Mathematical Programming Study 14 (1981) 1–22.
North-Holland Publishing Company

CHARACTERIZATIONS OF ADJACENCY OF FACES OF POLYHEDRA*

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Received 30 May 1980
Revised manuscript received 1 September 1980

We generalize the classical concept of adjacency of vertices of a polytope to adjacency of arbitrary faces of a polyhedron. There are three standard ways to describe a polyhedron P, namely, P is given as the intersection of finitely many halfspaces, i.e., $P = P(A, b) = \{x \mid Ax \leq b\}$, as the convex and conical hull of finitely many vectors, i.e., $P = \text{conv}(V) + \text{cone}(E)$, or P is given by its face lattice $\mathcal{F}(P)$. The adjacency relation of faces is characterized by means of all these three descriptions of a polyhedron. Our main tools in case of the descriptions $P = P(A, b)$ resp. $P = \text{conv}(V) + \text{cone}(E)$ are "good" characterizations of the equality set and extreme set of a face, respectively. These "good" characterizations enable us to present polynomial algorithms to check adjacency of faces. As a by-product we also obtain polynomial algorithms to make an inequality system $Ax \leq b$ nonredundant and to find a minimal generating system (basis) of a polyhedron. All these algorithms are based on the ellipsoid method which checks emptiness resp. nonemptiness of polyhedra in polynomial time.

Key words: Theory of Polyhedra, Adjacency, Face Lattice, Redundancy, Ellipsoid Method.

1. Introduction

Good characterizations of adjacency of vertices of polyhedra have often led to successful algorithms (e.g. the Simplex algorithm; see also [4]) or have given more insight into the combinatorial structure of a polytope (cf. [3, 7, 9]). Since the combinatorial properties of a polyhedron are to a large extent reflected by the face lattice of the polyhedron, better knowledge of general properties of this face lattice will provide more effective tools e.g. for polyhedral combinatorics.

The concept of adjacency is usually developed for vertices of polytopes only. Since vertices constitute the smallest nontrivial elements of the face lattice of a polyhedron, it is natural to ask how this concept can be generalized. In this paper we consider arbitrary polyhedra and define an adjacency relation for any two faces of a polyhedron which is a generalization of adjacency of vertices.

This paper is organized as follows. In the rest of the introductory section we make the necessary definitions and notations used available. In section 2 we present characterizations of adjacency which are independent of a description of a polyhedron and use properties of the face lattice only. To obtain description

* Supported by SFB 21 (DFG), Institut für Operations Research, Universität Bonn, Bonn.

dependent results we show in Section 3 how equality sets and extreme sets of faces can be calculated. These theorems are then utilized in Section 4 to give polynomial algorithms to decide whether two faces are adjacent or not. Finally, the appendix serves as a reference for some technical results concerning polyhedral theory which we need in our proofs and which can be found elsewhere. As a by-product of the results of Section 3 we present polynomial algorithms for finding a nonredundant subsystem of an inequality system $Ax \leq b$ and finding a minimal generating system for a polyhedron.

A matrix $A = (a_{ij})$ (where $a_{ij} \in \mathbf{R}$, $i = 1, \dots, m$ and $j = 1, \dots, n$) with m rows and n columns is called an *(m, n)-matrix.* For simplicity we usually assume that $M = \{1, 2, \dots, m\}$ is the set of row indices and $N = \{1, 2, \dots, n\}$ is the set of column indices. We shall extensively use the following notations to denote submatrices of A. Let $I = (i_1, i_2, \dots, i_r)$ $(J = (j_1, j_2, \dots, j_s))$ be a vector of pairwise different row (column) indices, i.e., $\{i_1, \dots, i_r\} \subseteq M, \{j_1, \dots, j_s\} \subseteq N$, then $A_{I,J}$ or just A_{IJ} denotes the following submatrix of A

$$A_{IJ} = \begin{pmatrix} a_{i_1j_1}, & \dots, & a_{i_1j_s} \\ \vdots & & \vdots \\ a_{i_rj_1}, & \dots, & a_{i_rj_s} \end{pmatrix}.$$

In case $J = (1, 2, \dots, n)$ $(I = (1, 2, \dots, m))$ we write $A_{I.}$ or $A_{I,.}$ $(A_{.J}$ or $A_{.,J})$. If $I = (i)$ and $J = (1, 2, \dots, n)$ $(J = (j)$ and $I = (1, 2, \dots, m))$ we write $A_{i.}$ or $A_{i,.}$ $(A_{.j}$ or $A_{.,j})$, i.e., $A_{i.}$ is the i-th row of matrix A (in the sequel $A_{i.}$ will always be considered as a row vector) and $A_{.j}$ is the j-th column of A. Often the order of the components of I or J is completely unimportant. Therefore, if $I \subseteq M$ and $J \subseteq N$ we shall also write $A_{I,J}$ to denote a submatrix of A. But such a matrix is only defined up to row and column permutations.

A *polyhedron* $P \subseteq \mathbf{R}^n$ is the intersection of finitely many halfspaces, i.e., P can be represented in the form $P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$ where A is an (m, n)-matrix and $b \in \mathbf{R}^m$. If A is an (m, n)-matrix, $b \in \mathbf{R}^m$, we denote by $P(A, b)$ the polyhedron $\{x \in \mathbf{R}^n \mid Ax \leq b\}$. By the well-known theorem of Weyl there is an equivalent definition of polyhedra, namely $P = \operatorname{conv}(V) + \operatorname{cone}(E)$, where V and E are finite subsets of $\mathbf{R}^n$, $\operatorname{conv}(V)$ denotes the convex hull of the elements of V and $\operatorname{cone}(E)$ the conical hull of E (i.e. the set of all vectors which are nonnegative linear combinations of E, the linear hull of E is denoted by $\operatorname{lin}(E)$). For convenience we shall often consider V and E as matrices containing the elements of V, E resp. as its columns. A bounded polyhedron is called a *polytope.* A polyhedron P with the property $\lambda x \in P$ for all $x \in P$ and all $\lambda \geqq 0$ is called a *polyhedral cone.* A polyhedron $P \subseteq \mathbf{R}^n$ such that $x + P$ is a cone for some $x \in \mathbf{R}^n$ is called a *polyhedral cone in general position.*

For any set $S \subseteq \mathbf{R}^n$, $\operatorname{rec}(S) := \{y \in \mathbf{R}^n \mid x + \lambda y \in S$ for all $x \in S$ and all $\lambda \geqq 0\}$ denotes the *recession cone* of S. It is well-known that if $P = P(A, b) = \operatorname{conv}(V) + \operatorname{cone}(E)$ is a polyhedron, then $\operatorname{rec}(P) = P(A, 0) = \operatorname{cone}(E)$. For any set

$S \subseteq \mathbf{R}^n$ we denote by lineal $(S) := \{y \in \operatorname{rec}(S) \mid -y \in \operatorname{rec}(S)\}$ the *lineality space* of S. Again, if $P = P(A, b) = \operatorname{conv}(V) + \operatorname{cone}(E)$, then $\operatorname{lineal}(P) = \{x \in \mathbf{R}^n \mid Ax = 0\} = \operatorname{cone}(\{e \in E \mid -e \in \operatorname{cone}(E)\})$.

By $\mathbf{1}$ we denote a vector all whose components are one. Its dimension is always clear from the context.

Let $P \subseteq \mathbf{R}^n$ be a polyhedron. We call $cx \leqq c_0$ a *valid inequality* for P if $cx \leqq c_0$ holds for all $x \in P$. A subset $F \subseteq P$ is a *face* of P if $F = \{x \in P \mid cx = c_0\}$ holds for some valid inequality $cx \leqq c_0$ of P, and this face F is *proper* if $F \neq P$. The empty set is a face, called the *empty face* of P. It is obviously the smallest face contained in any face of P. A proper face which is nonempty is called *nontrivial*. It is clear from the definition that the intersection of any number of faces of P is again a face of P. A nontrivial face which is not contained in any other proper face of P is called a *facet* of P. Every face itself is a polyhedron, hence we can consider faces of facets of P. A maximal proper face of a facet of P is called a *subfacet* of P. Note that a facet is by definition never empty, but a subfacet may be empty.

For deriving lattice theoretical results about polyhedra the object we would like to deal with is the collection of nonempty faces $\mathcal{F}'(P)$ of P. However, $\mathcal{F}'(P)$ does not necessarily contain a minimal element, i.e. a nonempty face which is contained in any nonempty face of P, as the example of polytopes shows. Thus, in order to get a lattice we take the smallest face of P

$$F_0 := \bigcap_{F \in \mathcal{F}'(P)} F$$

contained in all nonempty faces of P and add it to $\mathcal{F}'(P)$. The set $\mathcal{F}(P) := \mathcal{F}'(P) \cup \{F_0\}$ is a finite lattice under set inclusion called the *face lattice* of the polyhedron P and denoted by $(\mathcal{F}(P) \subseteq)$ or just $\mathcal{F}(P)$. As usual we denote by $F \vee G$ the *join* and by $F \wedge G$ the *meet* of F, $G \in \mathcal{F}(P)$, i.e., $F \vee G = \bigcap\{H \in \mathcal{F}(P) \mid F \cup G \subseteq H\}$ is the smallest face of P containing both F and G, and $F \wedge G = F \cap G$ is the largest face contained in both F and G. Two faces F, G are called *noncomparable* if neither $F \subseteq G$ nor $G \subseteq F$ holds. A face G is called a *cover* of a face F if $F \neq G$, $F \subseteq G$ and there is no face H different from F and G with $F \subseteq H \subseteq G$.

The *dimension* of a face $F \in \mathcal{F}(P)$, denoted by $\dim(F)$, is the maximal number of affinely independent points in F minus 1. The dimension of the empty face is -1. The dimension of a facet of P is $\dim(P) - 1$, and the dimension of a nonempty subfacet of P $\dim(P) - 2$. A face of dimension 0 is called a *vertex* and a face of dimension 1 an *edge*. In the following we shall denote a vertex $\{x\}$ of P just by x for ease of notation. A polyhedron which has a vertex is called *pointed*. If P is a pointed polyhedron and E an edge of P, then it is well-known that E has either one vertex or two vertices. In case E has one vertex, say x, then $E = x + \operatorname{cone}(e)$ holds for some vector $e \in \mathbf{R}^n$, in case E has two vertices, say x and y, then $E = \operatorname{conv}(\{x, y\})$.

Let $F_1, F_2 \in \mathcal{F}(P)$ be two faces of a polyhedron P and assume that $\dim(F_1) \leqq \dim(F_2)$, then F_1 and F_2 are called *adjacent* on P if

$$F_1 \vee F_2 \text{ is a cover of } F_2, \tag{1.1}$$

$$F_1 \text{ is a cover of } F_1 \wedge F_2. \tag{1.2}$$

If $F, G \in \mathcal{F}(P)$ are faces of a polyhedron and if G covers F then it is well-known that $\dim(F) = \dim(G) - 1$ holds if either $F \neq \emptyset$ or if P is pointed. Therefore, if $F_1, F_2 \in \mathcal{F}(P)$, $\dim(F_1) \leqq \dim(F_2)$, and P is pointed or $F_1 \wedge F_2 \neq \emptyset$, then F_1 and F_2 are adjacent if and only if

$$\dim(F_1 \vee F_2) = \dim(F_2) + 1, \tag{1.3}$$

$$\dim(F_1 \wedge F_2) = \dim(F_1) - 1. \tag{1.4}$$

In case P is not pointed, then the dimension of any cover of the empty face equals the dimension of the lineality space of P, thus if $F_1, F_2 \in \mathcal{F}(P)$, $\dim(F_1) \leqq \dim(F_2)$, P is not pointed and $F_1 \wedge F_2 = \emptyset$, then F_1 and F_2 are adjacent if and only if

$$\dim(F_1 \vee F_2) = \dim(F_2) + 1, \tag{1.5}$$

$$\dim(F_1) = \dim(\text{lineal}(P)). \tag{1.6}$$

Note that the definition given above of adjacency of faces is based on properties of the face lattice only and therefore does not rely on descriptions of a polyhedron P of the form $P = P(A, b)$ or $P = \text{conv}(V) + \text{cone}(E)$.
To give an example; in the unbounded, nonpointed polyhedron P in Fig. 1.1 any two noncomparable faces of P are adjacent except F_1 and F_3 because $F_1 \wedge F_3 = \emptyset$ and $\dim(F_1) = \dim(F_3) = 2$ but $\dim(\text{lineal}(P)) = 1$, i.e., (1.6) is not satisfied. For a further example consider the pyramid P shown in Fig. 2.1 and the subsequent discussion of the adjacency relations on P.

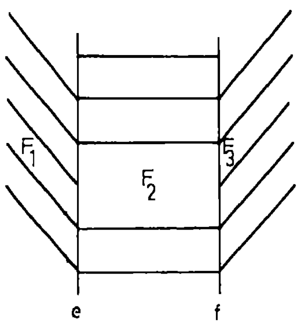

Fig. 1.1.

2. Characterization of adjacency by means of the face lattice

To give a motivation of our definition of adjacency we start by proving three simple observations

Proposition 2.1. (a) *If x and y are different vertices of a polyhedron P, then x and y are adjacent if and only if* $\text{conv}(\{x, y\})$ *is an edge of P.*

(b) *If F and G are different facets of a polyhedron P, then F and G are adjacent if and only if $F \cap G$ is a subfacet of P.*

(c) *If F and G are noncomparable faces of a polyhedron P such that F is a facet and G is a cover of the minimal face of P (e.g. G is a vertex if P is pointed), then F and G are adjacent.*

Proof. (a) If x and y are adjacent, then $E := x \vee y$ has dimension 1, thus E is an edge of P. Since x and y are vertices of P and contained in E, they are also vertices of E, hence $E = \text{conv}(\{x, y\})$ holds. It is clear from the definition of "join" that $\text{conv}(\{x, y\}) \subseteq x \vee y$ holds. If $\text{conv}(\{x, y\})$ is an edge of P, then $\text{conv}(\{x, y\})$ is a face of P of dimension 1 containing x and y, hence $x \vee y \subseteq \text{conv}(\{x, y\})$. Therefore $x \vee y$ covers x and since x covers $\emptyset = x \wedge y$, x and y are adjacent.

(b) If F and G are adjacent facets of P, then by definition F is a cover of $F \wedge G = F \cap G$, hence $F \cap G$ is a subfacet of P. Conversely, if F and G are different, then obviously P is a cover of F and G. If $F \cap G$ is a subfacet of P, then F and G are covers of $F \wedge G$. Therefore F and G are adjacent.

(c) Clearly $F \vee G = P$ and $F \wedge G = M$, where M is the minimal face of P, hence $F \vee G$ covers F, and G covers $F \wedge G$, i.e., F and G are adjacent.

For a polytope P, the condition $\text{conv}(\{x, y\})$ is an edge of P is clearly equivalent to the condition that every $z \in \text{conv}(\{x, y\})$ has a unique representation as a convex combination of vertices of P. It is this kind of uniqueness characterization that is usually used to define adjacency of vertices on polytopes, thus our adjacency relation covers the well-known concepts. The following example serves to illustrate the adjacency relationship geometrically.

Example 2.2. In the pyramid P, shown in Fig. 2.1, vertex 1 is adjacent to every other vertex, but vertices 2 and 4 resp. 3 and 5 are not adjacent since their join (the ground facet) has dimension 2. The ground facet is adjacent to every other facet, but the facets with vertices 1, 2, 3 and 1, 4, 5 are not adjacent since their meet has dimension zero. The edges with vertices 2, 3 and 4, 5 are not adjacent since their meet has dimension zero, and the edges with vertices 1, 3 and 1, 5 are not adjacent since their join is P. The facet with vertices 1, 2, 3 and the edge

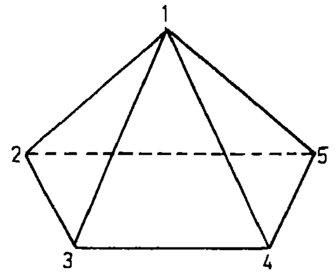

Fig. 2.1.

with vertices 3, 4 are adjacent, but this facet is not adjacent to the edge with vertices 4, 5.

The following lemma states one of the most important properties of the face lattice of a polyhedron.

Lemma 2.3. *If a subfacet F of a polyhedron P is the intersection of facets of P, then F is the intersection of exactly two facets of P. In particular, if a subfacet F is nonempty, then it is the intersection of exactly two facets of P.*

For a proof see e.g. [10, Theorem 2.13. 9, p. 71].

Since a subfacet which is the meet of facets is the meet of two unique facets we can rewrite Proposition 2.1(b) as follows: two facets of a polyhedron which have a nonempty meet are adjacent if and only if there are no two other facets with the same meet. Similarly, Proposition 2.1(a) can be rephrased as: two vertices of a polytope are adjacent if and only if there are no two other vertices which form the same join. This motivates our next definition.

Definition 2.4. Two faces F_1, F_2 of a polyhedron P are called *join-meet ambiguous* if there exist faces $F_3, F_4 \in \mathscr{F}(P) \setminus \{F_1, F_2, F_1 \vee F_2, F_1 \wedge F_2\}$ such that $F_3 \vee F_4 = F_1 \vee F_2$ and $F_3 \wedge F_4 = F_1 \wedge F_2$ holds, otherwise F_1 and F_2 are called *join-meet unique.*

We shall show in the sequel how this concept can be used to characterize adjacency of faces (cf. Theorem 2.10). The main tool for our proofs will be the following

Theorem 2.5. *Let $F \subseteq G \subseteq H$ be nonempty faces of a polyhedron P, then there exists a face $\bar{G}$ of P, called the 'relative complement' of G, with*

(*a*) $G \vee \bar{G} = H$, $G \wedge \bar{G} = F$, *and*

(*b*) $\dim(G) + \dim(\bar{G}) = \dim(H) + \dim(F)$.

This theorem can be proved by repeated application of Lemma 2.3. (cf. [10], p. 71), the statement of Theorem 2.5 here is a little more general than in [10].

Lemma 2.6. *Let $F \subseteq H$ be two faces of a polyhedron P and assume that F is the intersection of facets of H. Then for every cover $G \subseteq H$ of F there exists a facet $\bar{G}$ of H, called the 'complement' of G, such that $G \wedge \bar{G} = F$ and $G \vee \bar{G} = H$.*

Proof. Let $\{F_1, F_2, \dots, F_k\}$ be the set of all facets of H containing G; since G covers F, G is nonempty, therefore $G = \bigcap_{i=1}^{k} F_i$. Now let $F_{k+1}, F_{k+2}, \dots, F_r$ be all other facets of H containing F. By assumption $r > k$ and $F = \bigcap_{i=1}^{r} F_i$ holds. We are now looking for a smaller representation of F as the intersection of facets F_i, and consecutively proceed as follows. We delete F_r, F_{r-1}, F_{r-2} etc. until we obtain an index j such that $F = \bigcap_{i=1}^{j+1} F_i$ and $F \neq \bigcap_{i=1}^{j} F_i$. Since $G = \bigcap_{i=1}^{k} F_i$, we have $j \geqq k$. Suppose $j > k$, then $F \neq \bigcap_{i=1}^{j} F_i \subseteq G$, but since G covers F, $G = \bigcap_{i=1}^{j} F_i$ has to hold, therefore $j = k$ by the choice of k. Set $\bar{G} = F_{j+1}$, then $G \wedge \bar{G} = F$, and since $\bar{G}$ is a facet of H and $G \not\subseteq \bar{G}$, $G \vee \bar{G} = H$ holds.

A converse of Lemma 2.6 obviously also holds, namely, if there are proper faces G, G' of H with $F = G \wedge G'$ and $H = G \vee G'$, then F is the intersection of facets of H.

If the empty face F is the intersection of facets of H, we conclude from the lemma above that for every cover $G \subseteq H$ of F we can find a complement $\bar{G}$. Note however that such a complement does not necessarily exist for a face $G \subseteq H$ which is not a cover of F. Consider the following.

Example 2.7. In Fig. 2.2 the empty face is the intersection of all facets of P. Now take $F = \emptyset$, $G := F_2$, $H := P$, then there is no face $\bar{G}$ of H with $\bar{G} \wedge G = F$, hence G has no complement.

The next observation shows that join-meet uniqueness is not very helpful in case of faces of different dimension.

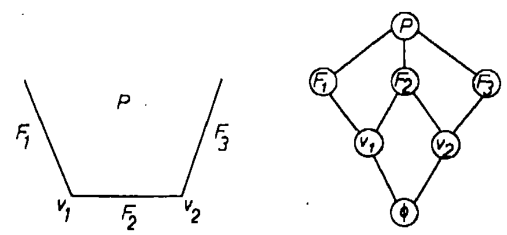

Fig. 2.2.

Proposition 2.8. *If F_1 and F_2 are noncomparable faces of different dimension of a polyhedron P, then F_1 and F_2 are join-meet ambiguous.*

Proof. Let $F := F_1 \wedge F_2$ and $H := F_1 \vee F_2$. Since F_1 and F_2 are noncomparable they are nonempty, this implies that F_1 and F_2 are intersections of facets of H and therefore F is also the intersection of facets of H. Assume $\dim(F_1) < \dim(F_2)$, then F_1 is not a facet of H and there exists a proper face G of F_2 which is a cover of F. By Lemma 2.6 there is a facet $\bar{G}$ of H such that $G \wedge \bar{G} = F$ and $G \vee \bar{G} = H$, by construction $G, \bar{G} \in \mathcal{F}(H) \setminus \{F_1, F_2, F, H\}$.

Note that in the case where F_1 and F_2 are adjacent and have different dimension the proofs of Lemma 2.6 and Proposition 2.8 yield that we can find $F_3, F_4 \in \mathcal{F}(P) \setminus \{F_1, F_2, F_1 \vee F_2, F_1 \wedge F_2\}$ such that $\dim(F_1) = \dim(F_3)$, $\dim(F_2) = \dim(F_4)$ and $F_3 \wedge F_4 = F_1 \wedge F_2$, $F_3 \vee F_4 = F_1 \vee F_2$ holds.

Proposition 2.9. *If $F_1, F_2 \in \mathcal{F}(P)$ are different nonadjacent faces of a polyhedron P of equal dimension such that $F_1 \wedge F_2 \neq \emptyset$, then F_1 and F_2 are join-meet ambiguous.*

Proof. Let us assume that $\dim(F_1 \wedge F_2) \leqq \dim(F_1) - 2$, then there is a face G such that $F_1 \wedge F_2 \subsetneq G \subsetneq F_1$ and by Theorem 2.5 there exists a relative complement $\bar{G}$ of G with respect to $F_1 \wedge F_2$ and $F_1 \vee F_2$; clearly $\bar{G} \neq F_1$. If $F_2 \neq G$ we are done, otherwise let F, H be faces such that $F_1 \wedge F_2 \subseteq F \subseteq H \subseteq F_1 \vee F_2$, H covers F_2, and F_2 covers F. By Lemma 2.3 there exists a unique facet F_3 of H with $F_2 \wedge F_3 = F$, $F_2 \vee F_3 = H$. By Theorem 2.5 there is a relative complement $\bar{F}_3$ of F_3 with respect to $F_1 \wedge F_2$, $F_1 \vee F_2$. By the dimension formula (b) of Theorem 2.5 $\dim(F_3) < \dim(F_1)$, and thus F_3 and $\bar{F}_3$ are the desired faces which show that F_1, F_2 are join-meet ambiguous. In case $\dim(F_1 \vee F_2) \geqq \dim(F_1) + 2$ the proof is analogous to the one above.

Note, that Proposition 2.9 does not hold without the assumption $F_1 \wedge F_2 \neq \emptyset$. Consider the polyhedron P of Fig. 2.2, here the facets F_1 and F_3 are nonadjacent, of equal dimension, and $F_1 \wedge F_3 = \emptyset$, but F_1, F_2 are join-meet unique.

Theorem 2.10. *Let $F_1, F_2 \in \mathcal{F}(P)$ be two different faces of a polyhedron P of equal dimension and $F_1 \wedge F_2 \neq \emptyset$. Then F_1 and F_2 are adjacent if and only if they are join-meet unique.*

Proof. "$\Leftarrow$" by Proposition 2.9.

"$\Rightarrow$" If F_1 and F_2 are adjacent, of equal dimension and $F_1 \wedge F_2 \neq \emptyset$, then $F_1 \wedge F_2$ is a subfacet of $F_1 \vee F_2$ and hence by Lemma 2.3 is the unique intersection of two facets of $F_1 \vee F_2$.

Corollary 2.11. *If P is a polyhedral cone in general position and $F_1, F_2 \in \mathcal{F}(P)$ are two different faces of equal dimension, then F_1 and F_2 are adjacent if and only if F_1 and F_2 are join-meet unique.*

Proof. Two faces of a polyhedral cone in general position never have an empty intersection.

To show the same result for polytopes we introduce the *τ-homogenization* of a set. Let $S \subseteq \mathbb{R}^n$ be a set and $\tau \in \{-1, 0, 1\}$ then

$$\tau\text{-hog}(S) := \left\{ \begin{pmatrix} x \\ \tau \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in S \right\}^{00} \tag{2.1}$$

where for $T \subseteq \mathbb{R}^{n+1}$, $T^0 := \{y \in \mathbb{R}^{n+1} \mid xy \leqq 0 \; \forall x \in T\}$ denotes the *polar cone* of T. If P is a polyhedron, then τ-hog(P) is a polyhedral cone.

Lemma 2.12. *If $\tau \in \{-1, 1\}$ and $P \subseteq \mathbb{R}^n$ is a polytope, then the face lattices $\mathcal{F}(P)$ and $\mathcal{F}(\tau$-hog$(P))$ are isomorphic.*

Proof. See Theorem 2.14.4 in [10].

The existence of a lattice isomorphism between $\mathcal{F}(P)$ and $\mathcal{F}(\tau$-hog$(P))$ immediately implies

Corollary 2.13. *If $F_1, F_2 \in \mathcal{F}(P)$ are two different faces of equal dimension of a polytope P, then F_1 and F_2 are adjacent if and only if F_1, F_2 are join-meet unique.*

Our discussion shows how the concepts of join-meet uniqueness and adjacency are related to each other; it turns out that the adjacency characterization in Proposition 2.1(a) and (b) can only be generalized for faces of equal dimension and that join-meet uniqueness is of no use in case of faces of different dimension.

One might be tempted to strengthen the join-meet uniqueness condition a little bit to include also faces of different dimension. One concept we believed to be appropriate is the following: Call two noncomparable faces F_1, F_2 of a polyhedron P "join-meet ambiguous" if there exist faces $F_3, F_4 \in \mathcal{F}(P) \setminus \{F_1, F_2, F_1 \vee F_2, F_1 \wedge F_2\}$ such that $F_1 \vee F_2 = F_3 \vee F_4$, $F_1 \wedge F_2 = F_3 \wedge F_4$ and $\dim(F_1) = \dim(F_3)$, $\dim(F_2) = \dim(F_4)$ holds, otherwise call F_1, F_2 "join-meet unique". However, this concept does not help characterizing adjacency, because Theorem 2.10 does not hold any more with the restricted definition of join-meet uniqueness. Consider the polytope P which is obtained by gluing two equally sized regular tetrahedra together (see Fig. 2.3). P has two vertices u and v which are not adjacent and are join-meet unique in the sense of the restricted definition,

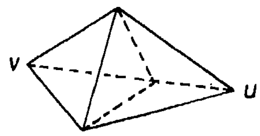

Fig. 2.3.

since there are no two other vertices of P having P as join. This polytope P is a counter-example to the validity of Corollary 2.13 in the case of the restricted join-meet uniqueness definition and the 1-homogenization of P gives a counterexample to Theorem 2.10.

3. Calculating the equality set and the extreme set of a face

Since the adjacency relationship is defined by means of properties of the join and the meet of faces, we have to show how these can be calculated given some description of the polyhedron. With the help of these results we can then deduce the description dependent adjacency characterizations.

The main tools we shall use to derive our results utilize the following concepts. If $P = P(A, b) \subseteq \mathbb{R}^n$ is a polyhedron and $M = \{1, 2, \dots, m\}$ is the row index set of A then we introduce the mapping (2^S denotes the power set of S)

$$\begin{cases} \mathrm{eq}: 2^P \to 2^M, \\ P \supseteq F \mapsto \{i \in M \mid A_{i.}x = b_i \text{ for all } x \in F\} \subseteq M, \end{cases} \tag{3.1}$$

thus eq(F) is the set of all row indices of A such that the corresponding inequalities are binding for F, eq(F) is called the *equality set of F*. The mapping

$$\begin{cases} \mathrm{fa}: 2^M \to 2^P, \\ M \supseteq I \mapsto \{x \in P \mid A_{i.}x = b_i \text{ for all } i \in I\} \end{cases} \tag{3.2}$$

associates with every set I of row indices of A the subset F of P such that all points in F satisfy the inequalities given by I with equality. Clearly, $fa(I)$ is a face of P, called the *face defined by I*.

If the polyhedron P is given as $P = \mathrm{conv}(V) + \mathrm{cone}(E) \subseteq \mathbb{R}^n$ we can define similar mappings as follows: Given a vector $x \in \mathrm{conv}(V) + \mathrm{cone}(E)$, then we say that $u \in V$ *convexly supports* x with respect to (V, E), if x has a representation $x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$ such that $\lambda_u > 0$, and we say that $f \in E$ *conically* supports x if x has a representation $x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$ such that $\mu_f > 0$. We define for $F \subseteq P$

$$\begin{cases} \mathrm{ex}_V(F) := \{v \in V \mid v \text{ supports some vector } x \in F \\ \qquad\qquad\quad \text{convexly with respect to } (V, E)\}, \\ \mathrm{ex}_E(F) := \{e \in E \mid e \text{ supports some vector } x \in F \\ \qquad\qquad\quad \text{conically with respect to } (V, E)\}, \end{cases} \tag{3.3}$$

and combining these notions we define the mapping

$$\begin{cases} \text{ex} : 2^P \to 2^V x 2^E, \\ P \supseteq F \mapsto \text{ex}(F) := (\text{ex}_V(F), \text{ex}_E(F)) \subseteq (V, E). \end{cases} \tag{3.4}$$

The set ex(F) is called the *extreme set* of F. Clearly, if F is a face of P, then $\text{ex}_V(F)$ contains all vertices of P contained in F, and $\text{ex}_E(F)$ contains all extreme vectors of F, and thus $F = \text{conv}(\text{ex}_V(F)) + \text{cone}(\text{ex}_E(F))$. Note that for the empty face of P we have $\text{ex}_V(\emptyset) = \emptyset$, $\text{ex}_E(\emptyset) = \emptyset$, and that $F \subseteq P$ is empty if and only if $\text{ex}_V(F) = \emptyset$.

To define a mapping converse to ex we first set

$$\begin{cases} \text{gen} \;\; 2^V \times 2^E \to 2^P, \\ (V, E) \supseteq (S, T) \mapsto \text{conv}(S) + \text{cone}(T) \subseteq P. \end{cases} \tag{3.5}$$

Note that gen(S, T) is in general not a face of P, but using the mappings ex and gen we can obtain the desired mapping as follows:

$$\begin{cases} \text{sp} : 2^V \times 2^E \to 2^P, \\ (V, E) \supseteq (S, T) \mapsto \text{gen}(\text{ex}(\text{gen}(S, T))), \end{cases} \tag{3.6}$$

One can show that for any set (S, T), sp(S, T) is a face of P called the *span* of (S, T), and sp(S, T) is the smallest face F of P such that $S \subseteq \text{ex}_V(F)$ and $T \subseteq \text{ex}_E(F)$ holds. Note that gen(S, T) is empty if and only if S is empty. Furthermore, $\text{ex}(\text{sp}(S, T)) = \text{ex}(\text{gen}(S, T))$ holds.

Theorem 3.1. *Let $P = P(A, b)$ be a polyhedron, $I \subseteq M = \{1, 2, \ldots, m\}$, $K := M \setminus I$, and let $F = \{x \in P \mid A_{I.}x = b_I\}$ be a nonempty face of P.* Then for all $j \in M$, *$j \in \text{eq}(F)$ if and only if*

$$(A_{j.}, b_j)^{\mathrm{T}} \in \text{lin}((A_I, b_I)^{\mathrm{T}}) - \text{cone}((A_{K.}, b_K)^{\mathrm{T}}).$$

Proof. Define $c^{\mathrm{T}} := \sum_{i \in I} A_{i.}$, $c_0 := \sum_{i \in I} b_i$, then clearly $F = \{x \in P \mid cx = c_0\}$ holds. By (A.1), cf. appendix, we obtain

$$\text{eq}(F) = \{j \in M \mid \exists u \in \mathbb{R}_+^m \text{ such that } u^{\mathrm{T}}A = c^{\mathrm{T}},\ ub = c_0,\ u_j > 0\}.$$

Therefore, if $j \in \text{eq}(F)$, there exists $u \geqq 0$ with $u_j > 0$ such that

$$A_{j.} = -v^{\mathrm{T}}A_{M \setminus \{j\}.} + (1/u_j)\mathbf{1}^{\mathrm{T}}A_{I.} \quad \text{and} \quad b_{j.} = -v^{\mathrm{T}}b_{M \setminus \{j\}} + (1/u_j)\mathbf{1}b_I,$$

where $v^{\mathrm{T}} = (1/u_j)(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m)$. This implies

$$(A_{j.}, b_j)^{\mathrm{T}} \in \text{lin}((A_{I.}, b_I)^{\mathrm{T}}) - \text{cone}((A_{K.}, b_K)).$$

Conversely, suppose $(A_{j.}, b_j)^{\mathrm{T}} \in \text{lin}(A_{I.}, b_I)^{\mathrm{T}}) - \text{cone}((A_{K.}, b_K)^{\mathrm{T}})$, i.e., there exist $u \in \mathbb{R}_+^{|I|}$, $v \in \mathbb{R}_+^m$ such that $A_{j.} = u^{\mathrm{T}}A_{I.} - v^{\mathrm{T}}A$ and $b_j = u^{\mathrm{T}}b_I - v^{\mathrm{T}}b$ holds. Define $d := u^{\mathrm{T}}A_I$ and $d_0 := u^{\mathrm{T}}b_I$. The inequality $dx \leqq d_0$ is a conic combination of valid inequalities of P and therefore also valid for P. Set $G := \{x \in P \mid dx = d_0\}$, then by definition of G, $F \subseteq G$ holds. This clearly implies $\text{eq}(G) \subseteq \text{eq}(F)$, thus it suffices to show $j \in \text{eq}(G)$. Now, $A_{j.} = d^T - v^T A$, $b_j = d_0 - v^T b$ and, setting $\bar{v}_i = v_i$, for $i \neq j$, and $\bar{v}_j = v_j + 1$ we have $d^T = \bar{v}^T A$, $d_0 = \bar{v}^T b$. Since $\bar{v}_j > 0$, Proposition A.1 implies that $j \in \text{eq}(G)$ which proves the theorem.

A simple calculation shows that if $(A_j, b_j) = \lambda^T(A_{I.}, b_I) - \mu^T(A_{K.}, b_K)$, $\lambda \in \mathbb{R}^{|I|}$, $\mu \in \mathbb{R}_+^{|K|}$, then every index $k \in K$ for which $\mu_k > 0$ is also contained in the equality set of F, i.e., only those $k \in K$ can have a positive coefficient μ_k which are in the equality set of F. This way we have shown:

Corollary 3.2. *For all* $j \in M$, $j \in \text{eq}(F)$ *if and only if* $(A_{j.}, b_j) \in \text{lin}((A_{I.}, b_I)^T) - \text{cone}((A_{eq(F).}, b_{eq(F)})^T)$.

Example 3.3. Given the following pyramid $P = P(A, b) \subseteq \mathbb{R}^3$ (see Fig. 3.1):

(1) $-x_1 + x_3 \leqq 0$,

(2) $-x_2 + x_3 \leqq 0$,

(3) $x_1 + x_3 \leqq 2$,

(4) $x_2 + x_3 \leqq 2$,

(5) $-x_1 \leqq 0$,

(6) $-x_2 \leqq 0$,

(7) $-x_3 \leqq 0$.

The inequalities (5) and (6) are redundant, inequality (7) defines the ground facet, inequality (1) the left facet, (2) the front facet, (3) the right facet, (4) the back facet. Now consider the row index set $I = \{2, 4\}$, then $F = \{x \in P \mid A_{2.}x = 0, A_{4.}x = 2\}$ is the intersection of the front and back facet. Obviously, F is the vertex $(1, 1, 1)$. The vector $(A_{1.}, b_1)$ corresponding to inequality (1) can be obtained by adding up the vectors defining inequalities (2) and (4) and subtracting the vector corresponding to inequality (3); and (A_3, b_3) can be obtained by adding up the vectors defining inequalities (2) and (4) and subtracting $(A_{1.}, b_1)$. This implies that (1) and (4) are also in the equality set of F, and it is simple to see that (5) is not in the equality set of F, thus $\text{eq}(F) = \{1, 2, 3, 4\}$ (which is obvious from the picture).

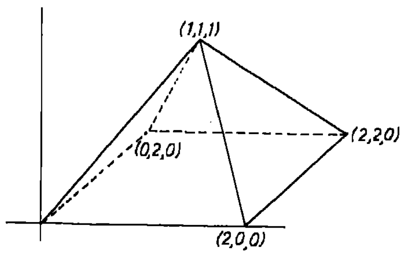

Fig. 3.1.

Corollary 3.4. *Let $P = P(A, 0)$ be a polyhedral cone, $I \subseteq M$, then for all $j \in M$, $j \in \mathrm{eq}(\mathrm{fa}(I))$ if and only if $A_{j.}^{\mathrm{T}} \in \mathrm{cone}((-A^{\mathrm{T}}, A_{I.}^{\mathrm{T}}))$.*

Proof. Note that $\mathrm{fa}(I) = \{x \in P \mid A_{I.}\, x = 0\}$ is a nonempty face of a polyhedral cone P.

If P is a polytope, then the face lattices of P and the 1-homogenization of P are isomorphic by Lemma 2.12, furthermore by (A.2) the equality set lattice of P and the face lattice of 1-hog(P) are anti-isomorphic. Applying Corollary 3.4 to the 1-homogenization of P and exploiting this latter anti-isomorphism we can get rid of the nonemptiness assumption in Theorem 3.1, i.e., we have

Corollary 3.5. *Let $P = P(A, b)$ be a polytope, $I \subseteq M$, $K := M \setminus I$, then for all $j \in M$, $j \in \mathrm{eq}(\mathrm{fa}(I))$ if and only if*

$$(A_{j.}, b_j)^{\mathrm{T}} \in \mathrm{lin}((A_{I.}, b_I)^{\mathrm{T}}) - \mathrm{cone}((A_{K.}, b_K)^{\mathrm{T}}).$$

With the help of the ellipsoid method, cf. [5, 8], we obtain

Theorem 3.6. *Given a polyhedron $P = P(A, b) \subseteq \mathbb{R}^n$ where A is a rational (m, n)-matrix and $b \in \mathbb{Q}^m$, and given $I \subseteq M$, then the equality set $\mathrm{eq}(\mathrm{fa}(I))$ can be determined in time polynomial in the length of a binary encoding of the data A, b and I.*

Proof. First determine whether $F := \mathrm{fa}(I) = \{x \in P \mid A_{I.}x = b_I\}$ is empty or not, this can be done in polynomial time with the ellipsoid method. If $F = \emptyset$, then $\mathrm{eq}(\mathrm{fa}(I)) = M$. Otherwise, for every $j \in K := M \setminus I$ determine whether the following polyhedron

$$P_j = \{(u^{\mathrm{T}}, v^{\mathrm{T}}) \in \mathbb{R}^m \mid u^{\mathrm{T}}(A_{I.}, b_I) - v^{\mathrm{T}}(A_{K.}, b_K) = (A_{j.}, b_j),\ v \geqq 0\}$$

is nonempty. If P_j is nonempty, then $j \in \mathrm{eq}(F)$ by Theorem 3.1, otherwise $j \notin \mathrm{eq}(F)$. Again, with the ellipsoid method the emptiness of P_j can be determined in polynomial time.

Using the polyhedron P defined by the valid inequalities with respect to P, the τ-homogenization of P and isomorphism relations between various lattices associated with these polyhedra we can derive

Theorem 3.7. *Let $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ be a nonempty polyhedron, $(S, T) \subseteq (V, E)$, and $F = \mathrm{sp}(S, T)$ be a face of P.*

(a) For all $v \in V$, $v \in \mathrm{ex}_V(F)$ if and only if there exists $\delta \geqq 0$ such that

$$v \in (1 + \delta)\mathrm{conv}(S) - \delta\mathrm{conv}(V) + \mathrm{lin}(T) - \mathrm{cone}(E \setminus T),$$

(b) For all $e \in E$, $e \in \mathrm{ex}_E(F)$ if and only if there exists $\lambda \geqq 0$ such that

$$e \in \lambda \operatorname{conv}(S) - \lambda \operatorname{conv}(V) + \operatorname{lin}(T) - \operatorname{cone}(E \setminus T)$$

or equivalently $e \in \operatorname{cone}((S - V) \cup (-E) \cup T)$.

Proof. By Proposition A.4 there exists an anti-isomorphism of face lattices $\sigma : \mathcal{F}((-1)\text{-hog}(P)) \to \mathcal{F}(P^\gamma)$. Define the mapping $\Phi: \mathcal{F}(P) \mapsto \mathcal{F}(P^\gamma)$ by $\Phi = \sigma \circ (-1)\text{-hog}$ and recall that for

$$Q \subseteq P^\gamma = P\left(\begin{pmatrix} V^T, & -\mathbf{1} \\ E^T, & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right),$$

cf.(A.3), γ-eq$(Q) = (V', E')$ where V' is the set of all $v \in V$ such that $cv = c_0$ for all $\binom{c}{c_0} \in Q^\gamma$ and E' is the set of all $e \in E$ such that $ce = 0$ for all $\binom{c}{c_0} \in Q^\gamma$ (cf. A.5). By (A.6), ex$(F) = \gamma$-ex$(\Phi(F))$, and

$$\Phi(F) = \left\{\begin{pmatrix} c \\ c_0 \end{pmatrix} \in P^\gamma \,\middle|\, \begin{pmatrix} S^T & -\mathbf{1} \\ T^T & 0 \end{pmatrix}\begin{pmatrix} c \\ c_0 \end{pmatrix} = 0\right\}.$$

Since P^γ is a cone, the equality set of $\Phi(F)$ and thus γ-eq$(\Phi(F))$ can be determined by Corollary 3.4, namely for $v \in V$, we have $v \in \gamma$-eq$(\Phi(F))$ if and only if there exists

$$\begin{pmatrix} v \\ -1 \end{pmatrix} \in \operatorname{cone}\left(\begin{pmatrix} -V, & -E, & S, & T \\ \mathbf{1}, & 0, & -\mathbf{1}, & 0 \end{pmatrix}\right) \tag{*}$$

and for all $e \in F$ we obtain

$$\begin{pmatrix} e \\ 0 \end{pmatrix} \in \operatorname{cone}\left(\begin{pmatrix} -V, & -E, & S, & T \\ \mathbf{1}, & 0, & -\mathbf{1}, & 0 \end{pmatrix}\right). \tag{**}$$

By simple calculations (*) is equivalent to

$$v \in (1+\delta)\operatorname{conv}(S) - \delta \operatorname{conv}(V) + \operatorname{lin}(T) - \operatorname{cone}(E \setminus T) \text{ for some } \delta \geqq 0$$

and (**) is equivalent to

$$e \in \lambda \operatorname{conv}(S) - \lambda \operatorname{conv}(V) + \operatorname{lin}(T) - \operatorname{cone}(E \setminus T) \quad \text{for some } \lambda \geq 0.$$

One can verify (a) and (b) of Theorem 3.7 alternatively by direct computation using the fact that $\mathrm{ex}_V(F) = \mathrm{ex}_V(\mathrm{gen}(S, T))$ and $\mathrm{ex}_E(F) = \mathrm{ex}_E(\mathrm{gen}(S, T))$. An argument of this kind will be given in the proof of Theorem 3.10. As usual polyhedral cones and polytopes behave better than general polyhedra, namely we obtain from Corollaries 3.4 and 3.5 immediately

Corollary 3.8. *Let $P = \operatorname{cone}(E)$ be a polyhedral cone, $T \subseteq E$ and $F = \mathrm{sp}(\{0\}, T)$, then for all $e \in E$, $e \in \mathrm{ex}_E(F)$ if and only if $e \in \operatorname{lin}(T) - \operatorname{cone}(E \setminus T)$.*

Corollary 3.9. *Let $P = \operatorname{conv}(V)$ be a polytope, $S \subseteq V$, and $F = \mathrm{sp}(S, \emptyset)$, then for* *all* $v \in V$, $v \in \mathrm{ex}_v(F)$ *if and only if* $v \in (1+\delta)\ \mathrm{conv}(S) - \delta\ \mathrm{conv}(V)$ *for some* $\delta \geqq 0$, *or equivalently* $v \in \mathrm{conv}(S) + \mathrm{cone}(S - V)$.

The results above show how one can determine all vertices and extreme vectors of the face spanned by some vertices and extreme vectors. The next theorem in addition shows that these vectors can be calculated in polynomial time too.

Theorem 3.10. *Given a polyhedron* $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ *where* V *and* E *are finite sets of rational vectors and given* $(S, T) \subseteq (V, E)$. *Let* $F = \mathrm{sp}(S, T)$ *be the face spanned by* (S, T), *then the vertices* $\mathrm{ex}_V(F)$ *of* F *and the extreme vectors* $\mathrm{ex}_E(F)$ *of* F *can be determined in time polynomial in the length of a binary encoding of the data* V *and* E.

Proof. For $w \in V$ we know that $w \in \mathrm{ex}_V(F)$ if and only if $w \in \mathrm{ex}_V(\mathrm{gen}(S, T))$. Now $x \in \mathrm{gen}(S, T)$ if and only if there are $\alpha \in \mathbb{R}_+^{|S|}$, $\beta \in \mathbb{R}_+^{|T|}$, $\sum_{s \in S} \alpha_s = 1$, with $x = \sum_{s \in S} \alpha_s s + \sum_{t \in T} \beta_t\, t$. Further, $w \in \mathrm{ex}(\mathrm{gen}(S, T))$ if and only if there exist $x \in \mathrm{gen}(S, T)$ and $\gamma \in \mathbb{R}_+^{|V|}$, $\delta \in \mathbb{R}_+^{|E|}$, $\sum_{v \in V} \gamma_v = 1$, $\gamma_w > 0$ such that $x = \sum_{v \in V} \gamma_v v + \sum_{e \in E} \delta_e e$. Therefore $w \in \mathrm{ex}_v(F)$ if and only if the set

$$P_w := \Big\{(\alpha^{\mathrm{T}}, \beta^{\mathrm{T}}, \gamma^{\mathrm{T}}, \delta^{\mathrm{T}}) \;\Big|\; \alpha \in \mathbb{R}_+^{|S|}, \sum_{s \in S} \alpha_s = 1, \beta \in \mathbb{R}_+^{|T|},$$
$$\gamma \in \mathbb{R}_+^{|V|}, \sum_{v \in V} \gamma_v = 1, \delta \in \mathbb{R}_+^{|E|},$$
$$\sum_{s \in S} \alpha_s s + \sum_{t \in T} \beta_t t = \sum_{v \in V} \gamma_v v + \sum_{e \in E} \delta_e e, \gamma_w > 0\Big\}$$

is nonempty. The emptiness resp. nonemptiness of P_w can be checked in time polynomial in an encoding of V and E with the ellipsoid method, hence for every $w \in V$ we can determine in polynomial time whether $w \in \mathrm{ex}_v(F)$ or not.

Defining a set P_e for all $e \in E$ in a similar manner and using the ellipsoid method to check $P_e = \emptyset$ we can decide in polynomial time whether $e \in E$ belongs to $\mathrm{ex}_E(F)$ or not.

We now show as a by-product that the algorithm presented in Theorem 3.6 can be utilized to check nonredundancy of an inequality system.

3.11. *Finding a nonredundant linear description of a polyhedron*

Given an inequality system $Ax \leqq b$ with rational (m, n)-matrix A and $b \in \mathbb{Q}^m$. Let $M = \{1, 2, \dots, m\}$ be the set of row indices and $P = P(A, b)$.

(3.11.1) Set $I = \emptyset$, then $\mathrm{fa}(I) = P$. Using the algorithm of Theorem 3.6, determine the equality set $\mathrm{eq}(P)$ of P.

(3.11.2) Using Gaussian elimination calculate the rank of $A_{\text{eq}(P).}$ and find an index set $I \subseteq \text{eq}(P)$ such that $A_{I.}$ has full row rank and $\text{rank}(A_{I.}) = \text{rank}(A_{\text{eq}(P).})$. (By construction, $\{x \in \mathbb{R}^n \mid A_{I.}x = b_I\}$ is a nonredundant representation of the affine hull of P.) Set $M' := M \setminus \text{eq}(P)$.

(3.11.3) For every $j \in M'$ calculate the equality set $\text{eq}(\text{fa}(\{j\}))$ by means of the algorithm of Theorem 3.6. Let all $j \in M'$ be unlabeled.

(3.11.4) If every $j \in M'$ is labeled $\rightarrow$ STOP.

(3.11.5) Otherwise, pick an unlabeled $j \in M'$. Check, whether there is a $k \in M'$ such that $\text{eq}(\text{fa}(\{k\}))$ is properly contained in $\text{eq}(\text{fa}(\{j\}))$.

(3.11.6) If yes, then remove j from M' and go to (3.11.4). (In this case, $A_{j.}x \leqq b_j$ is clearly a redundant inequality with respect to P.)

(3.11.7) If no, then label j, remove all indices contained in $\text{eq}(\text{fa}(\{j\})) \setminus \text{eq}(P)$ except j from M' and go to (3.11.4). (In this case $A_{j.}x \leqq b_j$ defines a facet of P. We remove all those inequalities which are equivalent, i.e., define the same facet of P as $A_{j.}x \leqq b_j$ does.)

After termination of the algorithm clearly

$$P = \{x \in \mathbb{R}^n \mid A_{I.}x = b_I,\ A_{M'.}x \leqq b_{M'}\}$$

holds, and by construction this representation of P is minimal, i.e., the removal of any equation $A_{i.}x = b_i$, $i \in I$, or any inequality $A_{j}x \leqq b_j$, $j \in M'$ results in a polyhedron which properly contains P.

Since the algorithm of Theorem 3.6 is called at most m times, only one rank calculation and some comparisons are performed, the overall running time of algorithm(3.11) is polynomial in any (binary) encoding of A and b.

Similarly, we can eliminate all superfluous vectors of a generating system of a polyhedron:

(3.12) *Construction of a minimal generating system of a polyhedron*

Given finite sets of vectors V, $E \subset \mathbb{Q}^n$, and let $P = \text{conv}(V) + \text{cone}(E)$ be the polyhedron generated by V and E. Let all vectors in V and E be unlabeled.

(3.12.1) If every $v \in V$ is labeled, go to (3.12.5).

(3.12.2) Otherwise, pick any unlabeled $w \in V$ and check with the ellipsoid method whether

$$P_w = \left\{(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}}) \mid \lambda \geqq 0,\ \mu \geqq 0,\ \sum_{v \in V \setminus \{w\}} \lambda_v = 1,\ w = \sum_{v \in V \setminus \{w\}} \lambda_v v + \sum_{e \in E} \mu_e e\right\}$$

is empty or not.

(3.12.3) If P_w is empty, label w and go to (3.12.1).

(3.12.4) If P_w is nonempty, set $V := V \setminus \{w\}$ and go to (3.12.1).

(3.12.5) If every $e \in E$ is labeled $\rightarrow$ STOP.

(3.12.6) Otherwise, pick any unlabeled $f \in E$ and check with the ellipsoid method whether $P_f = \{\lambda \mid \lambda \geq 0, f = \sum_{e \in E \setminus \{f\}} \lambda_e e\}$ is empty or not.

(3.12.7) If P_f is empty, then label f and go to (3.12.5).

(3.12.8) If P_f is nonempty, then set $E := E \setminus \{f\}$ and go to (3.12.5)

Clearly a vector $w \in V$ is superfluous in a generating system (V, E) if and only if w can be generated by the other vectors of a generating system, and similarly, a vector $e \in E$ is superfluous if and only if e is in the cone spanned by the other vectors of E. Therefore, by successively eliminating vectors $w \in V$ in Step (3.12.4) and vectors $f \in E$ in Step (3.12.8) we finally end up with a generating system which is minimal. Note however, that this system is not necessarily of minimum cardinality. Only in case P has a basis, i.e., P is pointed, it is guaranteed that after termination of the algorithm the final set (V, E) is of minimum cardinality, since if P has a basis, the basis is unique.

As the ellipsoid method is called exactly $|V| + |E|$ times and the ellipsoid method runs in time polynomial in an encoding of V and E, the overall running time of algorithm (3.12) is polynomial.

4. Polynomial algorithms to check adjacency

We now assume that the polyhedron P is given by $P = P(A, b)$ or $P = \text{conv}(V) + \text{cone}(E)$ (or both), and we want to show, how these descriptions of P can be utilized to characterize and check adjacency of faces of P. The following lemma is well-known.

Lemma 4.1. *Let $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ be a polyhedron and F a nonempty face of P. Define $I := \text{eq}(F)$ and $(S, T) = \text{ex}(F)$, then*

(*a*) $\dim(F) = n - \text{rank}(A_{I.})$,

(*b*) $\dim(F) = \text{arank}(S \cup (S + T)) - 1$.

Here arank(S) denotes the affine rank of S, i.e., the maximum number of affinely independent points in S.

Lemma 4.1 tells us how we can calculate the dimension of a face given some description of P resp. F. To check whether two faces are adjacent we have to determine the dimensions of these two faces and the dimensions of the join and the meet of these faces. To do this we shall utilize the results obtained in Section 3.

Lemma 4.2. *Let $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ be a nonempty polyhedron and F_1, F_2 two faces of P.*

(*a*) $\text{eq}(F_1 \vee F_2) = \text{eq}(F_1) \cap \text{eq}(F_2)$,

(*b*) *If* $\text{ex}_V(F_1) \cap \text{ex}_V(F_2) \neq \emptyset$, *then* $\text{ex}(F_1 \wedge F_2) = \text{ex}(F_1) \cap \text{ex}(F_2)$, *otherwise* $\text{ex}(F_1 \wedge F_2) = (\emptyset, \emptyset)$.

Proof. (a) By Proposition A.2 the mapping $\mathrm{eq}: \mathcal{F}(P) \to \mathrm{EQ}(A, b)$ from the face lattice $\mathcal{F}(P)$ of P into the equality set lattice $\mathrm{EQ}(A, b)$ of P is an anti-isomorphism which implies (a), since $\mathrm{eq}(F_1) \wedge \mathrm{eq}(F_2) = \mathrm{eq}(F_1) \cap \mathrm{eq}(F_2)$.

(b) Assume $\mathrm{ex}_V(F_1) \cap \mathrm{ex}_V(F_2) \neq \emptyset$. Recall the definitions of the mappings ϕ, σ and the γ-equality set γ-eq (cf. (A.5) resp. the proof of Theorem 3.7), then

$$\begin{aligned}
\mathrm{ex}(F_1 \wedge F_2) &= \gamma\text{-eq}(\phi(F_1 \wedge F_2)) && \text{(by Proposition A.6)}\\
&= \gamma\text{-eq}(\sigma \circ (-1)\text{-hog}(F_1 \wedge F_2))\\
&= \gamma\text{-eq}(\sigma((-1)\text{-hog}(F_1) \wedge (-1)\text{-hog}(F_2))) && \text{((}-1\text{)-hog is a homorphism)}\\
&= \gamma\text{-eq}(\sigma((-1)\text{-hog}(F_1)) \wedge \sigma((-1)\text{-hog}(F_2))) && (\sigma \text{ is an anti-isomorphism})\\
&= \gamma\text{-eq}(\phi(F_1) \vee \phi(F_2))\\
&= \gamma\text{-eq}(\phi(F_1)) \cap \gamma\text{-eq}(\phi(F_2)) && \text{(by part (a))}\\
&= \mathrm{ex}(F_1) \cap \mathrm{ex}(F_2) && \text{(by Proposition A.6).}
\end{aligned}$$

If $\mathrm{ex}_V(F_1) \wedge \mathrm{ex}_V(F_2) = \emptyset$, then $F_1 \wedge F_2 = \emptyset$ and thus $\mathrm{ex}(F_1 \wedge F_2) = (\emptyset, \emptyset)$.

Theorems 3.1 and 3.7 immediately imply

Theorem 4.3. *Let $P = P(A, b) = \mathrm{conv}(V) + \mathrm{cone}(E)$ be a polyhedron and F_1, F_2 two faces of P.*

(a) If $F_1 \wedge F_2 \neq \emptyset$ and $I := \mathrm{eq}(F_1) \cup \mathrm{eq}(F_2)$, then

$$\mathrm{eq}(F_1 \wedge F_2) = \left\{ i \in \{1, \dots, m\} \,\middle|\, (A_{i.}, b_i)^{\mathrm{T}} \in \mathrm{cone}\left(\begin{pmatrix} A_I, & b_I \\ -A, & -b \end{pmatrix}^{\mathrm{T}}\right)\right\}.$$

(b) Let $(S, T) = \mathrm{ex}(F_1) \cup \mathrm{ex}(F_2)$ then

$$\mathrm{ex}_V(F_1 \vee F_2) = \{v \in V \mid \exists \delta \geqq 0 \text{ with } v \in (1+\delta)\mathrm{conv}(S) - \delta\, \mathrm{conv}(V) + \mathrm{lin}(T) - \mathrm{cone}(E \setminus T)\}$$

$$\mathrm{ex}_E(F_1 \vee F_2) = \{e \in E \mid e \in \mathrm{cone}((S - V) \cup (-E) \cup T)\}.$$

Since adjacency can be defined by dimension formulas, cf. (1.3), (1.4) resp. (1.5), (1.6), we can determine the adjacency of faces F_1, F_2 by calculating the dimensions of F_1, F_2, $F_1 \vee F_2$, $F_1 \wedge F_2$. Lemma 4.1 tells us how the dimension of a face F can be obtained using the equality set $\mathrm{eq}(F)$ of F, Lemma 4.2 and Theorem 4.3 show how we can get the equality sets and extreme sets of the join resp. meet of faces.

The results above and the ellipsoid method enable us to check in polynomial time whether two faces of a polyhedron are adjacent or not. This can be done as follows:

4.4. Suppose an inequality description $P = P(A, b)$ is given, let $c^1x \leqq c_0^1$, $c^2x \leqq c_0^2$ be two valid inequalities with respect to P, and let $F_1 = \{x \in P \mid c^1x = c_0^1\}$, $F_2 = \{x \in P \mid c^2x = c_0^2\}$, and assume all data are rational.

(4.4.1) Add the inequality $c^1x \leqq c_0^1$ as constraint $m+1$ to the constraints $Ax \leqq b$ obtaining a system $A'x \leqq b'$ which also describes P. Setting $I := \{m+1\}$, calculate the equality set $I_1' \subseteq \{1, \dots, m, m+1\}$ of F_1 with respect to $P(A', b')$ by the algorithm given in Theorem 3.6. Since $c^1x \leqq c_0^1$ is redundant, $I_1 = I_1' \setminus \{m+1\}$ is the equality set of F_1 with respect to $P(A, b)$. In the same way calculate the equality set I_2 of F_2 with respect to $P(A, b)$. If one of the equality sets I_1, I_2 is empty or is the whole index set M, then stop, since one of the faces is equal to P or minimal and therefore F_1 and F_2 are not adjacent.

(4.4.2) Calculate the rank, say k_1, of A_{I_1}. and the rank, say k_2, of A_{I_2}. We may assume that $k_2 \leqq k_1$ holds, thus by Lemma 4.1, $n - k_1 = \dim(F_1) \leqq \dim(F_2) = n - k_2$.

(4.4.3) Let $J := I_1 \cap I_2$, then by Lemma 4.2 J is the equality set of $F_1 \vee F_2$. Calculate the rank j of A_J. If $j \neq k_2 - 1$, F_1 and F_2 are not adjacent $\rightarrow$ STOP.

(4.4.4) Let $K := I_1 \cup I_2$, then $F_1 \wedge F_2 = \{x \in \mathbb{R}^n \mid A_{K.}x = b_K,\ Ax \leqq b\}$. Check whether $F_1 \wedge F_2$ is empty or not using the ellipsoid method.

(4.4.5) If $F_1 \wedge F_2 = \emptyset$, then calculate the rank k of A. The number $n - k$ is the dimension of the lineality space of P. If $k_1 = k$, then F_1 and F_2 are adjacent, since $\dim(F_1) = \dim(\text{lineal}(P))$, otherwise F_1 and F_2 are not adjacent. $\rightarrow$ STOP.

(4.4.6) If $F_1 \wedge F_2 \neq \emptyset$, then using the algorithm of Theorem 3.6 for A, b and K determine the equality set L of $F_1 \wedge F_2$. Calculate the rank ℓ of A_L. If $\ell = k_1 + 1$, then F_1 and F_2 are adjacent, otherwise not. STOP.

Since we have used the algorithm of Theorem 3.6, the ellipsoid method and rank calculations only the algorithm above runs in time polynomial in a binary encoding of the data A, b, c^1, c^2, c_0^1, c_0^2. Next we show that adjacency can also be checked in polynomial time if the polyhedron is given by its vertices and extremals.

4.5. We assume that P is given by $P = \text{conv}(V) + \text{cone}(E)$ where V and E are finite sets of rational vectors in $\mathbb{R}^n$. We also assume that we have two faces F_1 and F_2. These could be given in two ways:

(4.5.1.a) Suppose two nonempty sets $(S_i', T_i') \subseteq (V, E)$ are given and $F_i = \text{sp}(S_i', T_i')$, $i = 1, 2$. Using the algorithm of Theorem 3.10 we can determine $(S_i, T_i) = (\text{ex}_V(F_i), \text{ex}_E(F_i))$, $i = 1, 2$ in polynomial time.

(4.5.1.b) Suppose two valid inequalities $c^1x \leqq c_0^1$ and $c^2x \leqq c_0^2$ are given and $F_i := \{x \in P \mid c^ix = c_0^i\}$, $i = 1, 2$. Then $v \in V$ belongs to $\text{ex}_V(F_i)$ if and only if $c^iv = c_0^i$. In case $\text{ex}_V(F_i) = \emptyset$ we know that $\text{ex}_E(F_i) = \emptyset$. If $\text{ex}_V(F_i) \neq \emptyset$ then $e \in E$ belongs to $\text{ex}_E(F_i)$ if and only if $c^ie = 0$. Thus we can determine $\text{ex}(F_1)$ and $\text{ex}(F_2)$ in time polynomial in the input length.

If one of the $\text{ex}_V(F_i)$ is empty then one of the faces is empty and we can stop with F_1, F_2 not adjacent.

(4.5.2) Calculate the affine ranks, say k_i, of $S_i \cup (S_i + T_i)$, $i = 1, 2$, which by Lemma 4.1 determine the dimensions of F_i. We may assume that $k_1 \leqq k_2$ holds.

(4.5.3) We now calculate the dimension of $F_1 \vee F_2$ as follows. Set $(P', Q') = \text{ex}(F_1) \cup \text{ex}(F_2)$, then by Theorem 4.3 using the algorithm of Theorem 3.10 we can calculate $(P, Q) = \mathrm{ex}(F_1 \vee F_2)$ in polynomial time. The affine rank, say j, of $P \cup (P+Q)$ is the dimension of $F_1 \vee F_2$, hence if $j \neq k_2+1$, F_1 and F_2 are not adjacent and we can stop.

(4.5.4) Set $(R, S) = \mathrm{ex}(F_1) \cap \mathrm{ex}(F_2)$.

(4.5.5) If $R \neq \emptyset$, then $F_1 \wedge F_2 = \emptyset$ and we have to calculate the dimension of the lineality space of P. For every $e \in E$ we check with the ellipsoid method whether $P_e = \{\lambda \geqq 0 \mid e = -E\lambda\}$ is empty or not. The set L of $e \in E$ for which P_e is nonempty generates the lineality space of P. We calculate the rank ℓ of L, in case $L = \emptyset$ set $\ell = 0$. If $\ell = k_1$ then F_1 and F_2 are adjacent, otherwise not, and we can stop.

(4.5.6) If $R = \emptyset$ then by Lemma 4.2, $(R, S) = \mathrm{ex}(F_1 \wedge F_2)$ and we calculate the affine rank of $R \cup (R+S)$, say k. If $k = k_1 - 1$, then F_1 and F_2 are adjacent, otherwise not. STOP!

The above algorithm uses the ellipsoid method, the algorithm of Theorem 3.10, rank and affine rank calculations only, therefore the overall computational time is bounded by a polynomial in a binary encoding of V and E, resp. c^i, c_0^i if the faces F_i are given by inequalities.

Appendix A

In the following we list some concepts and results of polyhedral theory which are partly well-known or can be found in [2], see also [1, 6 and 10].

We assume in the following that a polyhedron $P \subseteq \mathbb{R}^n$ is given and a linear description $P = P(A, b)$ with an (m, n)-matrix A and $b \in \mathbb{R}^m$ is known, and also that we know finite sets V, $E \subseteq \mathbb{R}^n$ such that $P = \mathrm{conv}(V) + \mathrm{cone}(E)$. We set $M = \{1, 2, \dots, m\}$.

Proposition A.4. *The mapping $F \to (-1)$-hog(F) is an injective homomophism from exists $u \in \mathbb{R}^m, u \geqq 0$ with $u^{\mathrm{T}}A = c^{\mathrm{T}}$, $ub \leqq c_0$. If for this u, $I = \{i \in M \mid u_i > 0\}$, then for every $x \in P$ the following holds:*

$$cx = ub \quad \text{if and only if} \quad A_{i.}x = b_i \quad \text{for all } i \in I.$$

Moreover, if $F = \{x \in P \mid cx = c_0\} \neq \emptyset$, then

$$\mathrm{eq}(F) = \{i \in M \mid \exists u \geqq 0,\ u^{\mathrm{T}}A = c^{\mathrm{T}},\ ub = c_0,\ u_i > 0\}.$$

On the set M of row indices of A we define the so called *equality set lattice* $\mathrm{EQ}(A, b) \subseteq 2^M$ as follows

$$\mathrm{EQ}(A, b) = \{I \subseteq M \mid \exists F \in \mathcal{F}(P) \text{ such that } I = \mathrm{eq}(F)\}.$$

$\mathrm{EQ}(A, b)$ is obviously a lattice under set inclusion, moreover we have:

Proposition A.2. *The equality set lattice* EQ(A, b) *is anti-isomorphic to the face lattice* $\mathcal{F}(P)$. *Furthermore, if P is a polytope, then* EQ(A, b) *is anti-isomorphic to the face lattice* $\mathcal{F}(\tau\text{-hog}(P))$ *of the* τ*-homogenization of P for* $\tau \in \{-1, 1\}$.

For any $S \subseteq \mathbb{R}^n$ we define the γ*-polar* of S to be

$$S^\gamma := \left\{ \begin{pmatrix} c \\ c_0 \end{pmatrix} \in \mathbb{R}^{n+1} \mid cx \leqq c_0 \text{ for all } x \in S \right\}.$$

Thus, S^γ is the set of all vectors $\binom{c}{c_0}$ defining a valid inequality $cx \leqq c_0$ for S. In case S is the polyhedron P we have:

Proposition A.3. *The following equation holds.*

$$\begin{aligned} P^\gamma &= ((-1)\text{-hog}(P))^0 \\ &= \operatorname{cone}\begin{pmatrix} A^T & 0 \\ b^T & 1 \end{pmatrix} = P\left(\begin{pmatrix} V^T & -1 \\ E^T & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right). \end{aligned}$$

In particular P^γ is a polyhedral cone. (Note that V and E are considered as matrices here where the elements of V and E are the columns).

The γ-polar P^γ of P serves as a connection between the two representations $P = P(A, b)$ and $P = \operatorname{conv}(V) + \operatorname{cone}(E)$ and is the main tool for deriving the following results.

Proposition A.4. *The mapping* $F \to (-1)\text{-hog}(F)$ *is an injective homomorphism from* $\mathcal{F}(P)$ *to* $\mathcal{F}((-1)\text{-hog}(P))$. *Furthermore there exists an anti-isomorphism*

$$\sigma : \mathcal{F}((-1)\text{-hog}(P)) \to \mathcal{F}(P^\gamma).$$

The composite function $\Phi = \sigma \circ (-1)\text{-hog}$ *is an injective anti-homomorphism*

$$\Phi : \mathcal{F}(P) \to \mathcal{F}(P^\gamma).$$

With respect to P^γ it is notationally convenient to define the mapping eq and the equality set lattice in a slightly different way. Since by (A.3)

$$P^\gamma = P\left(\begin{pmatrix} V^T & -1 \\ E^T & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

we set:

Definition A.5.

$$\gamma\text{-eq} : P^\gamma \to 2^V \times 2^E$$

$$P^\gamma \supseteq Q \to (S, T)$$

where

$$S = \left\{ v \in V \mid vy = z \text{ for all } \binom{y}{z} \in Q \right\},$$

$$T = \left\{ e \in E \mid ey = 0 \text{ for all } \binom{y}{z} \in Q \right\}.$$

The difference between the mappings eq and γ-eq is that in case of eq we use sets of row indices as images and that for γ-eq we use the row vectors (without the last component) corresponding to the row indices as images and that these in addition are split into two parts. It is therefore obvious that the equality set lattice of P^γ and the γ-equality set lattice (defined analogously) are isomorphic.

The following result justifies the use of the γ-equality mapping.

Proposition A.6. *For every face $F \in \mathcal{F}(P)$ we have*

$$\mathrm{ex}(F) = -\ \gamma\text{-eq}(\Phi(F)),$$

in particular, if $(S, T) = \mathrm{ex}(F)$, then

$$\Phi(F) = \left\{ \binom{c}{c_0} \in P^\gamma \mid \begin{pmatrix} S^{\mathrm{T}} & -1 \\ T^{\mathrm{T}} & 0 \end{pmatrix} \binom{c}{c_0} = 0 \right\}.$$

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