4.4) and (4.6) are a strengthening of the tournament constraints (4.2) associated with all the infeasible paths of the form $(\Phi[Q], v_k)$ and $(v_1, \Phi[S], v_k)$, respectively.

It is not easy to decide whether all the paths of the form $(\Phi[Q], v_k)$ and $(v_1, \Phi[S], v_k)$ are infeasible, as required for the validity of inequalities (4.4) and (4.6). Easily checkable sufficient conditions are given by the next lemma; tighter conditions can be derived in a similar way.

Lemma (4.7).

(*a*) *Take any $Q \subset V$ and $v_k \in V \backslash Q$. If*

$$\min_{v_i \in Q} \{r_{v_i}\} + \sum_{v_i \in Q} \min\{\vartheta_{v_i v_j} | v_j \in Q \cup \{v_k\}\} > d_k,$$

then all the paths of the form $(\Phi[Q], v_k)$ are infeasible.

(*b*) *Take any $S \subset V$ and $v_1, v_k \in V \backslash S, v_1 \neq v_k$. If*

$$r_{v_1} + \min\{\vartheta_{v_1 v_j} | v_j \in S\} + \sum_{v_i \in S} \min\{\vartheta_{v_i v_j} | v_j \in S \cup \{v_k\}\} > d_k,$$

then all the paths of the form $(v_1, \Phi[S], v_k)$ are infeasible.

Proof. Obvious from the definitions. ∎

4.1.2. Generalized Tournament Constraints The tournament constraint (4.2) can, in some cases, be generalized by using a clique-lifting technique akin to that described in Balas et al. [11, 12], in which each node v_i is replaced by a clique S_i of "clones." We obtain the following result:

Theorem (4.8). *Let $S_1, \dots, S_k$ be $k \geq 2$ disjoint node sets, and assume that any path of the form*

$(\Phi[S_1], \ldots, \Phi[S_k])$ *is infeasible. Then, the inequality*

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x(S_i : S_j) + \sum_{i=1}^{k} x(A(S_i)) \le k - 2 + \sum_{i=1}^{k} (|S_i| - 1) = \sum_{i=1}^{k} |S_i| - 2$$

is valid for P_{TW}.

Proof. The inequality can only be violated by a point $x \in P_{TW}$ if $x(A(S_i)) = |S_i| - 1$ for $i = 1, \ldots, k$, and $\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x(S_i : S_j) = k - 1$, where this latter condition implies that $x(S_1 : S_2) = \cdots = x(S_{k-1} : S_k) = 1$. But this would imply that the support graph of x contains an infeasible path of the form $(\Phi[S_1], \ldots, \Phi[S_k])$, a contradiction. ∎

Figure 5 gives an illustration of a generalized tournament constraint based on infeasible paths on four cliques.

A simple case in which the assumption of Theorem 4.8 is satisfied arises when the triangle inequality (1.1) on ϑ holds, and there exists $v_i \in S_i$ $(i = 1, \ldots, k)$ such that the path $P = (v_1, \ldots, v_k)$ is infeasible because of condition (*i*) in Lemma 2.1 or because of condition (*ii*) in the same lemma (provided that $w \notin \cup_{i=1}^{k} S_i$).

A different generalization of tournament constraints can be obtained along the following lines: Suppose that we are given a family $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ of node disjoint simple paths, and let ω be any permutation of the indices of $\mathcal{P}$. The path $P = (P_{\omega(1)}, P_{\omega(2)}, \ldots, P_{\omega(k)})$ is called a *concatenation* of the paths in $\mathcal{P}$. Now it may happen that the paths $P_1, P_2, \ldots, P_k$ are feasible in themselves, but that there is no way to connect them in a feasible way. This observation leads to the following result:

Theorem (4.9). *Let* $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ *be a family of node-disjoint simple paths and assume that the triangle inequality* (1.1) *on* ϑ *is satisfied. If* $\vartheta(P) > d_w$ *for any concatenation* $P = (\ldots, w)$ *of the paths in* $\mathcal{P}$, *then the inequality*

$$\sum_{i=1}^{k} x([P_i]) \le \sum_{i=1}^{k} |P_i| - 1$$

is valid for P_{TW}.

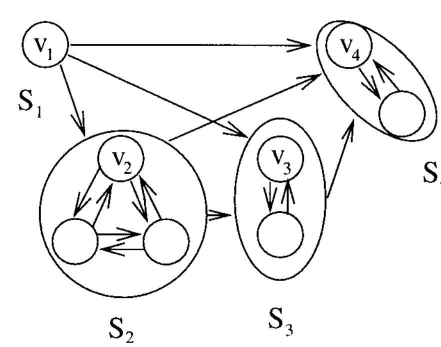

FIG. 5. A generalized tournament constraint.

Proof. To violate the inequality, a feasible solution $x \in P_{TW}$ must satisfy $x([P_i]) = |P_i|$ for all $i = 1, \ldots, k$. As observed in the proof of Theorem 4.1, this implies that $x_{uv} = 1$ for all $(u, v) \in P_i$ and $i = 1, \ldots, k$, a contradiction. ∎

4.1.3. Other Lifted Path Inequalities Given an infeasible path $P = (v_1, \ldots, v_k)$, one possible way of strengthening the basic infeasible path constraint $x(P) \le |P| - 1$ into a tournament constraint $x([P]) \le |P| - 1$ has been already presented. There are, however, several other ways to lift the basic infeasible path constraint, whose validity is based only on the assumption of the infeasibility of path $P = (v_1, \ldots, v_k)$.

Theorem (4.10). *If* $P = (v_1, v_2, \ldots, v_k)$ *is an infeasible path, then the following inequalities are valid for* P_{TW} :

$$\begin{array}{lll}
(a) & x(P) + \sum_{j=1}^{k-2} x_{v_j v_k} + \sum_{j=2}^{k-2} \sum_{l=1}^{j-1} x_{v_j v_l} & \le k - 2 \\
(b) & x(P) + \sum_{j=3}^{k} x_{v_1 v_j} + \sum_{j=3}^{k-1} \sum_{l=j+1}^{k} x_{v_l v_j} & \le k - 2 \\
(c) & x(P) + \sum_{j=2}^{k-1} \sum_{l=1}^{j-1} x_{v_j v_l} & \le k - 2 \\
(d) & x(P) + \sum_{j=2}^{k-1} \sum_{l=j+1}^{k} x_{v_l v_j} & \le k - 2.
\end{array}$$

Proof. Let $ax \le k - 2$ be any of the inequalities (*a*)–(*d*) above. Since P is an infeasible path, for any $x \in P_{TW}$, there must be some h such that $x_{v_h v_{h+1}} = 0$. We have to show that $ax \le k - 2$ holds for all cases (*a*)–(*d*) above:

(a) If $h = k - 1$, then $ax \le \sum_{i=1}^{k-2} x(\delta^+(v_i)) \le k - 2$. Otherwise, $ax \le x(A(\{v_1, \ldots, v_h\}) + \sum_{i=h+1}^{k-2} x(\delta^+(v_i)) + x(\delta^-(v_k)) \le (h - 1) + (k - 2 - h) + 1 = k - 2$.
(b) If $h = 1$, then $ax \le \sum_{i=3}^{k} x(\delta^-(v_i)) \le k-2$. Otherwise, $ax \le x(A(\{v_{h+1}, \ldots, v_k\}) + \sum_{i=3}^{h} x(\delta^-(v_i)) + x(\delta^+(v_1)) \le (k - h - 1) + (h - 2) + 1 = k - 2$.
(c) We have $ax \le x(A(\{v_1, \ldots, v_h\})) + \sum_{i=h+1}^{k-1} x(\delta^+(v_i)) \le (h - 1) + (k - 1 - h) = k - 2$.
(d) We have $ax \le x(A(\{v_{h+1}, \ldots, v_k\})) + \sum_{i=2}^{h} x(\delta^-(v_i)) \le (k - h - 1) + (h - 1) = k - 2$.

∎

Figure 6 gives examples of the inequalities (*a*)–(*d*) for the infeasible path $P = (v_1, \ldots, v_5)$. As in the case of tournament constraints, these inequalities can further be lifted when other infeasible paths through the nodes of P exist (see Ascheuer [3] for details).

4.2. A Lifting Procedure

In this section, we describe a lifting procedure, called *V-lifting*, that can be used to construct new families of valid infeasible path elimination inequalities for the ATSP-TW.

Suppose that we are given two valid inequalities with integer coefficients, say $\alpha x \le \alpha_0$ and $\beta x \le \beta_0$, such that $\beta_0 = \alpha_0 + 1$ and $\beta_{ij} \ge \alpha_{ij}$ for all $(i, j) \in A$. Furthermore, assume that there exist three distinct nodes, say u, w, and h, such that $\beta_{uh} \ge \alpha_{uh} + 1$ and $\beta_{hw} \ge \alpha_{hw} + 1$. By adding up and then rounding the following valid in-

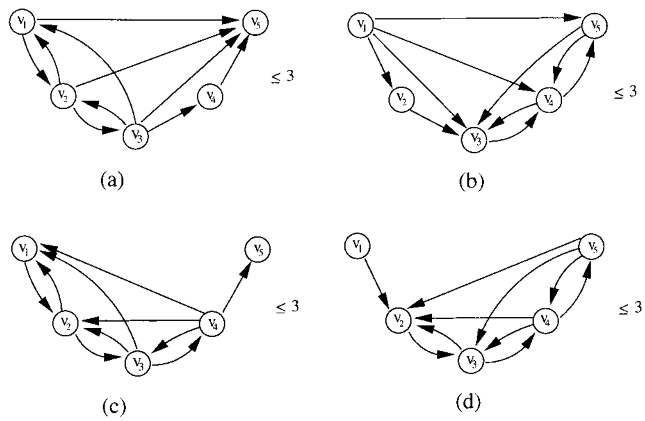

FIG. 6. Lifted infeasible path elimination constraints on $k = 5$ nodes.

equalities weighted by $\frac{1}{2}$,

$$\begin{aligned} \alpha x &\leq \alpha_0 \\ \beta x &\leq \beta_0 = \alpha_0 + 1 \\ x_{uh} + x_{hw} + 2x_{uw} &\leq 2, \end{aligned}$$

one obtains the new valid inequality

$$\alpha x + x_{uh} + x_{hw} + x_{uw} \leq \alpha_0 + \left\lfloor \frac{3}{2} \right\rfloor = \alpha_0 + 1,$$

in which the α-coefficient of the three arcs $(u, h), (h, w)$, and (u, w) is increased by 1 at the "expense" of an increase of 1 of the right-hand side α_0.

Notice that the correctness of the construction depends only on the validity of the inequality $x_{uh} + x_{hw} + 2x_{uw} \leq 2$ (which follows from the degree inequalities); hence, it holds for the unconstrained ATSP polytope as well.

V-lifting can easily be applied to tournament constraints, as illustrated in Figure 7. To this end, let $\alpha x := x([P]) \leq \alpha_0 := |P| - 1$ be the tournament constraint associated with any infeasible path $P = (v_1, \ldots, v_k)$. Take a node $h \notin \{v_1, \ldots, v_k\}$ and assume that the path $P' = (v_1, \ldots, v_i, h, v_{i+1}, \ldots, v_k)$ is infeasible for a certain index i (when ϑ satisfies the triangle inequality, this is likely to be the case for any choice of h and i). Then, let $\beta x \leq \beta_0$ be the tournament inequality $x([P']) \leq |P'| - 1$ associated with P'; Define $u := v_i$ and $w := v_{i+1}$ and obtain through V-lifting the new inequality

$$x([P]) + x_{v_i h} + x_{h v_{i+1}} + x_{v_i v_{i+1}} \leq |P|$$

in which the coefficient of arc (v_i, v_{i+1}) is raised to 2. By iterating the procedure, one can obtain V-lifted tournament inequalities with several coefficients 2.

4.3. Strengthened Predecessor/Successor-Inequalities

In Balas et al. [12], the so-called (π, σ)-inequalities (for predecessor–successor inequalities) were presented for the precedence-constrained ATSP. These inequalities can be strengthened for ATSP-TW by taking time windows into account explicitly.

The (π, σ)-inequalities can be described as follows: We are given two disjoint node sets $X, Y \subset V$ such that $i < j$ for all pairs $i \in X, j \in Y$. Furthermore, we are given a node set $S \subset V$ such that X is contained in S and Y in its complement $\bar{S} := V \backslash S$. In order not to violate the precedence relationships among the nodes in X and Y, a feasible path cannot cross the cut $(S : \bar{S})$ only through arcs incident with $W := \pi(X) \cup \sigma(Y)$; see Section 2 for the definition of $\pi(\cdot)$ and $\sigma(\cdot)$. This observation leads to the (π, σ)-*inequality*

$$x(S \backslash W : \bar{S} \backslash W) \geq 1. \qquad (4.11)$$

Due to the time-window restrictions, some paths from $S \backslash W$ to $\bar{S} \backslash W$ might be infeasible, which may be employed to reduce some left-hand side coefficients.

Theorem (4.12). *Let X and Y be two disjoint node sets such that $i < j$ for all $i \in X$ and $j \in Y$, and define $W := \pi(X) \cup \sigma(Y)$. Assume that the triangle inequality* (1.1) *on ϑ is satisfied and define*

$$\tilde{W} := W \cup \{k \in V \backslash (X \cup Y) \mid \exists i \in X \text{ and } j \in Y \text{ s.t. } \vartheta(i, k, j) > d_j\}$$

and

$$Q := \{(u, v) \in \delta^+(S) \mid \exists i \in X \text{ and } j \in Y \text{ s.t. } \vartheta(i, u, v, j) > d_j\}.$$

Then, for all $S \subset V$ such that $X \subseteq S$ and $Y \subseteq \bar{S}$, the inequality

$$x((S \backslash \tilde{W} : \bar{S} \backslash \tilde{W}) \backslash Q) \geq 1 \qquad (4.13)$$

is valid for P_{TW}.

Proof. For any feasible Hamiltonian path, the subpath P from the node of X visited last, say node i^*, to the node of Y visited first, say node j^*, cannot traverse any node of W without violating a precedence relationship. Furthermore, P cannot traverse any node $k \in \tilde{W} \backslash W$, since, otherwise, it would contain an infeasible path of the form $(i, \ldots, i^*, \ldots, k, \ldots, j^*, \ldots, j)$, where $i \in X$ and $j \in Y$ are the two nodes whose existence is required in the definition of $\tilde{W}$. Finally, P cannot use any arc $(u, v) \in Q$, since all paths of the form $(i, \ldots, i^*, \ldots, u, v, \ldots, j^*, \ldots, j)$ are infeasible. Thus, an arc in $(S \backslash \tilde{W} : \bar{S} \backslash \tilde{W}) \backslash Q$ has to be used to leave S, from which the validity of the inequality follows. ∎

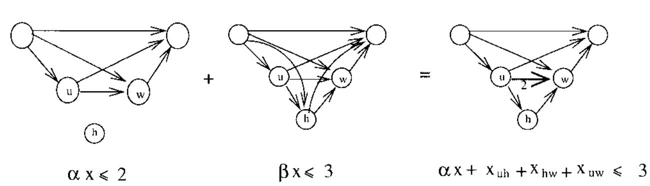

FIG. 7. V-lifted tournament constraint.

4.4. Separation

In general, solving the separation problem for the general classes of valid inequalities introduced above is hard. Among the new classes of inequalities, tournament constraints (4.2) can be separated in polynomial time and, in addition, special versions of the strengthened (π, σ)-inequalities (4.13). We developed several heuristic separation routines for all the inequalities above which are described in our companion paper [5]. We also reported in [5] which of the inequalities valid for P_{TW} are helpful in practical computation. It turns out that the use of infeasible path elimination constraints is important, in particular, tournament constraints (4.2) and inequalities (4.6) did a "good job" on our particular testbed.

5. CONCLUSIONS

The asymmetric traveling salesman problem with time windows (ATSP-TW) is a very important basic model for scheduling and routing applications. We studied the polyhedral structure of a possible formulation of the problem, in which only 0/1 arc variables are considered. This has the advantage of avoiding additional variables and the associated (typically very ineffective) linking constraints. A drawback is that the model cannot accommodate objective functions depending on makespan or waiting times easily.

Time windows are modeled using infeasible path elimination constraints which forbid the occurrence of paths leading to a violation of some deadlines. Similar constraints can also be used to model any other kind of path infeasibility. We described the basic form of these constraints and we introduced some possible strengthenings. Several other classes of cuts were described as well, derived from related ATSP problems.

We also studied the ATSP-TW polytope, P_{TW}, whose vertices are the characteristic vectors of the Hamiltonian paths satisfying the time-window restrictions. To our knowledge, this polytope has never been studied before. Even on a complete graph with triangular setup times, finding a feasible ATSP-TW solution is a difficult problem. Hence, deciding whether P_{TW} is nonempty is an $\mathcal{NP}$-complete problem in the general case. We then studied a very simple special case of the problem, in which only one of the time windows is active. Rather unexpectedly, even under these assumptions, the determination of the polytope dimension is far from trivial, in that the polytope lies in a family of hyperplanes having no counterpart in the pure ATSP case. Moreover, we showed that the exact determination of the polytope dimension is a difficult problem as it requires the solution of a min-cost Hamiltonian path problem. As a consequence, a deep analysis of the facial structure of P_{TW} appears a very difficult task.

In the companion paper [5], the polyhedral analysis herein presented is used to design a branch-and-cut algorithm for the exact solution of ATSP-TW, whose performance is evaluated computationally on several real-world test problems. An outcome is that the new approach outperforms some previous LP-based methods on loosely constrained classes of problem instances.

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