PART I
CHAPTER 2

New Aspects of Polyhedral Theory

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In this paper we survey several topics of polyhedral theory which are of particular interest for mathematical programming. In Section 1 we introduce our notation and some concepts of linear algebra which are needed in the sequel. In Section 2 we show that an algorithm, namely the Fourier–Motzkin elimination algorithm, can be used as the mainstay of a large portion of the results of polyhedral theory. We review in Section 3 the ellipsoid method due to Shor and discuss Khachian's proof that this algorithm can be used to solve linear programming problems in polynomial time. In Section 4 we study faces of polyhedra, in particular we give characterizations of valid inequalities, facets and vertices. We use these results in Section 5 to characterize minimal representations of polyhedra and demonstrate that redundancy can be checked in polynomial time. In Sections 6 and 7 we introduce the concepts of homogenization and polarization and indicate their usefulness for streamlining and unifying proofs in polyhedral theory. Finally Section 8 summarizes results on lattices associated with polyhedra.

0. Introduction

The theory of convex polyhedra dates back to the ancient Greeks. Euclid for example studied polyhedra in 2- and 3-dimensional spaces and found some volume formulas. The first essential post-greek contribution to polyhedral theory is probably due to Euler who established the famous relationship between the number of vertices, edges and faces of polytopes in 3-space, cf. Grünbaum (1967). Euler's results, however, seem to have been partially known to Descartes a hundred years before. His manuscript was unfortunately lost, but a partial copy made by Leibnitz was discovered in 1860, cf. Steinitz–Radermacher (1976). Detailed remarks on the beginning of polyhedral theory can be found in Coxeter (1963), Grünbaum (1967), Minkowski (1911), and Schläfli (1901).

A first state-of-the-art survey of the theory of polyhedra was presented by Steinitz and Rademacher in 1934 (new edition 1976). Today, Grünbaum (1967) is one of the best references with respect to the combinatorial investigation of polyhedra. An interesting new axiomatic approach to the theory of convex sets was recently presented by Prenowitz–Jantosciak (1979).

Around 1950 the theory of polyhedra received a new impetus from outside. Economists formulated optimization models whose feasible solutions are given by means of linear inequalities and whose objective functions are linear. To solve such problems the Simplex algorithm and many variants as well as several relaxation methods were designed. The development of the so-called theory of linear programming, its algorithmic implementation and economic interpretation is (among others) mainly due to A. Charnes, G. B. Dantzig, D. Gale, A. J. Hoffman, L. Kantorovich, T. C. Koopmans, H. W. Kuhn, T. Motzkin, A. W. Tucker and J. von Neumann, cf. the bibliography in Kuhn–Tucker (1956) pp. 305–322.

Beginning with these activities polyhedral theory was newly developed from a mathematical programming point of view, cf. Kuhn–Tucker (1956). Emphasis was layed on characterizing properties of polyhedra by means of a given description of a polyhedron and on developing fast algorithms for the solution of linear programming problems. The Simplex algorithm due to G. B. Dantzig turned out to be an extremely fast and reliable procedure to solve real-world problems, although Klee–Minty (1972) and others proved that this method may have a bad worst-case behaviour.

Whether linear programming problems can be solved in polynomial time was an outstanding open problem until recently Khachian (1979) showed that the ellipsoid method due to Shor (1977) can be used to check the nonemptiness of polyhedra and to solve linear programming problems in polynomial time.

In the sequel we shall survey the theory of polyhedra from the mathematical programming point of view. We emphasize unifying concepts which can be used to give short and elegant proofs of the key theorems of polyhedral theory, and we point out important computational aspects of polyhedral theory, e.g. indicate how the ellipsoid method can be used to solve various problems of polyhedral theory in polynomial time.

1. Notation

By $\mathbb{R}$ we denote the real numbers, and $\mathbb{R}^n$ is the usual vector space of n-tuples $x = (x_1, x_2, \ldots, x_n)^T$. Vectors will always be considered as *column vectors* and 'T' denotes transposition. For convenience everything takes place in $\mathbb{R}^n$, although the whole theory could also be developed in finite dimensional vector spaces over (archimedian) ordered fields.

A matrix $A = (a_{ij})_{i=1,\ldots,m;j=1,\ldots,n}$ (where $a_{ij} \in \mathbb{R}$) with m rows and n columns is called an *(m, n)-matrix*. For simplicity we usually assume that $M = \{1, 2, \ldots, m\}$ is the set of row indices and $N = \{1, 2, \ldots, n\}$ is the set of column indices. We shall extensively use the following notations to denote submatrices of A. Let $I = (i_1, i_2, \ldots, i_r)$ $\big(J = (j_1, j_2, \ldots, j_s)\big)$ be a vector of pairwise different row (column) indices, i.e. $\{i_1, \ldots, i_r\} \subseteq M$, $\{j_1, \ldots, j_s\} \subseteq N$, then $A_{I,J}$ or just A_{IJ} denotes the following submatrix of A:

$$A_{IJ} = \begin{pmatrix} a_{i_1 j_1}, \ldots, a_{i_1 j_s} \\ \vdots \qquad \vdots \\ a_{i_r j_1}, \ldots, a_{i_r j_s} \end{pmatrix}.$$

In case $J = (1, 2, \ldots, n)$ $(I = (1, 2, \ldots, m))$ we write $A_{I.}$ or $A_{I,\cdot}$ ($A_{.J}$ or $A_{\cdot,J}$). If $I = (i)$ and $J = (1, 2, \ldots, n)$ $(J = (j)$ and $I = (1, 2, \ldots, m))$ we write $A_{i.}$ or $A_{i,\cdot}$ ($A_{.j}$ or $A_{\cdot,j}$), i.e. $A_{i.}$ is the i-th row of matrix A (in the sequel $A_{i.}$ will always be considered as a row vector) and $A_{.j}$ is the j-th column of A. Often the order of the components of I or J is completely unimportant. Therefore, if $I \subseteq M$ and $J \subseteq N$ we shall also write $A_{I,J}$ to denote a submatrix of A. But such a matrix is only defined up to row and column permutations.

A vector $x \in \mathbb{R}^n$ is called a *linear combination* of the vectors $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ if for some $\lambda \in \mathbb{R}^k$

$$x = \sum_{i=1}^{k} \lambda_i x^i.$$

If additionally

$$\left.\begin{array}{l} \lambda \geqslant 0 \\ \lambda \cdot \mathbb{1} = 1 \\ \lambda \cdot \mathbb{1} = 1, \quad \lambda \geqslant 0 \end{array}\right\} \text{ we call } x \text{ a } \begin{Bmatrix} \text{conic} \\ \text{affine} \\ \text{convex} \end{Bmatrix} \text{ combination}$$

of the vectors $x^1, x^2, \ldots, x^k$. These combinations are called *proper* if neither $\lambda = 0$ nor $\lambda = e_j$ for some $j \in \{1, 2, \ldots, k\}$. (Here $\mathbb{1}$ denotes a vector of appropriate dimension all whose components are one, and e_j denotes a vector of appropriate dimension all whose components are zero except the j-th which is one.) For a nonempty subset $S \subseteq \mathbb{R}^n$ we denote by

$$\left.\begin{array}{l} \text{lin}(S) \\ \text{cone}(S) \\ \text{aff}(S) \\ \text{conv}(S) \end{array}\right\} \text{ the } \begin{Bmatrix} \text{linear} \\ \text{conic} \\ \text{affine} \\ \text{convex} \end{Bmatrix} \text{ hull of elements of } S,$$

i.e. the set of all vectors which are linear (conic, affine, convex) combinations of finitely many vectors of S. For the empty set we define $\mathrm{lin}(\emptyset) := \mathrm{cone}(\emptyset) = \{0\}$, $\mathrm{aff}(\emptyset) := \mathrm{conv}(\emptyset) = \emptyset$. If A is an (m, n)-matrix then $\mathrm{lin}(A)$ $\big(\mathrm{cone}(A), \mathrm{aff}(A), \mathrm{conv}(A)\big)$ is the linear (conic, affine, convex) hull of the column vectors $A_{.j}$, $j = 1, \ldots, n$.

$$\text{A subset } S \subseteq \mathbb{R}^n \text{ is called } \begin{Bmatrix} \text{a linear space} \\ \text{a cone} \\ \text{an affine space} \\ \text{a convex set} \end{Bmatrix} \text{ if } \begin{Bmatrix} S = \mathrm{lin}(S) \\ S = \mathrm{cone}(S) \\ S = \mathrm{aff}(S) \\ S = \mathrm{conv}(S) \end{Bmatrix}.$$

A nonempty finite subset $S \subseteq \mathbb{R}^n$ is called *linearly* (resp. *affinely*) *independent* if none of its members is a proper linear (affine) combination of elements of S. Otherwise S is called *linearly* (resp. *affinely*) dependent. The empty set is affinely but not linearly independent. It is well known that a linearly (resp. affinely) independent subset of $\mathbb{R}^n$ contains at most n (resp. $n + 1$) elements. For any set $S \subseteq \mathbb{R}^n$, the *rank* of S (*affine rank of* S) denoted by $\mathrm{rank}(S)$ $\big(\mathrm{arank}(S)\big)$, is the cardinality of the largest linearly (affinely) independent subset of S. For any subset $S \subseteq \mathbb{R}^n$, the *dimension* of S, denoted by $\dim(S)$, is the cardinality of a largest affinely independent subset of S minus one, i.e. $\dim(S) = \mathrm{arank}(S) - 1$.

For any set $S \subseteq \mathbb{R}^n$

$$\mathrm{rec}(S) := \{y \in \mathbb{R}^n \mid x + \lambda y \in S \text{ for all } x \in S \text{ and all } \lambda \geqslant 0\}$$

denotes the *recession cone* of S. Intuitively, every vector of the recession cone represents a 'direction to infinity' in S. By

$$\mathrm{lineal}(S) := \{y \in \mathbb{R}^n \mid -y \in \mathrm{rec}(S)\}$$

we denote the *lineality space* of S. The lineality space is the largest linear subspace L of $\mathbb{R}^n$ such that $x + L \subseteq S$ for all $x \in S$.

A subset P of $\mathbb{R}^n$ is called a *polyhedron* if there exists an (m, n)-matrix A and a vector $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$. Thus a polyhedron is the set of real solutions of a finite number of linear inequalities, or equivalently the intersection of finitely many closed half-spaces $\{x \in \mathbb{R}^n \mid A_{i.}x \leqslant b_i\}$ $(i = 1, \ldots, m)$. The inequality system $Ax \leqslant b$ is called a *linear defining system*, an *inequality representation* or just a *representation* of P. Clearly, such a representation is not unique. The polyhedron defined by the linear inequality system $Ax \leqslant b$ is denoted by $P(A, b)$.

If a mixed system of equations and linear inequalities is given and if in addition some variables are sign-restricted such as the system

$$\begin{aligned} &Bx + Cy = c, \\ &Dx + Ey \leqslant d, \\ &x \in \mathbb{R}^{n_1}, \quad y \in \mathbb{R}^{n_2}, \quad x \geqslant 0, \end{aligned} \tag{$*$}$$

then, using the following matrix A and vector b,

$$A := \begin{pmatrix} B & C \\ -B & -C \\ D & E \\ -I & 0 \end{pmatrix}, \qquad b := \begin{pmatrix} c \\ -c \\ d \\ 0 \end{pmatrix},$$

the mixed system $(*)$ has the same set of solutions as $Ax \leqslant b$, hence also defines a polyhedron. A special kind of polyhedra will be used frequently and is therefore denoted by a particular symbol, namely

$$P^{=}(A, b) := \{x \in \mathbb{R}^n \mid Ax = b, x \geqslant 0\}.$$

A *polyhedral cone* is a cone which is also a polyhedron. A bounded polyhedron is called a *polytope*.

The recession cone and the lineality space of polyhedra can be easily described, namely if $P = P(A, b) \subseteq \mathbb{R}^n$, then

$$\begin{aligned} &\operatorname{rec}(P) = \{x \in \mathbb{R}^n \mid Ax \leqslant 0\}, \\ &\operatorname{lineal}(P) = \{x \in \mathbb{R}^n \mid Ax = 0\}. \end{aligned}$$

Therefore, the recession cone and lineality space of a polyhedron are polyhedral cones.

In case $V, E \subseteq \mathbb{R}^n$ are finite sets and $P = \operatorname{conv}(V) + \operatorname{cone}(E)$ (we shall see later that such a set is a polyhedron and that every polyhedron can be described in this way), then

$$\begin{aligned} &\operatorname{rec}(P) = \operatorname{cone}(E), \\ &\operatorname{lineal}(P) = \operatorname{cone}(\{e \in E \mid -e \in \operatorname{cone}(E)\}). \end{aligned}$$

The problem of maximizing a linear objective function $f: \mathbb{R}^n \to \mathbb{R}$ (i.e. $f(x) = c^{\mathrm{T}}x$ for some $c \in \mathbb{R}^n$) subject to the solution set of a linear inequality system $Ax \leqslant b$ is called a *linear program*. To shorten notation the linear program: find $\bar{x} \in P(A, b)$ such that $c^{\mathrm{T}}\bar{x} = \max\{c^{\mathrm{T}}x \mid x \in P(A, b)\}$ is denoted by

$$\begin{aligned} &\max c^{\mathrm{T}}x, \\ &Ax \leqslant b, \end{aligned} \quad \text{or} \quad \max\{c^{\mathrm{T}}x \mid x \in P(A, b)\}. \tag{1.1}$$

For a linear program such as (1.1) the following three outcomes are possible:

(1) $P(A, b) = \emptyset$, i.e. there is no feasible solution and hence no optimal one. In this case the linear program is called *infeasible.*

(2) $P(A, b) \neq \emptyset$ and for all $x \in P(A, b)$ there is an $\bar{x} \in P(A, b)$ such that $c^{\mathrm{T}}\bar{x} \geqslant c^{\mathrm{T}}\bar{x} + 1$. Such a linear program is called *unbounded.*

(3) $P(A, b) \neq \emptyset$ and there is $c_0 \in \mathbb{R}$ such that $c^{\mathrm{T}}x \leqslant c_0$ for all $x \in P(A, b)$. Such a linear program is called *bounded* and we shall show later that in this case there is a feasible solution $\bar{x} \in P(A, b)$ such that $c^{\mathrm{T}}\bar{x} \geqslant c^{\mathrm{T}}x$ for all $x \in P(A, b)$. Such a vector $\bar{x}$ is called *optimal solution* of the linear program (1.1).

Note that the objective function can also be added to the constraints, and then a linear program can be stated in the following form

$$\begin{aligned} &\max z, \\ &\begin{pmatrix} -c^{\mathrm{T}} \\ A \end{pmatrix} x \leqslant \begin{pmatrix} -z \\ b \end{pmatrix}. \end{aligned} \tag{1.2}$$

In other words, a linear program can be interpreted as the problem of increasing the first component of the right hand side of a linear inequality system as much as possible without obtaining a nonempty solution set, or equivalently, if we define for every $z \in \mathbb{R}$ the polyhedron

$$P_z := P\left(\begin{pmatrix} -c^{\mathrm{T}} \\ A \end{pmatrix}, \begin{pmatrix} -z \\ b \end{pmatrix}\right) \subset \mathbb{R}^n, \tag{1.3}$$

then we want to find the largest $z \in \mathbb{R}$ such that the polyhedron P_z is nonempty.

2. Characterization of nonemptiness of polyhedra: The Fourier–Motzkin elimination and its consequences

Formulation (1.3) of a linear programming problem shows that the optimal value z^* of the linear program has the property that the polyhedra P_z are nonempty for all $z \leqslant z^*$ while all polyhedra P_z are empty for $z > z^*$.

Thus, finding the optimal value of a linear program can be viewed as the problem of characterizing the nonemptiness (so called *primal problem*) and emptiness (*dual problem*) of polyhedra, or stated formally:

$$(\exists\, x \in \mathbb{R}^n \;\; Ax \leqslant b)? \qquad \text{primal problem,} \tag{2.1}$$

$$(\forall\, x \in \mathbb{R}^n \;\; Ax \nleqslant b)? \qquad \text{dual problem.} \tag{2.2}$$

Logically, the primal and the dual problem are equivalent in the sense that a characterization of emptiness of polyhedra yields by negation a characterization of nonemptiness of polyhedra and vice versa. However, from an algorithmic point of view the primal problem (2.1) and the dual problem (2.2) are fundamentally different. Suppose for example that we wish to solve the primal and the dual problem on a computer (as a computer has only finitely many symbols at its disposal we shall restrict ourselves to rational numbers $\mathbb{Q}$ here). Let $S(x)$ be a subroutine that checks for any given $x \in \mathbb{Q}^n$ whether $Ax \leqslant b$ holds or not. Now choose any enumeration of the rationals $x^1, x^2, x^3, \ldots$ and run the subroutine $S(x^i)$, $i = 1, 2, 3, \ldots$. If there is a rational vector $\bar{x}$ such that $A\bar{x} \leqslant b$ holds, then after a finite number of calls of $S(x^i)$ we will have a proof of the consistency of the inequality system. If however there is no rational vector $\bar{x}$ with $A\bar{x} \leqslant b$, the inconsistency of the system cannot be verified in a finite number of steps. The problem here obviously lies in the generalization quantifier '∀' of the dual statement.

The goal of a duality theory (not only for the special case of linear programming) is to find a logically equivalent reformulation of the dual problem (2.2) in which the quantifier '∀' is replaced by an existence quantifier '∃', i.e. which yields an 'easily checkable' statement $E(x)$ so that

$$(\forall\, x \in \mathbb{R}^n \;\; Ax \nleqslant b) \Leftrightarrow \big(\exists\, x \in \mathbb{R}^n \text{ satisfying } E(x)\big).$$

holds. Such a characterization for polyhedra was first given by Farkas (1902) although the way was prepared by Fourier (1826) much earlier.

The main idea for finding a positive criterion for the dual problem (2.2) consists in the construction of a special linear inequality system $Dx \leqslant d$ such that $P(D, d) = \emptyset$ if and only if $P(A, b) = \emptyset$ and such that $P(D, d) = \emptyset$ is easy to verify.

Suppose we are able to find a construction such that this matrix D is a zero-matrix, then clearly $P(D, d) = \emptyset$ if and only if there is an index i with $d_i < 0$. If furthermore the construction of the zero-matrix D and the vector

d can be carried out in such a way that every row of D (every component of d) is a conic combination of rows of A (components of b), then $P(D, d) = \emptyset$ is equivalent to the existence of a vector $u \in \mathbb{R}^m$ with $u \geqslant 0$, $u^T A = 0$, $u^T b < 0$. Such a construction would yield the desired result, namely

$$P(A, b) = \emptyset \;\Leftrightarrow\; \exists\, u \in \mathbb{R}^m \colon u \geqslant 0,\, u^T A = 0,\, u^T b < 0.$$

This construction can in fact be carried out, the method which accomplishes this is known as the Fourier–Motzkin algorithm, and the characterization of emptiness of $P(A, b)$ given above is the celebrated Farkas Lemma. The following algorithm is an analogue of the well-known Gauß–Jordan algorithm for equality systems and was introduced as a main tool in the theory of inequalities by Dines (1919) and Motzkin (1936). The algorithm has the following form:

Algorithm 2.1. *Fourier–Motzkin elimination* (*of one variable*).

Input. An (m, n)-matrix A, an m-vector b, and a column index $j \in \{1, 2, \ldots, n\}$ of A.

Output. An (r, n)-matrix D (r will be determined by the algorithm) and an r-vector d, such that the j-th column of D is a zero-vector.

Method

Step 1. Partition the set of row indices $M = \{1, 2, \ldots, m\}$ of A into

$$N = \{i \in M \mid a_{ij} < 0\},$$
$$Z = \{i \in M \mid a_{ij} = 0\},$$
$$P = \{i \in M \mid a_{ij} > 0\}.$$

(The set $Z \cup (N \times P)$ will be the row index set of the matrix D and the vector d.)

Step 2. Let $r = |Z \cup (N \times P)|$, $R = \{1, 2, \ldots, r\}$ and let $p \colon R \to Z \cup (N \times P)$ be a bijection, i.e. a canonical indexing of $Z \cup (N \times P)$.

Step 3. FOR $i = 1$ TO r DO

(a) IF $p(i) \in Z$ THEN $D_{i\cdot} := A_{p(i)\cdot}$, $d_i := b_{p(i)}$

(b) IF $p(i) = (s, t) \in N \times P$ THEN

$$D_{i\cdot} := A_{tj} A_{s\cdot} - A_{sj} A_{t\cdot}$$
$$d_i := A_{tj} b_s - A_{sj} b_t$$

END

Theorem 2.2. *Let A be an (m, n)-matrix, b be an m-vector and choose some column index $j \in \{1, \ldots, n\}$. Let D resp. d be the (r, n)-matrix resp. r-vector obtained by the Fourier–Motzkin elimination algorithm for this specified j. Then $P(A, b)$ is nonempty if and only if $P(D, d)$ is nonempty.*

Proof. Every row $D_{i.}$ of D is a conic combination of rows of A, i.e. $D_{i.} = u^{\mathrm{T}}A$ for some $u \in \mathbb{R}^m_+$. Hence for any vector $x \in \mathbb{R}^n$ which satisfies $Ax \leqslant b$ we have $D_{i.}x = u^{\mathrm{T}}Ax \leqslant ub = d_i$ and thus x is also a solution of $Dx \leqslant d$.

To show the converse we prove that given a vector x with $Dx \leqslant d$ we can find a nonempty interval $[L, U]$ of real numbers such that for any $\lambda \in [L, U]$ the vector $x^\lambda := x + \lambda e_j$ satisfies $Ax^\lambda \leqslant b$ and $Dx^\lambda \leqslant d$.

Since by construction $D_{.j} = 0$ holds, the vector $x^\lambda = x + \lambda e_j$ satisfies $Dx^\lambda \leqslant d$ for any $\lambda \in \mathbb{R}$. We may therefore assume that the initial vector x with $Dx \leqslant d$ satisfies $x_j = 0$. Furthermore, by construction, for every $i \in Z$ there is a $k \in R$ with $p(k) = i$ and $A_{i.} = D_{k.}$, $b_i = d_k$, hence $A_{i.}x^\lambda \leqslant b_i$ also holds for all $\lambda \in \mathbb{R}$ and all $i \in Z$.

To determine the desired interval we calculate the 'scaled' slacks of the remaining inequalities, i.e. for all $i \in P \cup N$ we set

$$y_i := A_{ij}^{-1}\Big(b_i - \sum_{k \neq j} A_{ik}x_k\Big) = A_{ij}^{-1}(b_i - A_{i.}x)$$

and define $U := +\infty$ if $P = \emptyset$, otherwise $U := \min\{y_i \mid i \in P\}$ and $L := -\infty$ if $N = \emptyset$, otherwise $L := \max\{y_i \mid i \in N\}$.

We first show that $L \leqslant U$. This is obvious if $P = \emptyset$ or $N = \emptyset$. So suppose that $s \in N$, $t \in P$ and $i \in R$ are chosen such that $y_s = L$, $y_t = U$ and that $p(i) = (s, t)$, then

$$A_{tj}A_{s.}x - A_{sj}A_{t.}x = D_{i.}x \leqslant d_i = A_{tj}b_s - A_{sj}b_t$$

which implies $A_{sj}(b_t - A_{t.}x) \leqslant A_{tj}(b_s - A_{s.}x)$ and further

$$y_t = A_{tj}^{-1}(b_t - A_{t.}x) \geqslant A_{sj}^{-1}(b_s - A_{s.}x) = y_s,$$

i.e. from the choice of s and t we get $U \geqslant L$.

It remains to prove that $A_{i.}x^\lambda \leqslant b_i$ holds for all $i \in P \cup N$ and all $\lambda \in [L, U]$. Let $i \in P$ then $U < +\infty$ and

$$\begin{aligned} A_{i.}x^\lambda = A_{i.}x + A_{ij}\lambda &\leqslant A_{i.}x + A_{ij}U \\ &\leqslant A_{i.}x + A_{ij}y_i = b_i. \end{aligned}$$

The same follows for $i \in N$ similarly, and we are done.

The Fourier–Motzkin elimination as stated above is an algorithm which can be applied to an (m, n)-Matrix A and an m-vector b. No use is made of the relation holding between A and b. So, instead of considering an inequality system $Ax \leqslant b$ as in Theorem 2.2 we could also analyze the result of the elimination procedure applied to a strict inequality system $Ax < b$. By carefully checking the steps of the proof of Theorem 2.2, one can easily see how a statement equivalent to "$\{x \in \mathbb{R}^n \mid Ax < b\}$ is nonempty" should look. In the following corollary we formulate this equivalence for the case of a mixed system of weak and strict inequalities.

Corollary 2.3. *Let A be an (m, n)-matrix and b an m-vector, and let $j \in \{1, \ldots, n\}$ be any column index. Let D be the (r, n)-matrix and d be the r-vector obtained by the Fourier–Motzkin elimination algorithm for this particular j. Furthermore, let $I, J \subset M$ with $I \cap J = \emptyset$, $I \cup J = M$ be a partition of the row index set M of A, and let $E \cup F = R$ be a partition of the row index set R of D defined as follows:*

$$E := p[(Z \cap I) \cup [(N \times P) \cap (I \times I)]], \quad F := R \setminus E.$$

Then the system

$$A_{I.}x \leqslant b_I,$$
$$A_{J.}x < b_J$$

has a solution $x \in \mathbb{R}^n$ if and only if the system

$$D_{E.}x \leqslant d_E,$$
$$D_{F.}x < d_F$$

has a solution $x \in \mathbb{R}^n$.

Remark 2.4. After successive elimination of all variables x_j $(j = 1, \ldots, n)$ the final zero-matrix D may have a tremendously large number of rows. In the worst case A has no zero element in the first column to be eliminated and $|P| = |N| = m/2$ holds, so using Algorithm 2.2 we obtain a matrix D with $m^2/4$ rows. Assuming that this worst case holds in any further elimination process, we finally obtain a matrix D (a vector d) with

$$m^{2^n}/2^{2^{n+1}-2}$$

rows (components). Unfortunately, an exponential growth like this quite often occurs making the Fourier–Motzkin elimination of questionable practical value.

A direct consequence of Theorem 2.2 obviously is

Corollary 2.5 (Farkas lemma). *The polyhedron $P(A, b)$ is empty if and only if there exists an $u \in \mathbb{R}^m_+$ with $u^T A = 0$ and $ub < 0$.*

We may also formulate Corollary 2.5 as a theorem of the alternative.

Corollary 2.6. *Either $Ax \leqslant b$ has a solution $x \in \mathbb{R}^n$ or there exists an $u \in \mathbb{R}^m_+$ with $u^T A = 0$ and $ub < 0$ but not both.*

Note that in the formulation of the Farkas Lemma in Corollary 2.5 the set whose nonemptiness is equivalent to the emptiness of $P(A, b)$ is not a polyhedron, however, it is easy to see that

$$\exists\, u \geqslant 0 \quad u^T A = 0 \text{ and } ub < 0$$

is equivalent to

$$\exists\, u \geqslant 0 \quad u^T A = 0 \text{ and } ub = -1,$$

i.e. the Farkas Lemma can also be stated in the following form:

Corollary 2.7. *Let A be an (m, n)-matrix and b an m-vector, then the following holds:*

$$\{x \in \mathbb{R}^n \mid Ax \leqslant b\} \neq \emptyset$$

or

$$\{u \in \mathbb{R}^m \mid u \geqslant 0, u^T A = 0;\ u^T b = -1\} \neq \emptyset$$

but not both.

Therefore, the Farkas Lemma can be viewed as a theorem relating the emptiness resp. nonemptiness of the polyhedra

$$P(A, b) \quad \text{and} \quad P^{=}\left(\begin{pmatrix} b^T \\ A^T \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

to each other.

2.1. Theorems of the alternative

One of the most important aspects of the Farkas Lemma is its form of an alternative: one statement holds or another but never both. This version has found many generalizations to various kinds of polyhedral and non-polyhedral sets. Most of these theorems of the alternative can however be shown to be quite easy consequences of the Farkas Lemma (resp. Corollary 2.3 for mixed systems) and vice versa. These theorems are very useful as technical tools e.g. for characterizing valid inequalities for polyhedra, proving strong complementary slackness theorems and the like. For a survey of such results see Mangasarian (1969). We only state four of these theorems which are of particular interest resp. are used later.

Theorem 2.8. *Exactly one of the two following statements holds:*

(2.8.1) $\exists\, x$ *with* $Ax \leqslant a,\ Bx < b$;

(2.8.2) (a) $\exists\, (u^{\mathrm{T}}, v^{\mathrm{T}}) \geqslant 0$ *with* $v \neq 0,\ u^{\mathrm{T}}A + v^{\mathrm{T}}B = 0,\ u^{\mathrm{T}}a + v^{\mathrm{T}}b \leqslant 0$;

(b) $\exists\, u \geqslant 0$ *with* $u^{\mathrm{T}}A = 0$ *and* $u^{\mathrm{T}}a < 0$.

Corollary 2.9. *Let* $P(A, a)$ *be a nonempty polyhedron, then exactly one of the two following statements holds:*

(2.9.1) $\exists\, x$ *with* $Ax \leqslant a,\ Bx < b$;

(2.9.2) $\exists\, (u^{\mathrm{T}}, v^{\mathrm{T}}) \geqslant 0$ *with* $v \neq 0,\ u^{\mathrm{T}}A + v^{\mathrm{T}}B = 0,\ u^{\mathrm{T}}a + v^{\mathrm{T}}b \leqslant 0$.

A version of the Farkas Lemma relating orthogonal subspaces of $\mathbb{R}^n$ to each other, which has particular applications in network flow theory (cf. Ford–Fulkerson (1962)) is the following.

Theorem 2.10. *Let* L *be a linear subspace of* $\mathbb{R}^n$ *and let* $I := \times_{i=1}^{n} I_i$ *be the cartesian product of* n *nonempty intervals* I_i *of* $\mathbb{R}$. *Then* $L \cap I \neq \emptyset$ *if and only if for each* $y \in L^{\perp}$ *(orthogonal complement of* L*) there exists an* $x \in I$ *with* $x^{\mathrm{T}}y = 0$.

This theorem yields as an immediate consequence:

Corollary 2.11. *Let* A *be an* (m, n)*-matrix and* $I = \times_{i=1}^{n} I_i$ *be the cartesian product of* n *nonempty intervals* I_i *of* $\mathbb{R}$. *Then exactly one of the following two alternatives holds:*

(2.11.1) $\exists\, x \in \mathbb{R}^n$ *with* $Ax \in I$;

(2.11.2) $\exists\, u \in \mathbb{R}^m$ *with* $u^{\mathrm{T}}A = 0,\ u^{\mathrm{T}}z < 0$ *for all* $z \in I$.

2.2. Transformation of polyhedra

Let $P = P(A, b)$ be a polyhedron in $\mathbb{R}^n$. For $k, r \in \mathbb{N}$ with $n = k + r$ and $\tilde{A}$ a matrix obtained from A by permuting some columns, the set

$$Q := \left\{x \in \mathbb{R}^k \mid \exists\, y \in \mathbb{R}^r \begin{pmatrix} x \\ y \end{pmatrix} \in P(\tilde{A}, b)\right\}$$

is called a *projection* of P (onto some of its coordinates). It is often useful to know that any projection of a polyhedron is again a polyhedron. This observation follows from:

Theorem 2.12. *Let $P = P(A, b) \subseteq \mathbb{R}^n$ be a polyhedron, D be a (k, n)-matrix, $d \in \mathbb{R}^k$ and $f: \mathbb{R}^n \to \mathbb{R}^k$ be the affine mapping defined by $f(x) = Dx + d$. Then the image $f(P)$ of P under f is a polyhedron.*

Corollary 2.13. *Any projection of a polyhedron $P(A, b)$ onto some of its coordinates is a polyhedron, i.e. for any $I \subseteq \{1, \ldots, n\}$*

$$P^I := \{x_I \in \mathbb{R}^{|I|} \mid x \in P(A, b)\}$$

is a polyhedron.

This corollary implies a result which is of particular interest.

Corollary 2.14. *The linear (affine, conic, convex) hull of finitely many vectors is a polyhedron, i.e. for any (m, n)-matrix A*

$$\left.\begin{array}{l} \text{lin}(A) \\ \text{aff}(A) \\ \text{cone}(A) \\ \text{conv}(A) \end{array}\right\} \text{ is a polyhedron.}$$

The fact that every finitely generated cone is a polyhedron was first observed and proved by Weyl (1935). This result is therefore often called 'Weyl's Theorem' in the literature. Appropriately applied, it has far reaching consequences. Corollary 2.13 also easily yields

Corollary 2.15. *The sum $P = P_1 + P_2$ of two polyhedra $P_1, P_2 \subseteq \mathbb{R}^n$ is also a polyhedron. In particular, if A is an (m, n)-matrix and B an (m, n')-*

matrix, then

$$\text{conv}(A) + \text{cone}(B)$$

is a polyhedron.

2.3. *The polar cone*

For any subset $S \subseteq \mathbb{R}$ the set $S^0 \subseteq \mathbb{R}^n$ defined by

$$S^0 := \{y \in \mathbb{R}^n \mid y^T x \leqslant 0 \ \forall\, x \in S\}$$

is called the *polar cone* of S, i.e. S^0 is the set of all $y \in \mathbb{R}^n$ which form an obtuse angle with all $x \in S$. It is very simple to see that S^0 is indeed a cone. For polyhedral cones the polar cone can be easily described:

Theorem 2.16. *Let $P(A, 0)$ be a polyhedral cone. Then*

$$P(A, 0)^0 = \text{cone}(A^T).$$

Proof. Note that $P(A, 0)$ is never empty. Now, $b \in P(A, 0)^0$ if and only if $Ax \leqslant 0$ implies $bx \leqslant 0$, or equivalently the system $Ax \leqslant 0$, $-bx < 0$ is inconsistent. By Corollary 2.9 the alternative statement (2.9.2) holds, i.e. there exist $u \geqslant 0$, $v > 0$ such that $u^T A - vb^T = 0$, or equivalently $v^{-1}A^T u = b$. Hence $b \in P(A, 0)^0$ is equivalent to $b \in \text{cone}(A^T)$.

Note that for a polyhedral cone $P(A, 0)$, the equality $P(A, 0) = \text{cone}(A^T)^0$ holds by definition. This observation and Theorem 2.16 imply

Corollary 2.17. *Let A be an (m, n)-matrix, then*

$$P(A, 0)^{00} = P(A, 0), \qquad \text{cone}(A)^{00} = \text{cone}(A).$$

The truth of the converse statement of the Theorem of Weyl was observed by Minkowski (and proved by Weyl):

Theorem 2.18 (Minkowski's Theorem). *Every polyhedral cone is finitely generated, i.e. for every (m, n)-matrix A there exists an (n, k)-matrix B with*

$$P(A, 0) = \text{cone}(B).$$

Proof. Using the Theorem of Weyl 2.14 and Theorem 2.16 we obtain

$$P(A, 0) = \text{cone}(A^{\text{T}})^0 = P(B^{\text{T}}, 0)^0 = \text{cone}(B).$$

We have seen in Corollary 2.15 that for finite sets V, E, the set $\text{conv}(V) + \text{cone}(E)$ is a polyhedron. Using homogenization techniques which we shall introduce in Section 6 one can easily derive from Minkowski's Theorem

Theorem 2.19. *Every polyhedron is finitely generated, i.e. for every polyhedron P there exist finite sets V, E such that*

$$P = \text{conv}(V) + \text{cone}(E).$$

2.4. Linear programming duality

We shall now derive the 'duality theorem of linear programming' from the Farkas Lemma and prove our claim of Section 1 that every bounded linear program has an optimal (feasible) solution. Consider the linear program

$$\begin{aligned} &\max c^{\text{T}}x, \\ &Ax \leqslant b \end{aligned} \tag{2.3}$$

where the set of feasible solutions is the polyhedron $P(A, b)$. As mentioned in (1.3) this program can be formulated in the form

$$\sup\{z \in \mathbb{R} \mid P_z \neq \emptyset\} \tag{2.4}$$

where P_z is the polyhedron

$$P\left(\begin{pmatrix} -c^{\text{T}} \\ A \end{pmatrix}, \begin{pmatrix} -z \\ b \end{pmatrix}\right) \subseteq \mathbb{R}^n.$$

Since we have called the nonemptiness-question of polyhedra (2.1) the primal problem, we call the linear program (2.3) resp. (2.4) the *primal program*, and similarly we call the problem

$$\inf\{z \in \mathbb{R} \mid P_z = \emptyset\} \tag{2.5}$$

the *dual program* of (2.4). Clearly.

$$\sup\{z \mid P_z \neq \emptyset\} = \inf\{z \mid P_z = \emptyset\}, \tag{2.6}$$

i.e. the primal and the dual program have the same optimal value. Moreover, if the polyhedron $P(A, b)$ is empty (i.e. $P_z = \emptyset$ for all $z \in \mathbb{R}$), then the dual program is clearly unbounded.

The dual program (2.5) is stated in a very uncommon fashion, but the Farkas Lemma can now be used to reformulate '$P_z = \emptyset$' in such a way that the dual program is seen to be a linear program. Namely, by Corollary 2.5 we have

$$P_z = \emptyset \;\Leftrightarrow\; (\exists\, u \geqslant 0, \alpha \geqslant 0 \text{ such that } u^{\mathrm{T}}A = \alpha c^{\mathrm{T}} \text{ and } u^{\mathrm{T}}b < \alpha z).$$

Thus, if $P(A, b)$ is nonempty (and hence no $u \geqslant 0$ with $u^{\mathrm{T}}A = 0$ and $u^{\mathrm{T}}b < 0$ exists), we may assume that $\alpha > 0$ or equivalently (by scaling) that $\alpha = 1$ holds for the right hand side of the above equivalence. Hence under the assumption $P(A, b) \neq \emptyset$

$$P_z = \emptyset \;\Leftrightarrow\; (\exists\, u \geqslant 0 \text{ such that } u^{\mathrm{T}}A = c^{\mathrm{T}} \text{ and } u^{\mathrm{T}}b < z).$$

And therefore the dual program (2.5) can be written as

$$\inf\{u^{\mathrm{T}}b \mid u^{\mathrm{T}}A = c^{\mathrm{T}}, u \geqslant 0\}. \tag{2.7}$$

Thus if $P(A, b) \neq \emptyset$ the equality (2.6) reads

$$\sup\{c^{\mathrm{T}}x \mid Ax \leqslant b\} = \inf\{u^{\mathrm{T}}b \mid u^{\mathrm{T}}A = c^{\mathrm{T}}, u \geqslant 0\}. \tag{2.8}$$

If both polyhedra $P(A, b)$ and $P^{=}(A^{\mathrm{T}}, c)$ are nonempty, then by (2.8) the linear and hence continuous functions $c^{\mathrm{T}}x$ resp. $u^{\mathrm{T}}b$ are bounded on $P(A, b)$ resp. $P^{=}(A, b)$ and (because of the completeness property of real numbers) therefore both attain their maximum resp. minimum. Moreover, (2.8) also yields that the primal and dual program have the same optimal value. Altogether we have shown

Theorem 2.20 (Duality Theorem). *Consider the primal linear program* (P) *and its dual linear program* (D), *where*

$$(\mathrm{P}) = \begin{cases} \max c^{\mathrm{T}}x, \\ Ax \leqslant b \end{cases} \quad \textit{and} \quad (\mathrm{D}) = \begin{cases} \min u^{\mathrm{T}}b, \\ u^{\mathrm{T}}A = c^{\mathrm{T}}, u \geqslant 0. \end{cases}$$

(a) *The following three statements are equivalent*:

(1) (P) *and* (D) *have optimal solutions* $\bar{x} \in \mathbb{R}^n$ *resp.* $\bar{u} \in \mathbb{R}^m$ *with* $c^{\mathrm{T}}\bar{x} = \bar{u}^{\mathrm{T}}b$.

(2) *One of the programs* (P) *or* (D) *has an optimal solution.*

(3) *Both the primal program* (P) *and the dual program* (D) *have a feasible solution.*

(b) *If the primal program is unbounded, then the dual program is infeasible.*

(c) *If the dual program is unbounded, then the primal program is infeasible.*

(d) *If the primal program is infeasible, then the dual program is either infeasible or unbounded.*

(e) *If the dual program is infeasible, then the primal program is either infeasible or unbounded.*

Note that the proof of the duality theorem we have just presented uses the completeness of the real numbers. The duality theorem itself however holds in a much more general setting.

3. Characterization of nonemptiness of polyhedra: The ellipsoid method

The Fourier–Motzkin elimination procedure discussed in Section 2 is an elegant algorithm to prove the emptiness resp. nonemptiness of polyhedra. It yields as a simple corollary the Farkas Lemma and has therefore far reaching theoretical consequences. Although finite, this method is not efficient in the sense of complexity theory, since it typically has exponential behaviour.

Recently, Khachian (1979) has shown that emptiness resp. nonemptiness of polyhedra can be checked in time which is bounded by a polynomial in the length of the data encoding. His proof is based on the ellipsoid method of Shor (1977), for detailed proofs cf. Gács–Lovász (1981), König–Pallaschke (1981), Padberg–Rao (1979 and 1980) and the bibliography Wolfe (1980).

We shall first state this algorithm and then discuss its behaviour, its underlying geometrical idea, and some of its consequences.

Algorithm 3.1. *Ellipsoid Method.*

Input. An integer (m, n)-matrix A, $b \in \mathbb{Z}^m$. Let L be the length of a binary encoding of A and b.

Output. The algorithm either finds a vector $x \in \mathbb{Q}^n$ with $Ax < b$ or terminates after N steps, where $N := (10n^2 + 5n)L + 5n^2$. In the second case, the set $\{x \in \mathbb{R}^n \mid Ax < b\}$ is empty.

Initialization: Set

$$x^0 := 0 \in \mathbb{R}^n, \qquad B^0 := 2^{2L} I_n$$

where I_n is the (n, n)-identity matrix.

FOR $k = 0$ TO N DO

(1) IF $Ax^k < b$ THEN STOP (a solution is found).

(2) Choose any $i \in \{1, 2, \ldots, m\}$ such that

$$A_{i.}x^k \geqslant b_i$$

where $A_{i.}$ is the i-th row vector of A.

(3) Calculate $\tilde{d} := \sqrt{A_{i.}B^k A_{i.}^{\mathrm{T}}}$ and set

$$d \approx \tilde{d}$$

where $\approx$ means that the left hand side is obtained by rounding the binary expansion of the right hand side after $15nL$ places behind the point.

(4)

$$x^{k+1} :\approx x^k - \frac{1}{d(n+1)} B^k A_{i.}^T,$$

$$B^{k+1} :\approx \frac{n^2}{n^2-1}\left(B^k - \frac{2}{d^2(n+1)}(B^k A_{i.}^T)(B^k A_{i.}^{\mathrm{T}})^{\mathrm{T}}\right).$$

END

Stated in the way above it is not at all clear what the ellipsoid method has to do with ellipsoids; the role or the number L needs to be discussed; and we have to explain why one can stop after the execution of N macro steps. The termination question is answered in

Theorem 3.2. *If the algorithm does not find a solution and terminates after N steps, then the set* $\{x \in \mathbb{R}^n \mid Ax < b\}$ *is empty.*

The proof of Theorem 3.2 is based on several nontrivial observations which we shall outline in the sequel. First of all, the number L comes in through a number-theoretic argument and an estimation via Cramer's rule, namely from the integrality of the data A and b one can conclude

Lemma 3.3. *If $P = \{x \in \mathbb{R}^n \mid Ax < b\}$ is nonempty, then P contains a simplex B of volume at least $2^{-(n+1)L}$ which itself is contained in the ball of radius 2^L around the origin.*

This lemma is fundamental for the proof of Theorem 3.2. To demonstrate this we now present the geometrical idea behind the ellipsoid method.

A sequence of ellipsoids E_k, $k = 0, 1, 2, \ldots$, is constructed which has the following properties:

(a) *If $P = \{x \in \mathbb{R}^n \mid Ax < b\}$ is nonempty, then the initial ellipsoid E_0 contains a simplex $B \subseteq P$ whose volume is at least $2^{-(n+1)L}$.*

(b) *If the center x^k of ellipsoid E_k, $k = 0, 1, 2, \ldots$, is not in P, then E_k is cut by a hyperplane through x^k into two pieces one of which contains the simplex B.*

(c) *A new ellipsoid E_{k+1} is constructed from E_k such that E_{k+1} contains that piece of E_k which contains B.*

(d) *The sequence $v(E_k)$ of volumes of the ellipsoids shrinks geometrically, i.e. the ratio of volumes satisfies*

$$\frac{v(E_{k+1})}{v(E_k)} < c^{-1/p(L)} < 1 \quad \text{for all } k = 0, 1, 2, \ldots .$$

where $c > 1$ and $p(\ )$ is a polynomial in the input length.

Clearly, if a method satisfies (a), …, (d), then there is an integer N which is polynomial in L (cf. the discussion after Lemma 3.4) such that the ellipsoid E_N has a volume which is at most $2^{-(n+1)L}$.

Now, if P is nonempty, then by Lemma 3.3, P contains a set B of volume at least $2^{-(n+1)L}$ which by construction is also contained in E_N. Since the volume of B is not smaller than that of E_N, B must equal E_N and hence, the center x^N of E_N must be feasible. Therefore, if x^N is not in P, we conclude that P is empty.

We still need to explain what the geometrical idea of shrinking ellipsoids has to do with the purely algebraic method described in Algorithm 3.1, i.e. we have to show in which of the formulas of Algorithm 3.1 the ellipsoids are hidden.

Ellipsoids with the desired properties can be obtained from the inverses of the matrices B^k defined in the ellipsoid method. The initial matrix $B^0 = 2^{2L} I_n$ is symmetric and positive definite; because of the update formula (4) all matrices B^k are easily seen to be symmetric, and one can also show that every such matrix is also positive definite. Therefore, the inverse matrices $(B^k)^{-1}$ are also positive definite, and hence the sets

$$E_k := \{x \in \mathbb{R}^n \mid (x - x^k)^T (B^k)^{-1} (x - x^k) \leqslant 1\} \tag{3.1}$$

are ellipsoids with centers x^k. The initial ellipsoid E_0 is by definition a ball of radius 2^L around the origin, thus by Lemma 3.3, E_0 contains a set B of feasible solutions with volume at least $2^{-(n+1)L}$, in case P is nonempty. I.e. condition (a) of our requirements for the method is satisfied.

In step (1) of the ellipsoid method 3.1 we test whether the center x^k of E_k is feasible. If not, we choose in step (2) a violated inequality $A_{i.}x \geqslant b_i$. Then the hyperplane

$$H_k := \{x \in \mathbb{R}^n \mid A_{i.}x = A_{i.}x^k\}$$

is used to cut E_k into two pieces. The set B of feasible solutions is clearly contained in

$$E'_k := E_k \cap \{x \in \mathbb{R}^n \mid A_{i.}x \leqslant A_{i.}x^k\}.$$

The update formulas in (3) and (4) are designed in such a way that the matrix B^{k+1} yields an ellipsoid E_{k+1} with center x^{k+1} such that $E'_k \subseteq E_{k+1}$ holds. The demonstration of this fact requires some nontrivial calculations. Having shown this we know that our method satisfies (b) and (c).

If we remove the rounding provisions in (3) and (4) of Algorithm 3.1 and work with perfect arithmetic, then the new ellipsoid E_{k+1} constructed from E_k via formulas (3) and (4) has an important property. Namely, among all ellipsoids which contain the (unique) point $z \in E_k$ with

$$A_{i.}z = \min\{A_{i.}x \mid x \in E_k\}$$

and the subellipsoid F_k of E_k defined by

$$F_k = \{x \in \mathbb{R}^n \mid A_{i.}x = A_{i.}x^k\}$$
$$\cap \{x \mid (x - x^k)^T (B^k)^{-1} (x - x^k) = 1\},$$

the ellipsoid E_{k+1} has minimal volume. This observation is useful to prove the crucial step for the convergence of the ellipsoid method which shows that Algorithm 3.1 satisfies requirement (d).

Lemma 3.4. *If $v(E_k)$ denotes the volume of the ellipsoid E_k defined in* (3.1), *then*

$$\frac{v(E_{k+1})}{v(E_k)} < e^{-1/5n} < 1$$

holds for $k = 0, 1, \ldots, N$.

The maximal number N of iterations of the ellipsoid method follows from the following estimations. The volume of the initial ball E_0 is given by

$$v(E_0) = \left(\pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1)\right) 2^{nL}$$

where $\Gamma(\cdot)$ denotes the gamma function. To show that we can stop after $N = (10n^2 + 5n)\,L + 5n^2$ steps of the ellipsoid method we have to show that the volume of the ellipsoid E_N is at most $2^{-(n+1)L}$. Because of Lemma 3.4 we get

$$v(E_N) < v(E_0)\, e^{-N/5n}.$$

By taking logarithms (basis 2) we obtain

$$\begin{aligned}\log\left(v(E_N)\right) &< \log\left(v(E_0)\right) - \frac{N}{5n}\log(e)\\ &\leqslant \tfrac{1}{2}n\log(\pi) - \log\left(\Gamma(\tfrac{1}{2}n + 1)\right) + nL - \frac{N}{5n}\\ &\leqslant n + nL - \left((2n + 1)\,L + n\right)\\ &= -(n + 1)\,L\end{aligned}$$

which gives the desired result.

Since every update of the matrices B^k and the vectors x^k can be done in polynomial time and since N, the maximum number of iterations, is also a polynomial in the length of the input, the ellipsoid method stops after a number of steps which is a polynomial in the length of the data encoding. To determine the maximal number N of steps and an estimate for the required precision we have used the number L denoting the length of a binary encoding of A and b. It has been shown that the ellipsoid method is polynomial in smaller numbers than L, e.g. Gács–Lovász (1981) base their analysis on

$$L_1 := \lceil \sum_{i=1}^{m} \left(\log(|b_i| + 1) + \sum_{j=1}^{n} \log(|a_{ij}| + 1) \right) + \log(nm) + 1 \rceil ,$$

while Padberg–Rao (1979) work with

L_2 := largest absolute value of the determinant of a nonsingular submatrix of (A, b).

Of course, depending on the number L' used as a lower bound for the input length and depending on the sharpness of the inequalities in the estimation formulas, the number N' of steps necessary to determine inconsistency and the required precision slightly vary. What can be shown in principle is the following:

There is a number L' which is not larger than the length of a binary encoding of A and b such that the following holds: If all calculations are performed in $O(nL')$ precision (i.e. in the binary expansion of any number we round after $O(nL')$ places behind the point), then at most $O(n^2L')$ iterations of the ellipsoid method suffice to determine whether $Ax < b$ is consistent or not.

We want to point out that the ellipsoid method (with slightly different parameters) also works if A and b have rational entries, cf. Grötschel–Lovász–Schrijver (1981). Since irrational numbers cannot be represented in a binary expansion of finite length, it makes no sense to speak of the input length of an irrational number. Therefore, the rounding and stopping rules given above cannot be applied to inequality systems with irrational entries. If we discard the rounding procedure, i.e. work with perfect precision, and run the ellipsoid method without upper bound N on the number of iterations, we obtain an iterative method to find a solution of $Ax < b$. Clearly, such a method is a theoretical algorithm only and cannot be implemented (exactly) on a computing device.

After Kachian's paper appeared (in particular the improved version of Gács–Lovász (1981)) a number of variants of the ellipsoid method have been suggested. Not surprisingly, various authors discovered the same modifications independently at the same time, and also not surprisingly, many of these improvements were already published in the Russian literature, e.g. Shor (1977), Shor–Gershovich (1979). These variations are mainly concerned with new update formulas which are numerically more stable or with the choice of different parameters which yield faster convergence in practice. However, the worst-case running time of $O(n^2L)$ could not be improved yet, and the numerical stability as well as the performance of the ellipsoid method in practical computations is presently not at all satisfactory. For further informations on modifications of the ellipsoid method see Goldfarb–Todd (1979), König–Pallaschke (1981), Grötschel–Lovász–Schrijver (1981), Schrader (1982).

The ellipsoid method as described in Algorithm 3.1 is defined for strict inequality systems $Ax < b$ only, i.e. does not apply to polyhedra $P(A, b)$. But there is a simple trick that yields the desired polynomial time characterization of emptiness resp. nonemptiness of polyhedra.

Theorem 3.5. *Let A be an integer (m, n)-matrix, $b \in Z^m$ and let L be the length of a binary encoding of A and b. Then $P(A, b)$ is nonempty if and only if*

$$\{x \in \mathbb{R}^n \mid Ax < b + 2^{-L}\mathbb{1}\}$$

is nonempty.

Note that the data of the set $Q = \{x \mid Ax < b + 2^{-L}\mathbb{1}\}$ can be encoded in such a way that the length of this encoding is at most $(m + 1)L$, i.e. the length of this encoding is polynomial in L. Therefore, by running the ellipsoid method for the set Q we can decide in time polynomial in L, whether or not Q is empty and thus whether or not $P(A, b)$ is empty.

There are various ways to apply the foregoing results to solve linear programs in polynomial time. Gács and Lovász (1981) suggest to use duality theory for solving the linear programming problem

$$\max c^{\mathrm{T}}x,$$

$$Ax \leqslant b, \qquad x \geqslant 0$$

by taking the dual linear program

$$\min b^{\mathrm{T}}u,$$

$$A^{\mathrm{T}}u \geqslant c, \qquad u \geqslant 0$$

and checking whether the primal and dual system together with the reversed weak duality inequality have a solution, i.e. whether or not the following system is consistent

$$c^{\mathrm{T}}x - b^{\mathrm{T}}u \geqslant 0,$$

$$Ax \leqslant b,$$

$$A^{\mathrm{T}}u \geqslant c,$$

$$x, u \geqslant 0.$$

If this system is consistent, then for every solution $(\bar{x}, \bar{u})$ the vector $\bar{x}$ is an optimal solution for the primal program and $\bar{u}$ is an optimal solution of the dual.

Grötschel–Lovász–Schrijver (1981) suggest the use of a *separation algorithm* to generalize the ellipsoid method to general convex programming problems. If $K \subseteq \mathbb{R}^n$ is a convex set, then a separation algorithm for K is an algorithm which for every $y \in \mathbb{R}^n$ concludes with one of the following: (i) asserting that $y \in K$ or (ii) finding a vector $d \in \mathbb{R}^n$ such that for every $x \in K$, $d^{\mathrm{T}}x \leqslant d^{\mathrm{T}}y$. Grötschel–Lovász–Schrijver (1981) show that if for a convex set K there is a polynomial separation algorithm, then the ellipsoid method can be modified in such a way that the optimization problem $\max\{c^{\mathrm{T}}x \mid x \in K\}$ can be solved in polynomial time. Since the separation problem can easily be solved for polyhedra $P(A, b)$, linear programs can be solved in polynomial time with this method.

Although these variations of the ellipsoid method result in a polynomial time algorithm for the solution of linear programming problems, it should be noted that computational experiences show that the simplex algorithm (which is not polynomial in theory but in most practical applications) is still by far superior with respect to numerical stability and actual computing times.

A very important consequence of the ellipsoid method is the fact that every problem or property which can be formulated as a linear program or which can be characterized by means of a nonemptiness question of a polyhedron can now also be solved in polynomial time. More precisely, if we restrict our attention to polyhedra, then the following statement holds.

Theorem 3.6. *Let π be a property defined for polyhedra and let $P(A, a)$ be a polyhedron. Suppose there exist matrices B, D and vectors b, d which can be constructed from A and a in polynomial time and suppose property π*

can be characterized with respect to $P(A, a)$ as follows:

$$P(A, a) \text{ has property } \pi \Leftrightarrow S = \{x \in \mathbb{R}^n \mid Bx \leqslant b,\, Dx < d\} \neq \emptyset.$$

Then there is a polynomial time algorithm to decide whether $P(A, a)$ has property π or not.

In the next paragraph we shall define several properties of polyhedra (e.g. validity of an inequality, dimension of a face, redundancy of a given inequality system) which in fact can be characterized as required in Theorem 3.6. We shall sometimes point out that a given characterization yields a polynomial time algorithm via Theorem 3.6, but the reader is also invited to check whether the various theorems presented here (or in other books about polyhedra) can be used in the way described above to obtain good algorithms to determine a given property.

4. Faces of polyhedra

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. An inequality $c^{\mathrm{T}}x \leqslant c_0$ is called *valid* with respect to P if $c^{\mathrm{T}}x \leqslant c_0$ holds for all $x \in P$. A subset $F \subseteq P$ is called a *face* of P if there exists a valid inequality $c^{\mathrm{T}}x \leqslant c_0$ with

$$F = P \cap \{x \in \mathbb{R}^n \mid c^{\mathrm{T}}x = c_0\}.$$

Since $P = \{x \in P \mid 0^{\mathrm{T}}x = 0\}$, the polyhedron P is a face of itself, and since $\emptyset = \{x \in P \mid 0^{\mathrm{T}}x = 1\}$, the empty set is a face of P, called the *empty face*. Obviously, a face of a polyhedron is itself a polyhedron. It is also clear from the definition that the intersection of any number of faces is itself a face, and evidently, P is the largest face of P while the empty face is the smallest face of P.

A face F of P is called *proper* if $F \neq P$, and a proper face which is nonempty is called *nontrivial*. A face G is called a *cover* of a face F if $F \subseteq G$, $F \neq G$ and if there is no proper face H of G which properly contains F. Two faces F, G of P are called *noncomparable* if neither $F \subseteq G$ nor $G \subseteq F$ holds.

There are several kinds of faces which are of special interest and therefore carry particular names. A nontrivial face of P which is not contained in any other proper face of P is called a *facet* of P. A maximal proper face of a facet of P is called a *subfacet* of P. Thus, if $F \subseteq G \subseteq P$, F is a subfacet and G a facet of P, then P covers G and G covers F. Note that a facet is by definition never empty but a subfacet may be empty. A face of dimension zero is called a *vertex*, and a face of dimension one an *edge*. For ease of notation we shall denote a vertex $\{x\}$ of P just by x. A polyhedron which has a vertex is called *pointed*. The edges of a pointed polyhedral cone are also called *extreme rays*. Every nonzero vector of an extreme ray is called *extreme vector*.

Faces are of particular importance in linear programming, namely, if a linear program max $c^{\mathrm{T}}x$, $Ax \leqslant b$ has an optimal solution, then the set of optimal solutions is a nonempty face of $P(A, b)$. This can be seen as follows. Let c_0 be the optimal value of the linear program, then $c^{\mathrm{T}}x \leqslant c_0$ is satisfied by all feasible solutions x, hence $c^{\mathrm{T}}x \leqslant c_0$ is valid with respect to $P(A, b)$. In addition, the set

$$F = \{x \in P(A, b) \mid c^{\mathrm{T}}x = c_0\}$$

is the nonempty set of optimal solutions, therefore by definition, F is a face of $P(A, b)$.

It is of course important to know whether a given inequality is valid with respect to some polyhedron. The following theorem characterizes 'validity' in case the polyhedron is given by some description.

Theorem 4.1. *Let* $P = P(A, b) = \mathrm{conv}(V) + \mathrm{cone}(E)$ *be a nonempty polyhedron and* $c^{\mathrm{T}}x \leqslant c_0$ *an inequality. Then the following conditions are equivalent:*

(1) $c^{\mathrm{T}}x \leqslant c_0$ *is valid with respect to* P.
(2) *There exists an* $u \geqslant 0$ *with* $u^{\mathrm{T}}A = c^{\mathrm{T}}$, $u^{\mathrm{T}}b \leqslant c_0$.
(3) $c^{\mathrm{T}}v \leqslant c_0$ *for all* $v \in V$ *and* $c^{\mathrm{T}}e \leqslant 0$ *for all* $e \in E$.

Our next goal is to characterize faces, facets and vertices in case a polyhedron is given by one of the standard descriptions. To shorten notation and obtain elegant formulations of the results, the use of the following mappings and other concepts proved to be helpful.

If $P = P(A, b) \subseteq \mathbb{R}^n$ is a polyhedron and $M = \{1, 2, \ldots, m\}$ is the row index set of A then we introduce the mapping

$$\begin{aligned} &\mathrm{eq}\colon 2^P \to 2^M \qquad (2^S \text{ denotes the power set of } S), \\ &P \supseteq F \mapsto \{i \in M \mid A_{i.}x = b_i \text{ for all } x \in F\} \subseteq M, \end{aligned} \tag{4.1}$$

thus eq(F) is the set of all row indices of A such that the corresponding inequalities are binding for F, eq(F) is called the *equality set of F*. The mapping

$$\begin{aligned} &\text{fa}: 2^M \to 2^P, \\ &M \supseteq I \mapsto \{x \in P \mid A_{i\cdot}x = b_i \text{ for all } i \in I\} \end{aligned} \tag{4.2}$$

associates with every set I of row indices of A the subset F of P such that all points in F satisfy the inequalities given by I with equality. Clearly, fa(I) is a face of P, called the *face defined by I*.

If the polyhedron P is given as $P = \text{conv}(V) + \text{cone}(E) \subseteq \mathbb{R}^n$ we can define similar mappings as follows: Given a vector $x \in \text{conv}(V) + \text{cone}(E)$, then we say that $u \in V$ *convexly supports* x with respect to (V, E) if x has a representation

$$x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$$

such that $\lambda_u > 0$, and we say that $f \in F$ *conically* supports x if x has a representation. $x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$ such that $\mu_f > 0$. We define for $F \subseteq P$

$$\begin{aligned} \text{ex}_V(F) &:= \{v \in V \mid v \text{ supports some vector } x \in F \\ &\qquad\qquad \text{convexly with respect to } (V, E)\}, \\ \text{ex}_E(F) &:= \{e \in E \mid e \text{ supports some vector } x \in F \\ &\qquad\qquad \text{conically with respect to } (V, E)\}, \end{aligned} \tag{4.3}$$

and combining these notions we define the mapping

$$\begin{aligned} &\text{ex}: 2^P \to 2^V \times 2^E, \\ &P \supseteq F \mapsto \text{ex}(F) := (\text{ex}_V(F), \text{ex}_E(F)) \subseteq (V, E). \end{aligned} \tag{4.4}$$

The set ex(F) is called the *extreme set* of F. One can show that, if F is a face, $F = \text{conv}(\text{ex}_V(F)) + \text{cone}(\text{ex}_E(F))$ holds. Note that for the empty face of P we have $\text{ex}_V(\emptyset) = \emptyset$, $\text{ex}_E(\emptyset) = \emptyset$, and that $F \subseteq P$ is empty if and only if $\text{ex}_V(F) = \emptyset$.

To define a mapping converse to ex we first set

$$\begin{aligned} &\text{gen}: 2^V \times 2^E \to 2^P, \\ &(V, E) \supseteq (S, T) \mapsto \text{conv}(S) + \text{cone}(T) \subseteq P. \end{aligned} \tag{4.5}$$

Note that gen(S, T) is in general not a face of P, but using the mappings ex and gen we can obtain the desired mapping as follows:

$$\begin{aligned} &\text{sp}: 2^V \times 2^E \to 2^P, \\ &(V, E) \supseteq (S, T) \mapsto \text{gen}\big(\text{ex}(\text{gen}(S, T))\big). \end{aligned} \tag{4.6}$$

One can show that for any set (S, T), sp(S, T) is a face of P called the span of (S, T) and sp(S, T) is the smallest face F of P such that $S \subseteq \text{ex}_V(F)$ and $T \subseteq \text{ex}_E(F)$ holds. Note that gen(S, T) is empty if and only if S is empty. Furthermore, $\text{ex}\big(\text{sp}(S, T)\big) = \text{ex}\big(\text{gen}(S, T)\big)$ holds.

To illustrate the concepts defined above we give an example.

Example 4.2. Consider the polyhedron $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ given by

$$A = \begin{pmatrix} A_{1\cdot} \\ A_{2\cdot} \\ A_{3\cdot} \\ A_{4\cdot} \\ A_{5\cdot} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ -3 & 4 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 6 \\ 12 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{aligned} &V = \{v_1, v_2, v_3, v_4\}, \\ &E = \{e\} \end{aligned}$$

with

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This polyhedron is shown in Fig. 4.1.

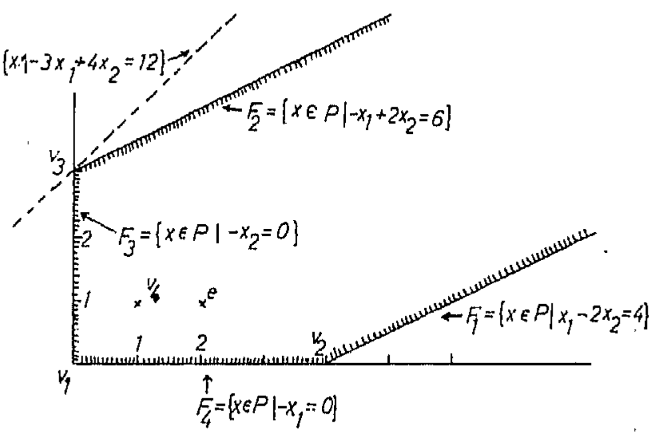

Fig. 4.1.

Note that both descriptions of P are redundant, namely the third inequality $A_{3.}x \leqslant b_3$, i.e. $-3x_1 + 4x_2 \leqslant 12$, can be removed without changing the polyhedron, and similarly the point v_4 is superfluous, i.e. for $V' = V \setminus \{v_4\}$ we still have $P = \text{conv}(V') + \text{cone}(E)$.

Examples of equality sets are:

$$\text{eq}(P) = \emptyset, \quad \text{eq}(F_1) = \{1\}, \quad \text{eq}(F_2) = \{2\}, \quad \text{eq}(F_3) = \{5\},$$
$$\text{eq}(F_4) = \{4\}, \quad \text{eq}(\{v_2\}) = \{1, 4\}, \quad \text{eq}(\{v_3\}) = \{2, 3, 5\},$$
$$\text{eq}(\{v_4\}) = \emptyset, \quad \text{eq}(\emptyset) = \{1, 2, 3, 4, 5\}.$$

Examples of faces defined by rows of the matrix A:

$$\text{fa}(\{1\}) = F_1, \qquad \text{fa}(\{1, 2\}) = \emptyset,$$
$$\text{fa}(\{2, 3, 5\}) = \text{fa}(\{2,5\}) = \text{fa}(\{3\}) = \{v_3\}.$$

Extreme sets of subsets of P are:

$$\text{ex}(F_1) = (\{v_2\}, \{e\}), \qquad \text{ex}(F_3) = (\{v_1, v_3\}, \emptyset).$$

Faces spanned by some vectors:

$$\text{sp}(\{v_1, v_2\}, \emptyset) = F_4, \qquad \text{sp}(\{v_1\}, \{e\}) = P,$$
$$\text{sp}(\{v_3\}, \{e\}) = F_2, \qquad \text{sp}(\{v_4\}, \emptyset) = P.$$

Let $S = \{x \in P \mid 3x_1 + 4x_2 = 12\}$, then the following holds:

$$\text{eq}(S) = \emptyset, \quad \text{ex}(S) = (V, E), \quad S = \text{gen}(\{v_2, v_3\}, \emptyset).$$

The mappings eq, fa, ex, gen, sp can be used to characterize faces in various ways.

Theorem 4.3. *Let $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ be a polyhedron and F a nonempty subset of P. Then the following conditions are equivalent:*

(1) *F is a face of P.*

(2) *There is a subset $I \subseteq \{1, 2, \ldots, m\}$ such that*

$$F = \text{fa}(I) = \{x \in P \mid A_{I.}x = b_I\}.$$

(3) $F = \text{fa}(\text{eq}(F))$.

(4) *There is a subset $(S, T) \subseteq (V, E)$, $S \neq \emptyset$ with $F = \text{sp}(S, T)$.*

(5) $F = \text{gen}(\text{ex}(F))$.

(6) $F = \text{sp}(\text{ex}(F))$.

Corollary 4.4. *Let* $c^{\mathrm{T}}x \leqslant c_0$ *be a valid inequality of the polyhedron* $P = \mathrm{conv}(V) + \mathrm{cone}(E)$, *and* $F = \{x \in P \mid c^{\mathrm{T}}x = c_0\}$. *Define the sets*

$$S := \{v \in V \mid c^{\mathrm{T}}v = c_0\}, \qquad T := \{e \in E \mid c^{\mathrm{T}}e = 0\}.$$

Then for the face F *the following holds*:
$F = \mathrm{gen}(S, T) = \mathrm{conv}(S) + \mathrm{cone}(T)$;
if $F \neq \emptyset$ *then* $\mathrm{ex}(F) = (S, T)$;
if $F = \emptyset$ *then* $\mathrm{ex}(F) = (\emptyset, \emptyset)$.

Note that condition (2) of Theorem 4.3 states that we do not need some unknown inequalities $c^{\mathrm{T}}x \leqslant c_0$ to get a face of a polyhedron P. All faces of P can be obtained by setting some of the inequalities $A_{i.}x \leqslant b_i$, $i \in \{1, \ldots, m\}$ to equality. This in particular implies that every polyhedron has a finite number of faces only. This observation also follows from condition (4), since there are only finitely many different sets $(S, T) \subseteq (V, E)$ and all faces are spanned by subsets of (V, E).

To streamline technical arguments in proofs the concept of interior points is very useful. Given a polyhedron P, then $x \in P$ is called an *interior point* of P if y is not contained in any proper face of P. (Note that occasionally (e.g. in topology) an interior point of P is called a relative interior point.)

Proposition 4.5. *Let* F *be a face of a polyhedron* $P = P(A, b) = \mathrm{conv}(V) + \mathrm{cone}(E)$ *and* $\bar{x} \in P$. *Then the following conditions are equivalent*:

(1) $\bar{x}$ *is an interior point of* F.

(2) $\mathrm{eq}(\{\bar{x}\}) = \mathrm{eq}(F)$.

(3) $\mathrm{ex}(\{\bar{x}\}) = \mathrm{ex}(F)$.

(4) *Let* $(S, T) := \mathrm{ex}(F)$, *then there are strictly positive* $\lambda \in \mathbb{R}^{|S|}$, $\mu \in \mathbb{R}^{|T|}$ *with* $\sum_{s \in S} \lambda_s = 1$, *such that*

$$\bar{x} = \sum_{s \in S} \lambda_s s + \sum_{t \in T} \mu_t t.$$

(5) *Let* $I = \mathrm{eq}(F)$ *and* $K = \{1, 2, \ldots, m\} \setminus I$, *then* $A_{I.}\bar{x} = b_I$ *and* $A_{K.}\bar{x} < b_K$.

Thus, Proposition 4.5 states that the equality set and extreme set of a face F are completely determined by any interior point of F.

The equality sets and extreme sets of faces can also be utilized to compute the dimension of faces.

Theorem 4.6. *Let F be a nonempty face of a polyhedron $P = P(A, b) =$ $\mathrm{conv}(V) + \mathrm{cone}(E)$, then the following holds*:

(a) *If $I = \mathrm{eq}(F)$, then $\dim(F) = n - \mathrm{rank}(A_{I.})$.*

(b) *If $(S, T) = \mathrm{ex}(F)$, then $\dim(F) = \mathrm{arank}(S \cup (S + T)) - 1$.*

The next theorem characterizes the facets of a polyhedron.

Theorem 4.7. *Let $P = P(A, b)$ be a polyhedron and F a nonempty face of P. Then the following conditions are equivalent*:

(1) *F is a facet of P.*

(2) *F is a maximal proper face of P.*

(3) *P covers F.*

(4) $\dim(F) = \dim(P) - 1$.

(5) *F contains exactly $\dim(P)$ affinely independent vectors.*

(6) *There exists a valid inequality $c^{\mathrm{T}}x \leqslant c_0$ with respect to P with the following properties*:

(a) $F = \{x \in P \mid c^{\mathrm{T}}x = c_0\}$.

(b) *There exists $\bar{x} \in P$ with $c^{\mathrm{T}}\bar{x} < c_0$.*

(c) *If any other valid inequality $d^{\mathrm{T}}x \leqslant d_0$ satisfies $F = \{x \in P \mid d^{\mathrm{T}}x = d_0\}$, then there are a vector $u \geqslant 0$ and a scalar $\alpha > 0$ with*

$$d^{\mathrm{T}} = \alpha c^{\mathrm{T}} + u^{\mathrm{T}}A_{\mathrm{eq}(P).}, \qquad d_0^{\mathrm{T}} = \alpha c_0 + u^{\mathrm{T}}A_{\mathrm{eq}(P)}.$$

Further interesting properties of facets are given in

Theorem 4.8. *Let $P = P(A, b)$ be a nonempty polyhedron and denote by $\mathrm{FA}(P)$ the set of all facets of P. Then*

(a) *For all $F_1, F_2 \in \mathrm{FA}(P)$, $F_1 \neq F_2$ we have*

$$\mathrm{eq}(F_1) \cap \mathrm{eq}(F_2) = \mathrm{eq}(P), \qquad \mathrm{ex}(F_1 \cup F_2) = \mathrm{ex}(P).$$

(b) $|\mathrm{FA}(P)| \leqslant m - |\mathrm{eq}(P)|$, *i.e. the number of facets of P is not greater than the number of inequalities of $Ax \leqslant b$.*

(c) *If $\mathrm{FA}(P) \neq \emptyset$ then there exists a row index set I with the properties*

(c_1) $I \subseteq \{1, 2, \ldots, m\} \setminus \mathrm{eq}(P)$,

(c_2) $|I| = |\mathrm{FA}(P)|$,

(c_3) *F is a facet of P if and only if $F = \mathrm{fa}(\{i\})$ for some $i \in I$.*

A row index set $I \subseteq M$ which satisfies all conditions (c_1), (c_2), (c_3) of Theorem 4.8 is called a *facet index set.*

One of the most important properties of facets is that any other face of P can be obtained as the intersection of facets of P, more precisely

Theorem 4.9. *Let P be a nonempty polyhedron.*

(a) *Every nonempty face of P is the intersection of facets of P.*

(b) *The empty face is the intersection of facets of P if and only if it is the intersection of nontrivial faces of P.*

(c) *If a subfacet of P is the intersection of facets of P, then it is the intersection of exactly two facets of P.*

Note that an empty subfacet is not necessarily the intersection of facets, consider e.g. a nonempty polyhedron $P = \{x \in \mathbb{R}^n \mid a^T x \leqslant b\}$, then the only nontrivial face of P is the facet $\{x \mid a^T x = a_0\}$. Thus the empty face is a subfacet of P but it is not the intersection of facets. In case $n \geqslant 2$, P is not pointed, i.e. polyhedra do not necessarily have vertices. But if a polyhedron is pointed, the next theorem shows how its vertices can be characterized.

Theorem 4.10. *If $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ is a polyhedron and $x \in P$ then the following conditions are equivalent:*

(1) *x is a vertex of P.*

(2) $\text{rank}(A_{\text{eq}(\{x\}).}) = n$.

(3) *x is not a proper convex combination of points of P.*

(4) $\text{ex}(\{x\}) = (\{x\}, \emptyset)$.

Suppose a polyhedron has the particular representation $P^{=}(A, b) = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$. If $P^{=}(A, b) \neq \emptyset$, then one can show that $P^{=}(A, b)$ is pointed and the vertices can be characterized as follows:

Theorem 4.11. *Let x be a point of a polyhedron $P = P^{=}(A, b)$ and*

$$J := \{j \in \{1, 2, \ldots, n\} \mid x_j \neq 0\}.$$

Then x is a vertex of P if and only if $\text{rank}(A_{.J}) = |J|$. *In other words, x is a vertex of P if and only if the column vectors $A_{.j}$, $j \in J$, are linearly independent.*

The simplex algorithm for solving linear programs (cf. Dantzig (1962)) is based on Theorem 4.11.

Assume that A is an (m, n)-matrix with full row rank and call an (m, m)-submatrix $A_{.J}$ of A a *basis* of A if $A_{.J}$ is nonsingular. If $A_{.J}$ is a basis and $\bar{x}_J := A_{.J}^{-1}b \geqslant 0$, then the vector $x \in \mathbb{R}^n$ with $x_j = \bar{x}_j$, $j \in J$, and $x_j = 0$ otherwise, is a vertex of $P^{=}(A, b)$, since the columns of A corresponding to nonzero components of x are linearly independent. On the other hand, if x is a vertex, then due to a result of linear algebra one can add to the columns of A, corresponding to positive components of x, further columns of A such that the resulting (m, m)-matrix $A_{.J}$ is nonsingular and $x_J = A_{.J}^{-1}b$ holds. This correspondence between vertices of $P^{=}(A, b)$ and bases of A is utilized in the Simplex algorithm to move from one vertex of $P^{=}(A, b)$ to another by computing a new basis of A from a given one.

We shortly summarize the computational aspects of some of the theorems of this section by applying Theorem 3.6. Theorem 4.1 implies that the validity of an inequality $cx \leqslant c_0$ with respect to $P(A, b)$ can be checked in polynomial time. Clearly, if P is given by $P = \text{conv}(V) + \text{cone}(E)$ then the validity of an inequality can also be checked in polynomial time in any encoding of V and E. We want to point out that the equality set and the extreme set of a face can be determined in polynomial time by giving an appropriate characterization as required in Theorem 3.6, cf. Bachem–Grötschel (1981). Since $\dim(F) = n - \text{rank}(A_{\text{eq}(F).})$ holds for a face F and since rank calculation is polynomially solvable, the dimension of a face can be computed in polynomial time by Theorem 4.3. Thus by Theorem 4.7 we can decide in polynomial time whether a given inequality defines a facet. By Theorem 4.5 we can check in polynomial time whether a point is an interior point of a given face and by Theorem 4.10 whether a given point is a vertex of a polyhedron.

5. Minimal representations of polyhedra

If $P = P(A, b)$ is a polyhedron and $c^{\mathrm{T}}x \leqslant c_0$ is a valid inequality for P, then $\{x \mid Ax \leqslant b, c^{\mathrm{T}}x \leqslant c_0\}$ is also a representation of P. Thus, a polyhedron has many representations in terms of the intersection of halfspaces. Similarly, a polyhedron P also has many representations of the form $P = \text{conv}(V) + \text{cone}(E)$. In the first part of this paragraph we shall study how a given representation $Ax \leqslant b$ can be reduced to a minimal one, in the second we shall show how a representation of the form $P = \text{conv}(V) + \text{cone}(E)$ can be made minimal.

Let $M := \{1, 2, \ldots, m\}$ and $J \subseteq M$. The partial inequality system $A_{J.}x \leqslant b_J$ is called *inessential* for $Ax \leqslant b$ if

$$P(A, b) = P(A_{M\setminus J,\cdot}, b_{M\setminus J}).$$

If the system $Ax \leqslant b$ has inessential inequalities it is called *redundant*, otherwise *irredundant*, see Example 4.2. An inequality $A_{i.}x \leqslant b_i$ is called *essential* if

$$P(A, b) \neq P(A_{M\setminus\{i\},\cdot}, b_{M\setminus\{i\}}),$$

thus a system $Ax \leqslant b$ is irredundant if and only if every inequality $A_{i.}x \leqslant b_i$ is essential.

With the help of the Farkas Lemma inessential inequalities can be easily characterized.

Proposition 5.1. *Let $P(A, b)$ be a nonempty polyhedron and let $i \in \{1, 2, \ldots, m\}$. Then the inequality $A_{i.}x \leqslant b_i$ is inessential for $P(A, b)$ if and only if there exists $u \geqslant 0$ with*

$$A_{i.} = u^{\mathrm{T}}A, \quad u^{\mathrm{T}}b \leqslant b_i \quad \text{and} \quad u_i = 0.$$

Note that 'essential' is defined relative to a system $Ax \leqslant b$ and that the removal of all inessential inequalities at the same time may result in a polyhedron larger than $P(A, b)$. To obtain an irredundant system for $P = P(A, b)$ we have to proceed as follows:

If there is no inessential inequality with respect to the present system, then STOP. *Otherwise determine an inessential inequality, remove it from the present system and continue.*

Note that by Proposition 5.1 an inequality $A_{i.}x \leqslant b_i$ is inessential if and only if the polyhedron

$$\{u \in \mathbb{R}^m \mid u \geqslant 0, u_i = 0, u^{\mathrm{T}}A = A_{i.}, u^{\mathrm{T}}b \leqslant b_i\}$$

is nonempty. With the ellipsoid method we can check the nonemptiness of this polyhedron in polynomial time, thus with the algorithm sketched above we can construct an irredundant system for P in polynomial time, since the ellipsoid method has to be applied at most m times.

The following theorems give necessary and sufficient conditions for a linear inequality system $Ax \leqslant b$ to be irredundant.

Theorem 5.2. *Let* $P = P(A, b)$ *be a nonempty polyhedron and* $M = \{1, 2, \ldots, m\}$. *Let* $J \subseteq M$ *with* $\text{eq}(P) \subseteq J$ *and* $P' := P(A_{J\cdot}, b_J)$. *Then* $P \neq P'$ *if and only if there exists a proper face* F *of* P *with* $\text{eq}_P(F) \nsubseteq J$.

Theorem 5.3. *Let* $P = P(A, b)$ *be a nonempty polyhedron, and* $I \subseteq \{1, 2, \ldots, m\} \setminus \text{eq}(P)$, $J \subseteq \text{eq}(P)$ *such that*

$$P = \{x \in \mathbb{R}^n \mid A_{I\cdot}x \leqslant b_I, A_{J\cdot}x = b_J\}.$$

Then this representation of P *is irredundant if and only if*

(a) I *is a facet index set of* P, *and*

(b) $A_{J\cdot}$ *has full row rank.*

Theorem 2.19 states that every polyhedron can be generated as the convex and the conical hull of finitely many vectors. If $P = \text{conv}(V) + \text{cone}(E)$, then (V, E) is called a *generating system* of P. A generating system (V, E) of P is called *minimal*, if $P \neq \text{conv}(V') + \text{cone}(E')$ for all proper subsets $(V', E') \subseteq (V, E)$, and a minimal generating system (V, E) is called a *basis* of P if it is of minimum cardinality, i.e. if $|V| + |E|$ is as small as possible. Every polyhedron clearly has a basis. Note that if (V, E) is a generating system of a polyhedron P, then there exists a subset of (V, E) which is a minimal generating system of P, however (V, E) does not necessarily contain a basis of P. Consider e.g. the polyhedron $P = \mathbb{R}^2$ with the generating system

$$V = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, \qquad E = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}.$$

This system (V, E) is minimal but not a basis of P, since (V, F) is a basis of P where

$$F = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}.$$

This misbehaviour does not occur if P is a pointed polyhedron.

Theorem 5.4. *Let* $P = \text{conv}(V) + \text{cone}(E)$ *be a pointed polyhedron and* $(S, T) \subseteq (V, E)$. *Then* (S, T) *is a basis of* P *if and only if the following holds:*

(a) S *is the set of vertices of* P.

(b) *The set of extreme rays of* $\text{rec}(P)$ *is* $R = \{\text{cone}(\{e\}) \mid e \in T\}$ *and* R *has cardinality* $|T|$.

In case P is a nonempty polytope and V is the set of vertices then $P = \text{conv}(V)$ holds. Since the recession cone of a polytope is $\{0\}$, Theorem 5.4 implies that $(V, \emptyset)$ is the unique basis of P. If P is a pointed polyhedral cone then P may have different bases but these bases are unique up to multiplication with a constant, namely, if (V, E) and (W, F) are two bases of a pointed polyhedral cone then $V = W = \{0\}$, $|E| = |F|$ and for every $e \in E$ there exists a unique $f \in F$ such that $e = \lambda f$ for some $\lambda > 0$. As the recession cone of a pointed polyhedron is also pointed, every pointed polyhedron has a unique basis (V, E) where E is unique up to multiplication by scalars.

We want to point out that Theorem 5.4 can be generalized to nonpointed polyhedra by introducing the concepts of pseudo vertices and pseudo rays. A *pseudo vertex* is a face which is a translate of the lineality space of P, and a *pseudo ray* of a polyhedral cone P is a face which is of the form $\text{cone}(\{z\}) + \text{lineal}(P)$ for some $z \neq 0$.

Note that if (V, E) is a rational generating system of a polyhedron P, then one can design an algorithm which repeatedly calls the ellipsoid method, runs in time polynomial in an encoding of V and E, and finds a minimal generating system (S, T) of P with $(S, T) \subseteq (V, E)$. In case P is pointed, the minimal generating system (S, T) will be a basis of P, cf. Bachem-Grötschel (1981).

6. Homogenization

Since in this survey we are not forced to prove all theorems we state, we have often grouped results in a way which would be different in case we would have given proofs of all statements and would have tried to streamline the proofs in an economic way. Many of the theorems stated in the preceding paragraphs for general polyhedra are quite hard to prove directly, while the corresponding results for polyhedral cones are often surprisingly simple to show. It would therefore be nice to have an apparatus which reduces the theory of polyhedra to the theory of polyhedral cones, i.e. a general technique which allows to deduce a statement on polyhedra from a result about polyhedral cones in a very simple way.

For this reason we introduce a proof technique which we call τ-homogenization (resp. τ-dehomogenization) and which adjoins to each polyhedron $P \subseteq \mathbb{R}^n$ a polyhedral cone called $\tau\text{-hog}(P) \subseteq \mathbb{R}^{n+1}$ such that P

is exactly the intersection of τ-hog(P) with a hyperplane 'parallel to $\mathbb{R}^n$' at level τ. We shall see that almost all informations about P are still contained in τ-hog(P) and that we can deduce results about P from results about τ-hog(P) by dehomogenization.

As an application we shall characterize various polarity relations by means of homogenization and dehomogenization. This proves, cf. Section 7, that polarity theory is essentially a duality theory. In Section 8 we shall introduce various lattices associated with polyhedra and we shall show that the face lattice of a polyhedron and the face lattice of the corresponding τ-homogenization are very close relatives.

Before we state a formal definition of homogenization let us discuss what we are looking for. Let $\tau \in \{-1, 1\}$, then we want the τ-homogenization of a polyhedron $P \subseteq \mathbb{R}^n$ to be a cone $C \subseteq \mathbb{R}^{n+1}$ with the property that P, more correctly

$$P_\tau = \left\{\begin{pmatrix} x \\ \tau \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in P\right\},$$

is exactly the intersection of C with the hyperplane $\{\binom{x}{z} \in \mathbb{R}^{n+1} \mid z = \tau\}$, cf. Fig. 6.1.

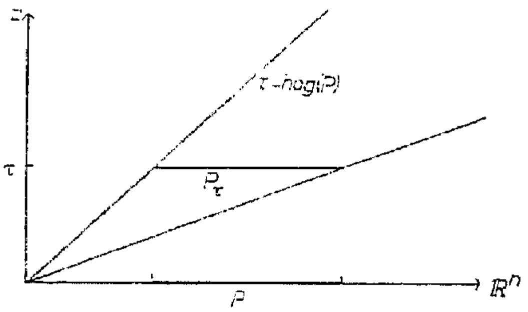

Fig. 6.1.

Intuitively, τ-hog(P) should be the intersection of all halfspaces $\{x \mid c^{\mathrm{T}}x \leqslant 0\}$ defined by inequalities $c^{\mathrm{T}}x \leqslant c_0$ which are valid for the polyhedron $P_\tau = \{\binom{x}{\tau} \mid x \in P\}$. By the definition of cone polarity, cf. Section 2, $c^{\mathrm{T}}x \leqslant 0$ is a valid inequality for P_τ if and only if $c \in (P_\tau)^0$. Thus τ-hog(P) should be the set

$$C := \bigcap_{c \in (P_\tau)^0} \{y \mid cy \leqslant 0\}.$$

One easily verifies that $C = (P_\tau)^{00}$ holds. Therefore, we now formally define the notions:

Definition 6.1. Let $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^{n+1}$ and $\tau \in \{-1, 0, 1\}$, then we call the set

$$\tau\text{-hog}(S) := \left\{ \begin{pmatrix} x \\ \tau \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in S \right\}^{00}$$

the *τ-homogenization* of S. The set

$$\tau\text{-dhog}(T) := \left\{ x \in \mathbb{R}^n \mid \begin{pmatrix} x \\ \tau \end{pmatrix} \in T \right\}$$

is called the *τ-dehomogenization* of T.

The idea of homogenization is very natural and has been implicitly considered in many papers. It was known to Minkowski and for instance employed by Goldman (1956), but we could not find out the first explicit use of it. To our knowledge Stoer–Witzgall (1970) were the first who developed homogenization techniques in a broader sense. We found some extensions of their notions very useful.

For sets $S, S_1, S_2 \subseteq \mathbb{R}^n$, $T, T_1, T_2 \subseteq \mathbb{R}^{n+1}$ and $\tau \in \{-1, 0, 1\}$, $\delta \in \{-1, 1\}$ the following calculation rules are obvious:

(a) *If* $S_1 \subseteq S_2$, *then* $\tau\text{-hog}(S_1) \subseteq \tau\text{-hog}(S_2)$.
(b) $\tau\text{-dhog}(T_1 \cap T_2) = \tau\text{-dhog}(T_1) \cap \tau\text{-dhog}(T_2)$.
(c) $\delta(\tau\text{-hog}(S)) = (\delta\tau)\text{-hog}(\delta S)$.
(d) $\delta(\tau\text{-dhog}(T)) = (\delta\tau)\text{-dhog}(\delta T)$.

Note that τ-hog and τ-dhog could have been defined for any $\tau \in \mathbb{R}$, but a moment's reflection shows that the cases $\tau \in \{-1, 0, 1\}$ are the essential ones and all other τ-homogenizations (τ-dehomogenizations) can be obtained from the above given ones by simple scaling.

A frequently used object in polyhedral theory is the so-called γ-polar of a polyhedron. For general sets $S \subseteq \mathbb{R}^n$ the *γ-polar* is defined as follows:

$$S^\gamma = \left\{ \begin{pmatrix} c \\ c_0 \end{pmatrix} \in \mathbb{R}^{n+1} \mid cx \leqslant c_0 \quad \forall\, x \in S \right\}.$$

Thus, one can consider S^γ as the set of all valid inequalities with respect to S, more correctly, the set of all vectors $\binom{c}{c_0}$ such that $cx \leqslant c_0$ is valid for S. The next observation shows that the γ-polar can be obtained by means of homogenization using cone polarity.

Theorem 6.2. *Let $S \subseteq \mathbb{R}^n$ be any set, then*

$$S^{\gamma} = \big((-1)\text{-hog}(S)\big)^{0}.$$

Corollary 6.3. *Let $P = P(A, b) = \text{cone}(V) + \text{cone}(E)$ be a nonempty polyhedron, then*

$$P^{\gamma} = \text{cone}\left(\begin{pmatrix} A^{T}, 0 \\ b^{T}, 1 \end{pmatrix}\right) = P\left(\begin{pmatrix} V^{T}, 1 \\ E^{T}, 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

Corollary 6.3 is extremely important, namely, it can be used as a vehicle to transform description-dependent results for polyhedra from one description into the other. E.g. consider Theorem 5.3 which gives a characterization of a minimal representation of a polyhedron as the intersection of halfspaces. Using the γ-polar, Theorem 5.4 which characterizes minimal generating systems can be easily derived from Theorem 5.3 via Corollary 6.3. Thus, in order to obtain description-dependent results for polyhedra one usually has to prove one such theorem only (e.g. for the description $P = P(A, b)$). The other description-dependent result (e.g. for $P = \text{cone}(V) + \text{cone}(E)$) is then an easy consequence by employing P^{γ} and Corollary 6.3.

The Farkas Lemma can be used to show that the τ-homogenization of a polyhedron is a polyhedral cone, moreover, given a description of a polyhedron one can also give a description of its τ-homogenization, namely:

Theorem 6.4. *Let $P = P(A, b) = \text{conv}(V) + \text{cone}(E)$ be a nonempty polyhedron and let $\tau \in \{-1, 1\}$ Then*

$$\begin{aligned}
\tau\text{-hog}(P) &= \left\{\begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{n+1} \mid Ax - \tau bz \leqslant 0, \tau z \geqslant 0\right\} \\
&= P(B_\tau, 0), \quad \textit{where } B_\tau := \begin{pmatrix} A, \tau b \\ 0, -\tau \end{pmatrix} \\
&= \text{cone}\left(\left\{\begin{pmatrix} v \\ \tau \end{pmatrix} \mid v \in V\right\} + \left\{\begin{pmatrix} e \\ 0 \end{pmatrix} \mid e \in E\right\}\right) \\
&= \text{cone}(D_\tau), \quad \textit{where } D_\tau = \begin{pmatrix} V, E \\ \tau \mathbb{1}^{T}, 0^{T} \end{pmatrix}.
\end{aligned}$$

The next result shows how information about P can be derived from information about τ-hog(P).

Theorem 6.5. *Let P be a nonempty polyhedron and $\tau \in \{-1, 1\}$, then*

$$P = \tau\text{-dhog}(\tau\text{-hog}(P)) \quad \textit{and} \quad \text{rec}(P) = 0\text{-dhog}(\tau\text{-hog}(P)).$$

Theorem 6.5 shows that τ-hog(P) in a sense contains both P and its recession cone rec(P). It is also possible to describe the faces of τ-hog(P) which correspond to faces of P, namely these are exactly those faces which have a nonempty intersection with $\{\binom{x}{z} \in \mathbb{R}^{n+1} \mid z = \tau\}$. A complete analysis of this correspondence will be given in Section 8.

7. Polarization

Recall that the polar cone S^0 of a set $S \subseteq \mathbb{R}^n$ is the set

$$S^0 := \{y \in \mathbb{R}^n \mid y^T x \leqslant 0 \quad \forall\, x \in S\},$$

while the γ-polar S^γ of S is

$$S^\gamma = \left\{\begin{pmatrix} c \\ c_0 \end{pmatrix} \in \mathbb{R}^{n+1} \mid c^T x \leqslant c_0 \quad \forall\, x \in S\right\}.$$

In a sense the polar cone S^0 represents all valid inequalities for S with right hand side zero while S^γ represents all valid inequalities. Theorem 2.16 shows that the concept of a polar cone is extremely useful for polyhedral cones, but for general polyhedra the restriction to a right hand side of zero limits its applicability. The γ-polar has the nice feature of making a transformation from one description of a polyhedron to another possible, but has the disadvantage that S^γ is in a space of higher dimension than S. We shall therefore study in this section polarity relations which overcome some of these disadvantages and provide further insights into the structure of polyhedra.

Definition 7.1. Let $S \subseteq \mathbb{R}^n$, $\alpha \in \{-1, 0, 1\}$, $\beta \in \{-1, 1\}$, then

$$S^{\alpha,\beta} := \{y \in \mathbb{R}^n \mid \beta y^T x \geqslant \alpha\beta, \ \forall x \in S\}$$

is called the *(α, β)-polar* of S.

We could of course have defined (α, β)-polars for all $\alpha \in \mathbb{R}$ (β only takes care of the direction of the inequality), but if $\alpha \neq 0$ then $S^{\alpha,\beta} = |\alpha|\, S^{\delta,\beta}$ for $\delta := \alpha/|\alpha| \in \{-1, 1\}$, thus up to scaling our definition captures all essential cases. Note also that $S^{0,-1}$ is nothing but the polar cone S^0 and that $S^{0,1}$ equals $-S^0$.

Polarity also is an old tool of mathematics, Minkowski was probably the first to use cone polarity in polyhedral theory. Later Fenchel (1951) recognized that polarity is a very helpful tool in duality theory. What we call (1, −1)-polar is the 'polar' usually considered in the literature and which is denoted by S^*, i.e.

$$S^* = S^{1,-1} = \{y \in \mathbb{R}^n \mid y^T x \leqslant 1 \quad \forall x \in S\}.$$

Some basic results concerning the polar P^* of a polyhedron P can be found in Grünbaum (1967) and Stoer–Witzgall (1970). More general polarity relations were studied by Araoz (1973), Balas (1974) and Griffin (1977).

The next theorem shows that the (α, β)-polar, $\alpha \neq 0$, can be computed with the help of the polar cone, homogenization and dehomogenization.

Theorem 7.2. *Let* $S \subseteq \mathbb{R}^n$, $\alpha, \beta, \tau \in \{-1, 1\}$. *Then*

$$S^{\alpha,\beta} = (-\alpha\tau)\text{-dhog}\big((\tau\text{-hog}(S))^{0,\beta}\big).$$

There are various other formulas with which $S^{\alpha,\beta}$ can be expressed. In essence they are all captured in Diagram 7.1 which tells us how we can com-

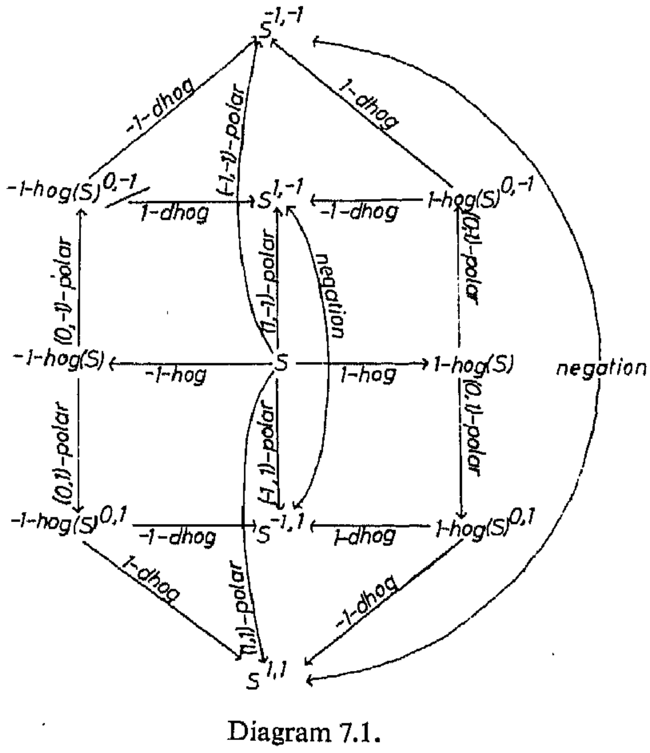

Diagram 7.1.

pute $S^{\alpha,\beta}$ be means of cone polarity, homogenization, dehomogenization and negation.

E.g. Diagram 7.1 says that $S^{1,-1}$ can be obtained from S by first going to 1-hog(S), then taking the usual polar cone of 1-hog(S) and then dehomogenizing at the -1-level; or $S^{1,-1}$ can be obtained by first going to -1-hog(S), taking the (0, 1)-polar, i.e. the negative of the polar cone of -1-hog(S), by dehomogenizing $((-1)\text{-hog}(S))^{0,1}$ at the -1-level and then multiplying every vector of this set with -1. We can now use Theorem 7.2 (resp. the results captured in Figure 7.1) in conjunction with Theorem 6.4 and Theorem 2.16 to characterize the (α,β)-polars of polyhedra.

For technical reasons we assume throughout this paragraph that the right hand sides of inequality systems are normalized. In particular, unless otherwise specified we assume that $\alpha \in \{-1, 0, 1\}$, $\beta \in \{-1, 1\}$ and that inequality systems are given in the form

$$\beta Ax \geqslant \beta b$$

where b is a vector whose components are 0, -1 or 1. Clearly, any inequality system can be written in such a way, and we shall say that such an inequality system is in *normal form.*

Moreover, we assume that A has m rows and n columns and that $M = \{1, 2, \ldots, m\}$. To shorten notation we use the following abbrevations:

$$I^- := \{i \in M \mid b_i = -1\} \quad \text{and} \quad A^- := (A_{I^-,\cdot})^T,$$

$$I^0 := \{i \in M \mid b_i = 0\} \quad \text{and} \quad A^0 := (A_{I^0,\cdot})^T,$$

$$I^+ := \{i \in M \mid b_i = 1\} \quad \text{and} \quad A^+ := (A_{I^+,\cdot})^T,$$

$$Q := \{x \in \mathbb{R}^n \mid x = A^+y + A^-z \text{ with } y \geqslant 0,\ z \geqslant 0 \text{ and } z^T\mathbb{1} \leqslant y^T\mathbb{1}\}.$$

With some (nontrivial) effort one can characterize the (α,β)-polars of polyhedra as follows:

Theorem 7.3. *Let P be a nonempty polyhedron defined by the linear inequality system $\beta Ax \geqslant \beta b$ in normal form. Then the (α,β)-polar $P^{\alpha,\beta}$ of P has the representation shown in Table* 7.1.

Table 7.1

	$\alpha\beta = -1$	$\alpha\beta = 0$	$\alpha\beta = 1$
$I^+ = \emptyset$ $I^- = \emptyset$	$\mathrm{cone}(A^T)$	$\mathrm{cone}(A^T)$	$\emptyset$
$I^+ = \emptyset$ $I^- \neq \emptyset$	$\mathrm{cone}(A^T)$	$\mathrm{cone}(A^T)$	$\mathrm{conv}(A^-) + \mathrm{cone}(A^T)$
$I^+ \neq \emptyset$ $I^- = \emptyset$	$\mathrm{conv}(A^+, 0) + \mathrm{cone}(A^0)$	$\mathrm{cone}(A^0)$	$\emptyset$
$I^+ \neq \emptyset$ $I^- \neq \emptyset$	$\mathrm{conv}(A^+, 0) + \mathrm{cone}(A^0) + Q$	$\mathrm{cone}(A^0) + Q$	$\mathrm{conv}(A^-) + \mathrm{cone}(A^0, A^-) + Q$

E.g. take $\alpha = 1$ and $\beta = -1$, $I^+ \neq \emptyset$ and $I^- = \emptyset$, then P has the form

$$\{x \in \mathbb{R}^n \mid Bx \leqslant \mathbb{1},\ Cx \leqslant 0\},$$

and by Theorem 7.3, the $(1, -1)$-polar of P is the polyhedron which is the sum of the convex hull of the rows of B and the zero vector plus the conical hull of the rows of C. Note also that the case $I^+ \neq \emptyset$ and $I^- \neq \emptyset$ causes some problems, because in the representation of every (α, β)-polar the polyhedron Q comes up. In all other cases, Theorem 7.3 shows that the (α, β)-polars of polyhedra can be characterized in a nice way.

The counterpart to Theorem 7.3 with respect to the description $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ is

Theorem 7.4. *Let* $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ *be a nonempty polyhedron and let* $\alpha \in \{-1, 0, 1\}$, $\beta \in \{-1, 1\}$. *Then*

$$P^{\alpha,\beta} = \{x \in \mathbb{R}^n \mid \beta V^T x \geqslant \beta\alpha,\ \beta E^T x \geqslant 0\}$$
$$= \left\{x \in \mathbb{R}^n \mid \beta \begin{pmatrix} V^T \\ E^T \end{pmatrix} x \geqslant \beta \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}\right\}.$$

For notational convenience in the above theorem, the sets V, E are also considered as matrices whose columns are the elements of V and E. It is rather surprising that $P^{\alpha,\beta}$ can be described in a much easier and more compact form if a polyhedron P is given by $\mathrm{conv}(V) + \mathrm{cone}(E)$ than if P is given by $P(A, b)$.

Note that (α, β)-polarity is not necessarily idempotent, e.g.

$$(\mathbb{R}^n)^{\alpha,\beta} = \{0\} \quad \text{if } \alpha\beta \in \{0, -1\}, \quad \text{and} \quad (\mathbb{R}^n)^{\alpha,\beta} = \emptyset \quad \text{if } \alpha\beta = 1,$$

while

$$\emptyset^{\alpha,\beta} = \mathbb{R}^n \quad \text{for all } \alpha, \beta,$$

and

$$\{0\}^{\alpha,\beta} = \mathbb{R}^n \quad \text{if } \alpha\beta \in \{0, 1\}, \quad \text{and} \quad \{0\}^{\alpha,\beta} = \emptyset \quad \text{if } \alpha\beta = -1.$$

Therefore, we call a polyhedron P *(α, β)-closed* if $P = (P^{\alpha,\beta})^{\alpha,\beta}$, e.g. $\mathbb{R}^n$ is (α, β)-closed for all α, β; $\emptyset$ is not (α, β)-closed for $\alpha\beta \in \{0, -1\}$; $\{0\}$ is not (α, β)-closed for $\alpha\beta = 1$. The following theorem gives a complete characterization of (α, β)-closedness.

Theorem 7.5. *Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron.*

(a) *In case $\alpha\beta = -1$, P is (α, β)-closed if and only if $0 \in P$.*

(b) *In case $\alpha\beta = 1$, P is (α, β)-closed if and only if $0 \notin P$ and* $\text{rec}(P) \subseteq P$.

(c) *In case $\alpha\beta = 0$, P is (α, β)-closed if and only if P is a polyhedral cone.*

Finally (α, β)-polarity can be used to characterize nonredundancy, cf. Section 5.

Theorem 7.6. *Let $\alpha \in \{-1, 0, 1\}$, $\beta \in \{-1, 1\}$ and let*

$$P = \{x \in \mathbb{R}^n \mid \beta Ax \geqslant \beta b\} \neq \mathbb{R}^n$$

be a nonempty fully dimensional polyhedron given in normal form, where $b \in \{0, \alpha\}^m$ and $b \neq 0$ in case $\alpha\beta \neq 0$. Let P be (α, β)-closed and assume further in case $\alpha\beta = 1$ that $\beta A \geqslant \beta b$ is not trivially redundant, i.e. $A_{i.} \neq \lambda A_{j.}$ for all $\lambda \geqslant 0$ and $i \neq j$. Then $\beta Ax \geqslant \beta b$ is an irredundant description of P if and only if

(a) *in case $\alpha\beta = 0$: $(\{0\}, A^{\mathrm{T}})$ is a basis of $P^{\alpha,\beta}$;*

(b) *in case $\alpha\beta = -1$: either (A^+, A^0) or $((A^+, 0), A^0)$ is a basis of $P^{\alpha,\beta}$;*

(c) *in case $\alpha\beta = 1$: (A^-, A^0) is a basis of $P^{\alpha,\beta}$.*

Example 7.7. Consider the polyhedron P of Example 4.2. We want to determine the $(1, -1)$-polar of P. The inequality system $\beta Ax \geqslant \beta b$ describing

P in normal form ($\beta = -1$) is

$$\begin{array}{lrl} (1) & 0.25x_1 - 0.5x_2 & \leqslant 1, \\ (2) & -0.1\overline{6}x_1 + 0.\overline{3}x_2 & \leqslant 1, \\ (3) & -0.25x_1 + 0.\overline{3}x_2 & \leqslant 1, \\ (4) & -x_1 \quad & \leqslant 0, \\ (5) & -x_2 & \leqslant 0. \end{array}$$

Therefore $I^+ = \{1, 2, 3\}$, $I^0 = \{4, 5\}$, $I^- = \emptyset$. By Theorem 7.3, we have

$$P^{1,-1} = \operatorname{conv}(\{A_{1\cdot}^T, A_{2\cdot}^T, A_{3\cdot}^T, 0\}) + \operatorname{cone}(\{A_{4\cdot}^T, A_{5\cdot}^T\}),$$

i.e.

$$P^{1,-1} = \operatorname{conv}\left(\frac{1}{12}\left\{\begin{pmatrix} 3 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}\right) + \operatorname{cone}\left\{\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}.$$

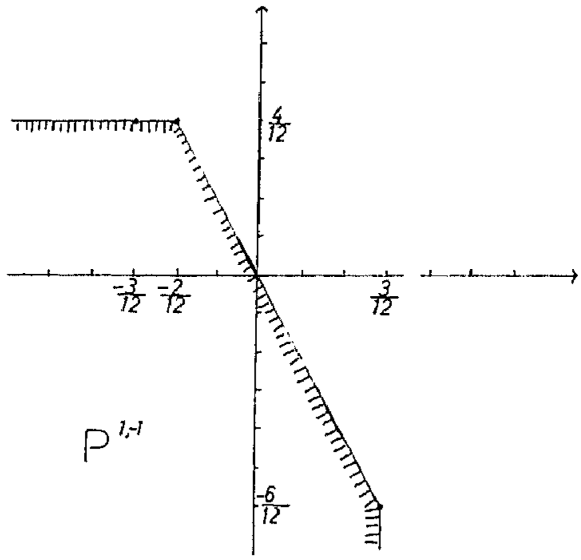

Fig. 7.1.

See Fig. 7.1. Since $0 \in P$, Theorem 7.5 implies that P is $(1, -1)$-closed, i.e. $P = (P^{1,-1})^{1,-1}$ holds. We can therefore apply Theorem 7.6 to check nonredundancy of P, i.e. we have to find out whether (A^+, A^0) or $((A^+, 0), A^0)$ is a basis of $P^{1,-1}$. First of all we see that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{2}{60}\begin{pmatrix} 3 \\ -6 \end{pmatrix} + \frac{3}{60}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{2}{5}A_{1\cdot}^T + \frac{3}{5}A_{2\cdot}^T,$$

therefore $0 \in \text{conv}(A^+)$ and we have to check whether (A^+, A^0) is a basis or not. But since

$$A_{3\cdot}^{\mathrm{T}} = \frac{1}{12}\begin{pmatrix}-3\\ 4\end{pmatrix} = \frac{1}{12}\begin{pmatrix}-2\\ 4\end{pmatrix} + \frac{1}{12}\begin{pmatrix}-1\\ 0\end{pmatrix} = A_{2\cdot}^{\mathrm{T}} + \frac{1}{12}A_{4\cdot}^{\mathrm{T}},$$

we obtain that (A^+, A^0) is not a basis of $P^{1,-1}$, which implies that our inequality system is redundant.

A second description of P is given in Example 4.2, namely $P = \text{conv}(V) + \text{cone}(E)$, where

$$V = \{v_1, v_2, v_3, v_4\} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

and $E = \{e\} = \{\binom{2}{1}\}$. This characterization of P gives a description of $P^{1,-1}$ by means of linear inequalities via Theorem 7.4 as follows:

$$\begin{aligned}
0x_1 + 0x_2 &\leqslant 1,\\
4x_1 \qquad &\leqslant 1,\\
3x_2 &\leqslant 1,\\
x_1 + \ x_2 &\leqslant 1,\\
2x_1 + \ x_2 &\leqslant 0.
\end{aligned}$$

Here the first and the fourth inequality can be removed to obtain a nonredundant description of $P^{1,-1}$.

8. Lattices associated with polyhedra

A lattice is a pair $L = (S, \leqslant)$ consisting of a set S (for our purposes S is always finite) and a partial order '$\leqslant$' on S which satisfies the following condition:

(L) For every $T \subseteq S$ there exist a least upper bound and a greatest lower bound for T.

The greatest lower bound of two elements $x, y \in S$ is called the *meet* of x and y and is denoted by $x \wedge y$, while the smallest upper bound of $x, y \in S$ is called the *join* of x and y and is denoted by $x \vee y$.

The ground sets S of lattices $L = (S, \leqslant)$ we consider will be of a particular form, namely S will always be a finite subset of the power set 2^M for some set M, and the partial order '$\leqslant$' in S will always be the usual set inclusion '$\subseteq$', i.e. if $X, Y \in S$ then X and Y are subsets of M and '$X \leqslant Y$' means nothing but '$X \subseteq Y$'. For notational convenience we shall therefore speak of a lattice S instead of $(S, \leqslant)$, keeping in mind that S is a lattice with respect to set inclusion '$\subseteq$'.

There is a 'natural lattice' one can associate with a polyhedron P, the so called face lattice $\mathbf{F}(P)$. The object we would like to deal with is

$$\mathbf{F}'(P) := \{F \subseteq P \mid F \text{ is a nonempty face of } P\}. \tag{8.1}$$

and the join and meet operation should clearly be

$$F \wedge G := F \cap G,$$

$$F \vee G := \cap \{H \in \mathbf{F}'(P) \mid F \cup G \subseteq H\},$$

for any two faces $F, G \in \mathbf{F}'(P)$. In case P is a polyhedral cone $\mathbf{F}'(P)$ has a minimal (nonempty) element, however, if P is not a polyhedral cone then $F \wedge G$ may be empty. Therefore we define

$$F_0 := \cap \{F \mid F \in \mathbf{F}'(P)\},$$

and set

$$\mathbf{F}(P) := \mathbf{F}'(P) \cup \{F_0\}. \tag{8.2}$$

Then $\mathbf{F}(P)$ is by construction a lattice called the *face lattice* of P. Note that this definition implies that the empty face is an element of the face lattice if and only if P has no minimal nonempty face. E.g. the empty face is not in the face lattice of a polyhedral cone, but is an element of the face lattice of a nonempty polytope.

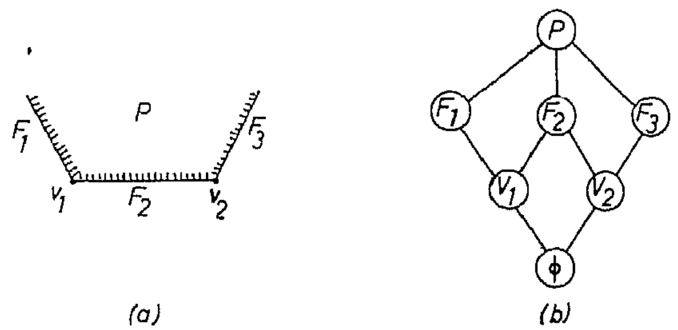

Fig. 8.1.

Consider for example the polyhedron $P \subseteq \mathbb{R}^2$ shown in Fig. 8.1(a). Its face lattice $\mathbf{F}(P)$ has seven elements, namely the empty face, two vertices v_1 and v_2, three facets F_1, F_2 and F_3, and P itself. The usual way to depict $\mathbf{F}(P)$ is by means of a diagram where faces of equal dimension are represented on one level and a line between faces on different levels means that the face on the higher level is a cover of the other. The lattice $\mathbf{F}(P)$ of the polyhedron P in Fig. 8.1(a) is shown in Fig. 8.1(b).

For another example consider a nonempty halfspace $P = \{x \in \mathbb{R}^n \mid a^{\mathrm{T}}x \leqslant a_0\}$. Here $\mathbf{F}(P)$ has only two elements, namely P and its unique facet $\{x \mid a^{\mathrm{T}}x = a_0\}$. Note that in this case the empty face is a subfacet of P but it is not an element of the face lattice $\mathbf{F}(P)$.

We now introduce several further lattices which can be associated with a polyhedron P, and we shall study the relation between these lattices.

If P is given in the form $P = P(A, b)$ where A is an (m, n)-matrix and $M = \{1, \ldots, m\}$, then for every subset S of P we have introduced in Section 4 the equality set

$$\mathrm{eq}(S) := \{i \in M \mid A_i x = b_i \ \ \forall\, x \in S\}.$$

The *equality set lattice* of $P(A, b)$ is then defined as follows:

$$\mathrm{EQ}(A, b) = \{I \subseteq M \mid \exists\, S \subseteq P \text{ with } I = \mathrm{eq}(S)\}. \tag{8.3}$$

Here and in the sequel it is easy to see (and left to the reader) that the object for which we claim that it is a lattice really is a lattice.

If a polyhedron P is given by means of $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ for some finite sets $V, E \subseteq \mathbb{R}^n$, then for every $S \subseteq P$ we have introduced in Section 4 the so called extreme set $\mathrm{ex}(S) := (\mathrm{ex}_V(S), \mathrm{ex}_E(S))$ which loosely speaking is the set of vertices and extreme vectors which support some vectors $x \in S$. The *extreme set lattice* of $P = \mathrm{conv}(V) + \mathrm{cone}(E)$ is defined as follows:

$$\mathrm{EX}(V, E) = \{(W, F) \subseteq (V, E) \mid \exists\, S \subseteq P \text{ with } (W, F) = \mathrm{ex}(S)\}. \tag{8.4}$$

Note that the lattice $\mathbf{F}(P)$ is description-independent while $\mathrm{EQ}(A, b)$ and $\mathrm{EX}(V, E)$ clearly depend on the given representation of P. We shall however see later, that the differences between the lattices associated with various desriptions are only superficial.

In Section 6 we have introduced the γ-polar S^γ of a set S and we have shown in Corollary 6.3 that the γ-polar P^γ of a polyhedron P has a particularly simple representation, namely if $P = P(A, b) = \mathrm{conv}(V) + \mathrm{cone}(E)$, then

$$P^\gamma = \operatorname{cone}\left(\begin{pmatrix} A^{\mathrm{T}} & 0 \\ b^{\mathrm{T}} & 1 \end{pmatrix}\right) = P\left(\begin{pmatrix} V^{\mathrm{T}} & -\mathbb{1} \\ E^{\mathrm{T}} & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right),$$

in other words, if

$$B = \begin{pmatrix} V^{\mathrm{T}} & -\mathbb{1} \\ E^{\mathrm{T}} & 0 \end{pmatrix} \quad \text{and} \quad D = \{(A_{i.}, b_i)^{\mathrm{T}} \mid i \in M\} \cup \left\{\begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}\right\},$$

then

$$P^\gamma = \operatorname{conv}(\{0\}) + \operatorname{cone}(D) = P(B, 0).$$

Thus, P^γ is a polyhedral cone and therefore we can consider the various lattices associated with P defined above. For notational convenience, however, we shall describe the equality set lattice and the extreme set lattice of P in terms of the original representation of P and thus change our notation slightly. To introduce these new symbols we define the following two mappings:

Let $R = \{1, 2, \ldots, r\}$ be the row index set of the matrix B defined above and set

$$\begin{aligned} &\mu\colon 2^R \to 2^V \times 2^E, \\ &R \supseteq I \mapsto (S, T) \end{aligned} \tag{8.5}$$

where $S := \{v \in V \mid \exists\, i \in I \text{ with } (v^{\mathrm{T}}, -1) = B_{i.}\}$, $T := \{e \in E \mid \exists\, i \in I \text{ with } (e^{\mathrm{T}}, 0) = B_{i.}\}$;

$$\begin{aligned} &\nu\colon D \to M' := M \cup \{m+1\} = \{1, 2, \ldots, m+1\}, \\ &(A_{i.}, b_i)^{\mathrm{T}} \mapsto i \in M, \\ &(0^{\mathrm{T}}, 1)^{\mathrm{T}} \mapsto m+1, \end{aligned} \tag{8.6}$$

In order to distinguish between the mappings eq, fa, etc. (cf. Section 4) with respect to P and P^γ we use a superscript 'γ', i.e. eq^γ, fa^γ etc., if we use these mappings with respect to P^γ. Now we define

$$\gamma\text{-eq}\colon 2^{P^\gamma} \to 2^V \times 2^E, \qquad \gamma\text{-eq} := \mu \circ \mathrm{eq}^\gamma, \tag{8.7}$$

$$\gamma\text{-fa}\colon 2^V \times 2^E \to 2^{P^\gamma}, \qquad \gamma\text{-fa} := \mathrm{fa}^\gamma \circ \mu^{-1}, \tag{8.8}$$

$$\gamma\text{-ex}\colon 2^{P^\gamma} \to 2^{M'}, \qquad \gamma\text{-ex} := \nu \circ \mathrm{ex}^\gamma, \tag{8.9}$$

$$\gamma\text{-gen}\colon 2^{M'} \to 2^{P^\gamma}, \qquad \gamma\text{-gen} := \mathrm{gen}^\gamma \circ \nu^{-1}, \tag{8.10}$$

$$\gamma\text{-sp}\colon 2^{M'} \to 2^{P^\gamma}, \qquad \gamma\text{-sp} := \mathrm{sp}^\gamma \circ \nu^{-1}. \tag{8.11}$$

These mappings γ-eq, ... etc. look rather awkward but they are essentially the same as the mappings eq, ... etc., we have only replaced the domains resp. image spaces by more convenient isomorphic ones.

With respect to the γ-polar P^γ of P we introduce the following lattices:

$$\gamma\text{-EQ}(V,E) := \{(W,F) \subseteq (V,E) \mid \exists\, S \subseteq P^\gamma \text{ with } (W,F) = \gamma\text{-eq}(S)\},$$

$$\gamma\text{-EQ}(A,b) := \{I \subseteq M' \mid \exists\, S \subseteq P^\gamma \text{ with } I = \gamma\text{-ex}(S)\}.$$

To give a comprehensive overview we have summarized all the mappings of interest in Table 8.1.

Table 8.1

mapping	P given by	domain	range	name
eq	$P(A, b)$	2^P	M	equality set
fa	$P(A, b)$	M	2^P	face
ex	conv(V) + cone(E)	2^P	$2^V \times 2^E$	extreme set
gen	conv(V) + cone(E)	$2^V \times 2^E$	2^P	generating set
sp	conv(V) + cone(E)	$2^V \times 2^E$	2^P	span
γ-eq	conv(V) + cone(E)	2^{P^γ}	$2^V \times 2^E$	γ-equality set
γ-fa	conv(V) + cone(E)	$2^V \times 2^E$	2^{P^γ}	γ-face
γ-gen	$P(A, b)$	$2^{M'}$	2^{P^γ}	γ-generating set
γ-sp	$P(A, b)$	$2^{M'}$	2^{P^γ}	γ-span
γ-ex	$P(A, b)$	2^{P^γ}	$2^{M'}$	γ-extreme set

We further want to consider the mappings eq, fa, ... etc. with respect to the recession cone rec(P) of P and with respect to a τ-homogenization τ-hog(P) of P. When these mappings are used with respect to

$$\text{rec}(P) = P(A, 0) = \text{cone}(E),$$

we use a superscript '0', when used with respect to

$$\tau\text{-hog}(P) = P(B_\tau, 0) = \text{cone}(D_\tau),$$

cf. Theorem 6.4, we add a superscript 'τ', i.e. let mp stand for one of the mappings eq, fa, ex, gen, sp, then mp* is used with respect to P if '$*$' does not appear, with respect to P^γ if $* = \gamma$, with respect to rec(P) if $* = 0$, with respect to τ-hog(P) if $* = \tau$.

Finally, we introduce the most important lattice of all, the so called *extended face lattice* of P. This lattice is obtained by adding the face lattice of the recession cone to the face lattice of P in the following way.

Let P be a nonempty polyhedron. A face of P and a face of rec(P) may set-theoretically be identical, therefore, in order to distinguish them we label all faces of P and rec(P), in particular let $\tau \in \mathbb{R}$ and $F \subseteq \mathbb{R}^n$ then

$$(F, \tau) := \left\{ \binom{x}{z} \in \mathbb{R}^{n+1} \mid x \in F,\ z = \tau \right\}$$

and for $\tau \in \{-1, 1\}$ set

$$\mathbf{F}_\tau(P) := \{(F, \tau) \mid F \in \mathbf{F}(P),\ F \neq \emptyset\}$$

and

$$\mathbf{F}_0(P) := \{(F, 0) \mid F \in \mathbf{F}(\mathrm{rec}(P))\}.$$

Thus all nonempty faces of $\mathbf{F}(P)$ carry a label $\tau \neq 0$ and all faces of rec(P) a label '0'. Set

$$\mathbf{X}_\tau(P) := \mathbf{F}_\tau(P) \cup \mathbf{F}_0(P).$$

Then $\mathbf{X}_\tau(P)$ is called the *τ-extended face lattice of P*, where the partial order in $\mathbf{F}_\tau(P)$ and $\mathbf{F}_0(P)$ is defined as usual by set-inclusion, i.e.

$$(F_1, \tau) \leqslant (F_2, \tau) \quad \text{if and only if} \quad F_1 \subseteq F_2,$$

$$(F_1, 0) \leqslant (F_2, 0) \quad \text{if and only if} \quad F_1 \subseteq F_2,$$

while for $(F, \tau) \in \mathbf{F}_\tau(P)$ and $(G, 0) \in \mathbf{F}_0(P)$ we set

$$(G, 0) \leqslant (F, \tau) \quad \text{if and only if} \quad G \subseteq \mathrm{rec}(F),$$

and

$$(F, \tau) \leqslant (G, 0)$$

never holds.

The maximal element of $\mathbf{X}_\tau(P)$ is clearly (P, τ), and the minimal element of $\mathbf{X}_\tau(P)$ is $(M, 0)$ where M is the minimal (nonempty) face of rec(P).

There are—of course—various relations between the lattices associated with a polyhedron P as defined above. To state and show these relations we need a rather long sequence of technical lemmas. Instead of giving the list of theorems we summarize our results in Diagrams 8.1 and 8.2 and just explain how these diagrams can be read in order to obtain valid theorems.

We assume that a nonempty polyhedron P is given as $P = P(A, b)$ where A is an (m, n)-matrix and $M = \{1, 2, \ldots, m\}$, $M' = M \cup \{m + 1\}$. For $\tau \neq 0$ we set

$$P_\tau := \left\{ \begin{pmatrix} x \\ \tau \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in P \right\}$$

and

$$P_0 := \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in \operatorname{rec}(P) \right\}.$$

Then

$$\operatorname{rec}(P) = P(A, 0), \qquad P^\gamma = \operatorname{cone}\left(\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}^{\mathrm{T}} \right),$$

$$\tau\text{-hog}(P) = P\left(\begin{pmatrix} A & -\tau b \\ 0 & -\tau \end{pmatrix}, 0 \right), \qquad \tau \in \{-1, 1\}.$$

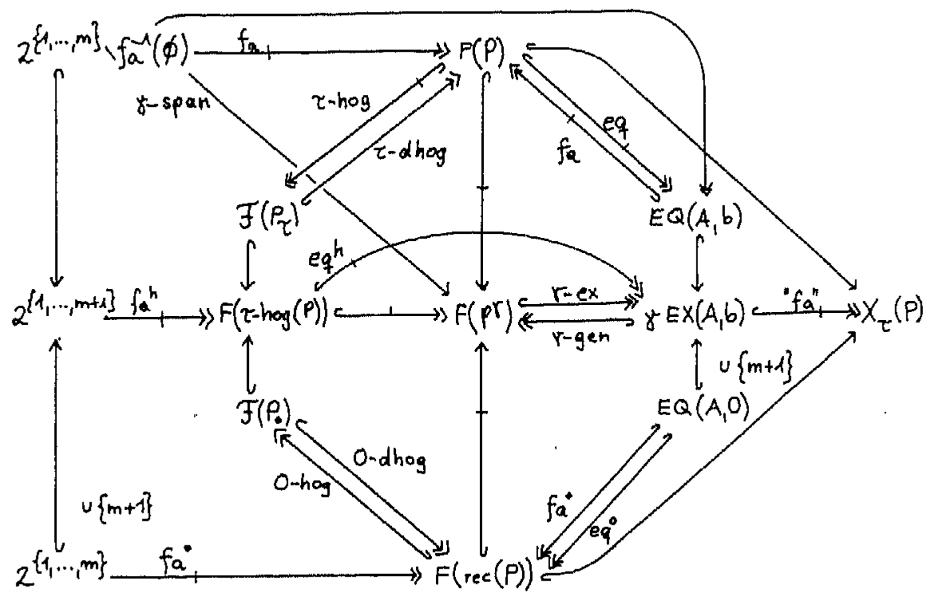

Diagram 8.1.

The arrows shown in Diagram 8.1 have the following meaning:

- → inclusion-preserving (homomorphism)
- ↛ inclusion-reversing (antihomomorphism)
- ↪ injective
- ↠ surjective

To give some examples, Diagram 8.1 states:

(a) $\mathbf{F}(\tau\text{-hog}(P))$ and $\mathbf{F}(P^\gamma)$ are anti-isomorphic, since there is an injective and surjective antihomomorphism.

(b) $\mathbf{F}(P^\gamma)$ and γ-EX(A, b) are isomorphic.

(c) γ-EX(A, b) and $\mathbf{X}_\tau(P)$ are anti-isomorphic.

(d) $\mathbf{F}(\tau\text{-hog}(P))$ and $\mathbf{X}_\tau(P)$ are isomorphic.

(e) There is an inclusion-preserving injective mapping from $\mathbf{F}(\text{rec}(P))$ to $\mathbf{F}(\tau\text{-hog}(P))$, and an inclusion-reversing injective mapping from $\mathbf{F}(\text{rec}(P))$ to γ-EX(A, b).

For the next diagram we assume that the nonempty polyhedron P is given by

$$P = \text{conv}(V) + \text{cone}(E)$$

where $V, E \subseteq \mathbb{R}^n$ are finite, sets; then

$$\text{rec}(P) = \text{conv}(\{0\}) + \text{cone}(E), \qquad P^\gamma = P\left(\begin{pmatrix} V, & E \\ -\mathbb{1}^{\mathrm{T}}, & 0\end{pmatrix}^{\mathrm{T}}, 0\right),$$

$$\tau\text{-hog}(P) = \text{cone}\left(\begin{pmatrix} V, & E \\ \tau\mathbb{1}^{\mathrm{T}}, & 0\end{pmatrix}\right), \qquad \tau \in \{-1, 1\}.$$

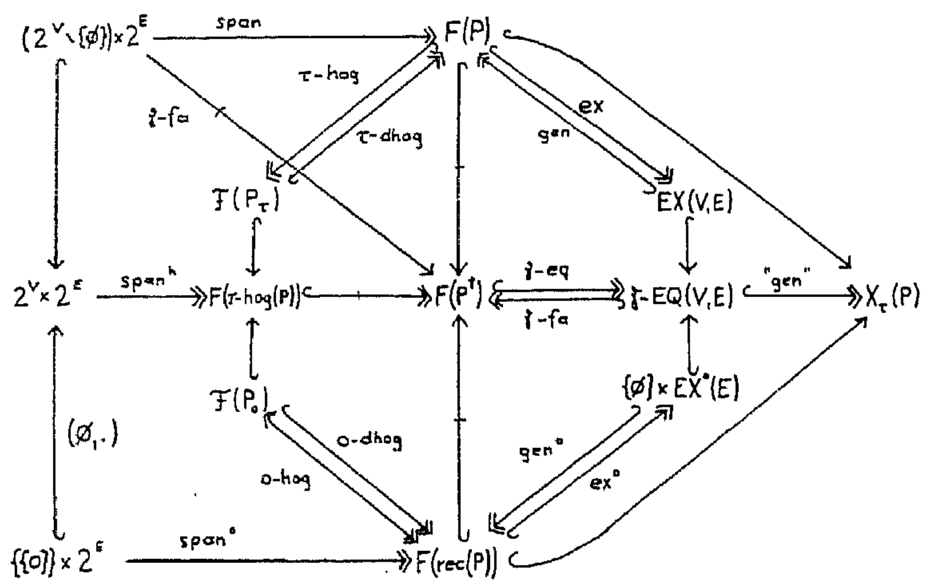

Diagram 8.2.

The arrows have to be interpreted as in Diagram 8.1. We also give some examples of how Diagram 8.2 can be read:

(a) γ-EQ(V, E) and $\mathbf{X}(P_\tau)$ are isomorphic.

(b) F(P) and EX(V, E) are isomorphic.

(c) F(P^γ) and γ-EQ(V, E) are antiisomorphic.

(d) There is an inclusion-preserving injective mapping from **F**(P) to γ-EQ(V, E).

The proofs for the results stated in Diagrams 8.1 and 8.2 will appear elsewhere.

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