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The Travelling Salesman Problem for Graphs not Contractible to $K_5 - e$

by

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Abstract

We study the travelling salesman problem for the class $\mathcal{K}$ of graphs not contractible to $K_5 - e$. A decomposition theorem of Wagner is used to characterize the travelling salesman polytope for every graph in $\mathcal{K}$ and to give a linear time algorithm for the travelling salesman problem for the graphs in $\mathcal{K}$.

Keywords: Travelling Salesman Problem, Polynomial Time Algorithms, Polyhedral Combinatorics

1. Introduction

The symmetric travelling salesman problem (TSP) is the task to find a shortest hamiltonian cycle in the complete graph with edge weigths. This problem is well-known to be $\mathcal{NP}$-hard. It is an interesting problem to determine classes $\mathcal{G}$ of graphs such that the TSP becomes easy when restricted to $\mathcal{G}$. The TSP restricted to the graphs in $\mathcal{G}$ is often called the **TSP for $\mathcal{G}$**. More exactly, if $\mathcal{G}$ is a class of graphs then an instance of the TSP for $\mathcal{G}$ is the following. Given an element $G = (V, E)$ of $\mathcal{G}$ with edge weights c_e for all $e \in E$, decide whether G contains a hamiltonian cycle, and if this is so, find a hamiltonian cycle T with $c(T) := \sum_{e \in T} c_e$ as small as possible.

If $\mathcal{G}$ is for instance the class of planar, or cubic or 3-connected graphs then the problem of deciding whether a graph $G \in \mathcal{G}$ has a hamiltonian cycle is $\mathcal{NP}$-complete (see GAREY & JOHNSON (1979)). So the TSP for these classes of graphs is $\mathcal{NP}$-hard. On the other hand there are some classes $\mathcal{G}$ of graphs known for which the TSP can be solved in polynomial time. Among them are wheels, Halin graphs and some generalizations of these, see CORNUEJOLS, NADDEF & PULLEYBLANK (1983, 1985). Moreover, in some of the above cases it was possible to give an explicit complete characterization of the convex hull of the incidence vectors of hamiltonian cycles for all $G \in \mathcal{G}$.

In this paper we will determine a further class of graphs for which the TSP is easy and for which the associated travelling salesman polytopes can be described explicitly. The graphs we investigate include the graphs not contractible to $K_5 - e$ (i. e. the complete graph on 5 nodes with one edge removed). Note that the class of graphs not contractible to K_5 contains the planar graphs, and so for these graphs the TSP is $\mathcal{NP}$-hard. This shows that the class we study is on the "boundary" between the hard and easy problems. It turns out that for the description of the travelling salesman polytope for a graph $G = (V, E)$ not contractible to $K_5 - e$ only trivial inequalities $0 \leq x_e \leq 1$, equations $x_f = 0$ for some $f \in E$ and equations $x(\delta(W)) = 2$ for some cuts $\delta(W)$, $W \subseteq V$, are needed. .

2. 1-Sums and 2-Sums

All graphs throughout this paper have no loops and no multiple edges. For a graph $G = (V, E)$ and an edge $e \in E$, $G - e$ denotes the graph obtained from G by **deleting** (or removing) the edge e. $G - W$ denotes the graph obtained by deleting the node set $W \subseteq V$. $G \cdot e$ denotes **contraction** of e, i. e. $G \cdot e$ is obtained from G by identifying the two endnodes of e, deleting e and, if parallel edges appear by this node identification, removing one edge of each pair of parallel edges. A graph G is said to be **contractible to** a graph H, if a graph isomorphic to H can be obtained from G by repeated (in any order) deletion and contraction of edges of G. Let us denote the class of all connected graphs which are not contractible to $K_5 - e$ (this graph is shown in Figure 1) by $\mathcal{K}$.

WAGNER (1960) gave a constructive characterization of the graphs in $\mathcal{K}$ which has been the stimulus for the results to be presented later. The graphs shown in Figure 2 and their subgraphs are not contractible to $K_5 - e$.

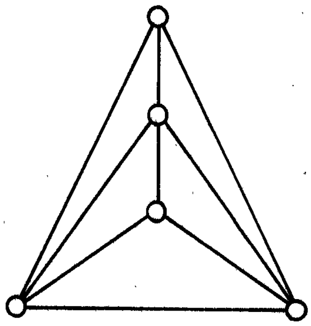

Fig. 1

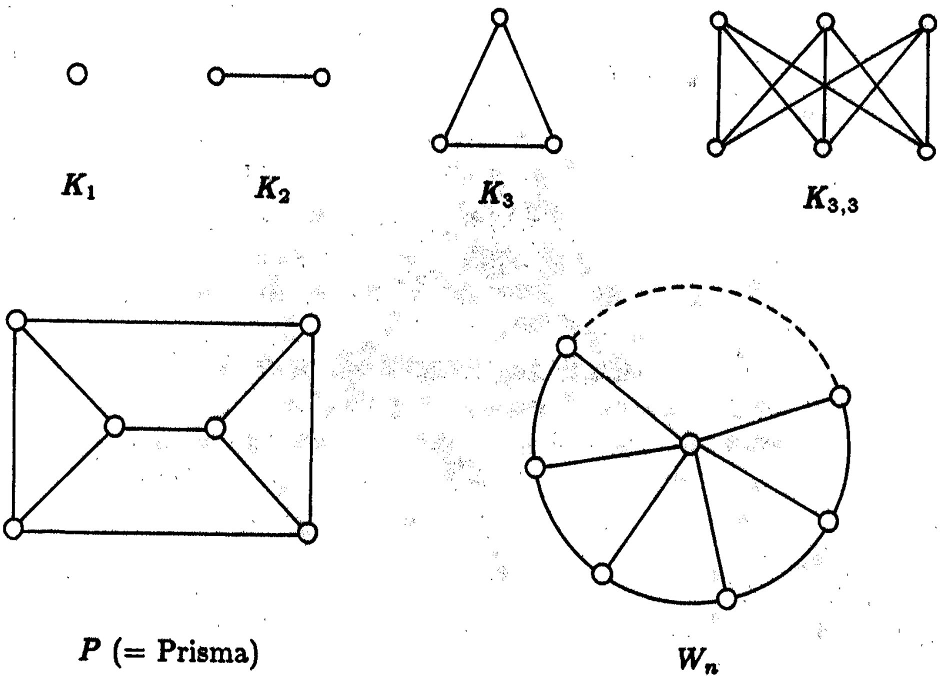

Fig. 2

The graph W_n (called ***n*-wheel**) consists of a cycle of length n and an additional node (the **center**) linked to all nodes of the cycle. Note that the 3-wheel is isomorphic to the complete graph K_4.

If G_1 and G_2 are node-disjoint graphs with at least two nodes, v_1 is a node of G_1, v_2 a node of G_2, then the **1-sum** G of G_1 and G_2 (with respect to v_1 and v_2) is obtained by identifying the nodes v_1 and v_2. This new node, say v, obtained by identifying v_1 and v_2 is an **articulation node** of G, i. e. $G - v$ has more components than G.

If e_1 is an edge of G_1 and e_2 an edge of G_2 then the **2-sum** G of G_1 and G_2 (with respect to e_1 and e_2) is obtained by identifying e_1 and e_2 (and of course, the endnodes of e_1 and e_2). This implies that the two identified nodes form an articulation set of G (provided G_1 and G_2 have at least three nodes).

WAGNER (1960) proved that each maximal graph G in $\mathcal{K}$ (i. e. by adding a further edge to G the new graph will be contractible to $K_5 - e$) can be obtained by starting with the graphs of Figure 2 and taking repeated 1-sums or 2-sums. Equivalently, if we have a graph $G \in \mathcal{K}$ we can decompose it into the graphs of Figure 2. Decomposition is done in two possible ways.

(2.1) *If G has an articulation node, say v, then let $V_1, V_2, \ldots, V_k$ be the node sets of the connected components of $G - v$. The* ***v*-components** *G_i of G are the subgraphs of G induced by $V_i \cup \{v\}$ for $i = 1, \ldots, k$. (So G is the 1-sum of the G_i with respect to v.)*

(2.2) *If G has an articulation set $\{u, v\}$ of size two, let $V_1, V_2, \ldots, V_k$ be the node sets of the connected components of $G - \{u, v\}$. Then the* **$\{u, v\}$-components** *G_i of G are the subgraphs induced by $V_i \cup \{u, v\}$ plus the edge uv.* (So G is the 2-sum of the G_i with respect to edge uv, where after taking the 2-sum the edge uv may have to be deleted.)

If we apply the procedures (2.1) and (2.2) recursively to a graph G we will end up with a list of graphs which cannot be decomposed any further. Let us call these graphs the **bricks** of G. Note that i. g. the bricks are not uniquely determined by G. The list of bricks depends on the order of choosing articulation sets $\{u, v\}$. Wagner's theorem states that, in whatever order (2.1) and (2.2) are performed, the bricks of the graphs in $\mathcal{K}$ are isomorphic to the graphs shown in Figure 2.

To give an example, the graph shown in Figure 3, is the 2-sum of two prisms, a K_3, a K_4 and a 4-wheel.

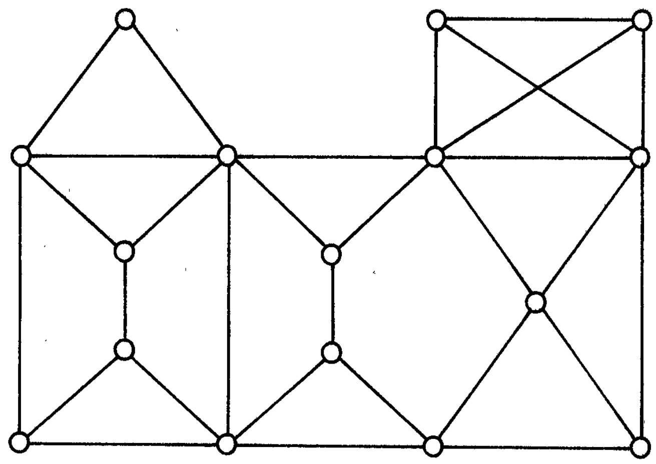

Fig. 3

3. A Recursive Algorithm for the TSP

The 2-sum composition immediately suggests a recursive procedure for finding a shortest hamiltonian cycle in a graph $G = (V, E)$ with edge weights c_e for all $e \in E$. If G is disconnected, G has no hamiltonian cycle. If G is the 1-sum of two graphs (with at least 2 nodes) then G is not 2-connected, so G does not contain a hamiltonian cycle. Hence we can solve the TSP for such graphs trivially.

If G is the 2-sum of two connected graphs G_1 and G_2 with respect to an edge $f = uv$, then each hamiltonian cycle of G (if one exists) is composed of a hamiltonian path from u to v in G_1 and a hamiltonian path from u to v in G_2. No hamiltonian cycle of G will use the edge uv. Therefore, the problem of determining a shortest hamiltonian cycle can be decomposed into the problems of finding shortest hamiltonian (u, v)-paths in G_1 and G_2.

These two problems can be solved as follows. First pick graph G_1, all edges e of G_1 keep the weight c_e except for the edge $f = uv$. The weight c_f of f is zero. Each hamiltonian path P from u to v in G_1 corresponds to a hamiltonian cycle C of G_1 containing f, and C and P have the same length $c(C) = c(P)$. Let c' be the length of a shortest hamiltonian cycle C_1 in G_1. Now consider G_2. Each edge $e(\neq f)$ of G_2 gets the weight c_e, and we set $c_f := c'$. Again, each hamiltonian (u, v)-path P in G_2 corresponds to a unique hamiltonian cycle C in G_2 containing f with $c(P) = c(C) - c'$. By construction, if C_2 is a shortest hamiltonian cycle of G_2 containing f then $c(C_2)$ is the length of the shortest hamiltonian cycle in G. Moreover, $(C_1 \cup C_2) \setminus \{f\}$ is the shortest hamiltonian cycle of G. Clearly if G_1 or G_2 have no hamiltonian cycle containing f then G has no hamiltonian cycle.

In this way we have reduced the TSP for G to two TSP's for the smaller graphs G_1, G_2, and we can construct a solution for G from the solutions for G_1 and G_2. Our algorithm thus works as follows.

Let $G = (V, E)$ be a graph with edge weights c_e for all $e \in E$. Decompose G into its bricks using procedures (2.1) and (2.2) and solve the TSP for the bricks. If at any stage (2.1) is successful, which implies that G is not 2-connected, we can stop and declare G nonhamiltonian. Moreover, if at any stage procedure (2.2) finds more than two $\{u, v\}$-components of G, we can stop and declare G nonhamiltonian. Therefore, the decomposition is only carried out in case procedure (2.2) finds an articulation set $\{u, v\}$ such that there are only tow $\{u, v\}$-components.

Observe that if a graph G with p nodes is decomposed in two $\{u, v\}$-components G_1, G_2 with q_1 resp. q_2 nodes then $q_1 + q_2 = p + 2$ and $3 \leq q_1 \leq p - 1$, $3 \leq q_2 \leq p - 1$. So if the initial graph has n nodes and if we assume that this decomposition is applied $n - 1$ times we have produced $n - 1$ graphs $G_1, \ldots, G_{n-1}$ with a total number of $3n - 3$ nodes where each graph G_i has at least 3 nodes. This is impossible, which shows that this decomposition can be carried out at most $n - 2$ times.

In fact, it is not necessary to carry out procedures (2.1) and (2.2) in sequence for each decomposed graph. By using depth-first-search one can check whether G is connected and find all articulation nodes in $O(|V| + |E|)$ time. If G is connected and has no articulation node then one can use the depth-first-search based procedure of HOPCROFT & TARJAN (1973) to determine a list of bricks of G in time $O(|E|)$. So in at most $O(|V| + |E|)$ steps a brick decomposition of a graph G can be obtained.

We still have to discuss how we solve the TSP for the bricks. For bricks of fixed (small) size, we solve the TSP by brute force enumeration. This is (in practice) probably the fastest method for K_k with, say, $k \leq 7$ and all subgraphs of K_k and requires only constant time. But a brick may have a large number of nodes (compared with n), and in such cases we need to know that such a brick belongs to a class of graphs for which the TSP is solvable in polynomial time. If this is so for all bricks our recursive algorithm runs in polynomial time. So — in principle — we could use all known polynomial time algorithms for the TSP for certain classes of graphs (as subroutines) and apply them to our bricks. If for each brick one such algorithm finds a shortest hamiltonian cycle our procedure is successful and has polynomial running time. As mentioned before there are a number of such classes of graphs known. Let us just mention how to treat wheels W_n.

Clearly each hamiltonian cycle H of W_n has to pass through the center, and it is obvious that the two other endnodes of the two edges of H containing the center must be adjacent on the outer cycle of W_n. This observation shows that a wheel contains exactly n hamiltonian cycles (the outer cycle minus an edge plus the two edges linking this path to the center) and that a shortest hamiltonian cycle in W_n can be found in $O(n)$ time. Since the bricks of the graphs in class $\mathcal{K}$ are those of Figure 2 we can conclude.

(3.1) Corollary. *The TSP for the class of graphs not contractible to* $K_5 - e$ *can be solved in polynomial time.*

□

Observe that the number of edges in a graph $G \in \mathcal{K}$ is linear in the number of nodes, and with a more careful analysis of the algorithm described above one can show that the TSP for the graphs in $\mathcal{K}$ can be solved in time linear in $|V|$.

4. Travelling Salesman Polyhedra and 2-Sums

Let $G = (V, E)$ be a graph and $F \subseteq E$, then $\chi^F \in \mathbb{R}^E$ denotes the incidence vector of F, i. e. $\chi^F_e = 1$ if $e \in F$, $\chi^F_e = 0$ otherwise. The travelling salesman polytope $Q_T(G)$ is the convex hull of the incidence vectors of the hamiltonian cycles (= tours) of G, i. e.

$$Q_T(G) = \mathrm{conv}\{\chi^T \in \mathbb{R}^E \mid T \subseteq E \text{ tour}\}.$$

This polytope has been intensively studied for the complete graph K_n, see GRÖTSCHEL & PADBERG (1985) for a survey, but complete descriptions of $Q_T(K_n)$ by means of linear inequalities are only known for $n \leq 7$. CORNUÉJOLS, NADDEF & PULLEYBLANK (1983, 1985) were able to describe $Q_T(G)$ for several classes of graphs including wheels and Halin graphs. We will present a linear characterization of $Q_T(G)$ for all graphs $G \in \mathcal{K}$.

First we give a description of $Q_T(G)$ for the basic graphs of Figure 2. For $W \subseteq V$ let $\delta(W) := \{uv \in E \mid u \in W, v \in V \setminus W\}$ denote the cut of W (we write $\delta(v)$ instead of $\delta(\{v\})$). The following proposition immediately follows from the known linear systems describing $Q_T(K_n)$, $n \leq 6$, resp. the fact that the travelling salesman polytope for all these graphs (except for the prisma P) is equal to the perfect 2-matching polytope. For any graph $G = (V, E)$ consider the inequalities resp. equations:

(4.1) $$0 \leq x_e \leq 1 \quad \text{for all } e \in E,$$

(4.2) $$x(\delta(v)) = 2 \quad \text{for all } v \in V.$$

Clearly every incidence vector of a tour satisfies (4.1) and (4.2).

(4.3) Proposition.

(a) $Q_T(K_1)$ *and* $Q_T(K_2)$ *are empty.*

(b) $Q_T(K_3)$,$Q_T(K_{3,3})$ *and* $Q_T(W_n)$ *for* $n \geq 3$ *are completely determined by (4.1) and (4.2).*

(c) $Q_T(P)$ *is given by (4.1), (4.2) and the equation*

$$x(\delta(W)) = 2 \tag{4.4}$$

where $\delta(W)$ *is the 3-edge cut (= perfect matching) of* P *linking the two triangles of* P*, i. e.* W *is the node set of one of the triangles.*

□

Since the matrix making up the left-hand side of the equation system (4.2) is the node-edge incidence matrix of G, and since this matrix is of full rank if and only if G is nonbipartite (otherwise the rank is $|V| - 1$) we obtain:

(4.5) Corollary.

$$\dim(Q_T(G)) = \begin{cases} |E| - |V| & \text{if } G = K_3 \text{ or } G = W_n, \\ |E| - |V| + 1 & \text{if } G = K_{3,3}, \\ |E| - |V| - 1 & \text{if } G = P. \end{cases}$$

□

Using the information of (4.5) one can give a more concise description of the polytopes $Q_T(G)$ characterized in (4.4) by removing redundant equations and inequalities. It is not difficult to see that

$$\begin{aligned} Q_T(K_3) &= \{x \in \mathbb{R}^E \mid x \text{ satisfies (4.2)}\}, \\ Q_T(P) &= \{x \in \mathbb{R}^E \mid x \text{ satisfies (4.2), (4.4) and} \\ &\qquad x_e \leq 1 \text{ for all } e \in \delta(W) \text{ (cf. (4.4))}\}, \\ Q_T(K_{3,3}) &= \{x \in \mathbb{R}^E \mid x \text{ satisfies any 5 of the six equations (4.2),} \\ &\qquad x_e \leq 1 \text{ for all } e \in E\}, \\ Q_T(W_n) &= \{x \in \mathbb{R}^E \mid x \text{ satisfies (4.2),} \\ &\qquad x_e \leq 1 \text{ for all edges } e \text{ of the outer cycle}\}, n \geq 3 \end{aligned}$$

are nonredundant characterizations of these travelling salesman polytopes.

Clearly, the travelling salesman polytope for any subgraph of the special graphs discussed above can be obtained by setting $x_e = 0$ for all edges e not contained in the subgraph. If we know $Q_T(G)$ and we want to determine the polytope of the incidence vectors of tours containing a fixed edge e, we just add the equation $x_e = 1$.

From this observation one can easily derive the following. Suppose $Q_T(G) = \{x \in \mathbb{R}^E \mid Ax \leq a\}$, $f = uv \in E$, and b is the column of A corresponding to f. Let $\overline{A}$ be the matrix derived from A by setting the column corresponding to f to zero, then

$$Q_P(G) := \{x \in \mathbb{R}^E \mid \overline{A}x \leq a - b\}$$

is the convex hull of the incidence vectors of the hamiltonian paths from u to v in G.

If G is the 2-sum of G_1 and G_2 with respect to an edge $f = uv$ then each hamiltonian cycle in G is the composition of a hamiltonian $[u, v]$-path in G_1 and a hamiltonian $[u, v]$-path in G_2. This implies that $Q_T(G)$ is the cartesian product of $Q_P(G_1)$ and $Q_P(G_2)$. So we can state the following result.

(4.6) Theorem. *Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs which have the edge f in common and let $G = (V, E)$ be the 2-sum of G_1 and G_2 with respect to f. Suppose $Q_T(G_i) = \{x \in \mathbb{R}^{E_i} \mid A_i x \le a_i\}$, $i = 1, 2$, and suppose the last column of A_1 and the first column of A_2 correspond to the edge f. Set*

$$A := \begin{array}{|c|c|} \hline A_1 & 0 \\ \hline 0 & A_2 \\ \hline \end{array} \qquad a := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and let b be the column of A corresponding to f. Let $\overline{A}$ be obtained by setting all entries of the column of A corresponding to f to zero, then

$$Q_T(G) = \{x \in \mathbb{R}^E \mid \overline{A}x \le a - b\}.$$

□

The theorem above applies to any graph. So whenever we decompose a graph G into its bricks by applying procedures (2.1) and (2.2) recursively and if we have a complete linear characterization of the travelling salesman polytope for each brick we can determine a complete linear inequality system for $Q_T(G)$. (Obviously, if G is disconnected or the 1-sum of two graphs, $Q_T(G)$ is empty.)

Let us apply this observation to the graphs not contractible to $K_5 - e$. The travelling salesman polytopes of the bricks of any graph $G \in \mathcal{K}$ are given in Proposition (4.3). Note that the linear systems consist of trivial inequalities (4.1) $0 \le x_e \le 1$ and equations only. All these equations are equations for certain cuts. Moreover, each polytope is defined by at most $2|E|$ inequalities and at most $|V|+1$ equations. Theorem (4.5) therefore implies that for each graph G not contractible to $K_5 - e$ the travelling salesman polytope $Q_T(G)$ is defined by at most $2|E|$ (trivial) inequalities and at most $|E|$ equations where all equations have 0/1-coefficients which can be determined in polynomial time recursively from the equations for the polytopes of the bricks. An easy induction shows that the equations that are obtained this way can be written either as $x_f = 0$ for some edges $f \in E$ or as cut equations with right-hand side 2, i. e. $x(\delta(W)) = 2$ for some $W \subseteq V$. More exactly, we have:

(4.7) Theorem. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two members of $\mathcal{K}$ which have an edge $f = uv$ in common, and let $G = (V, E)$ be the 2-sum of G_1 and G_2 with respect to f. Suppose there are sets $F_i \subseteq E_i$ and $\mathcal{W}_i \subseteq 2^{V_i}$ such that for $i = 1, 2$ the following holds*

$$\begin{aligned} Q_T(G_i) = \{x \in \mathbb{R}^{E_i} \mid 0 \le x_e &\le 1 \text{ for all } e \in E_i \setminus F_i \\ x_e &= 0 \text{ for all } e \in F_i \\ x(\delta(W)) &= 2 \text{ for all } W \in \mathcal{W}_i\}, \end{aligned}$$

then

$$\begin{aligned} Q_T(G) = \{x \in \mathbb{R}^E \mid 0 \le x_e &\le 1 \text{ for all } e \in E \setminus (F_1 \cup F_2 \cup \{f\}) \\ x_e &= 0 \text{ for all } e \in F_1 \cup F_2 \cup \{f\} \\ x(\delta(W)) &= 2 \text{ for all } W \in \mathcal{W}_1 \cup \mathcal{W}_2 \cup \{V_1, V_1 \cup \{u\}\}\}. \end{aligned}$$

$\square$

Since for each $G = (V, E) \in \mathcal{K}$ the linear equation and inequality system describing $Q_T(G)$ can be set up in polynomial time, the TSP for all graphs $G \in \mathcal{K}$ can be solved in polynomial time by any good linear programming algorithm.

To give an example consider the polytope $Q_T(G)$ for the graph $G = (V, E)$ shown in Figure 4. G is the 2-sum of two prisms. The edge $e = 89$ on which the 2-sum has been performed is deleted.

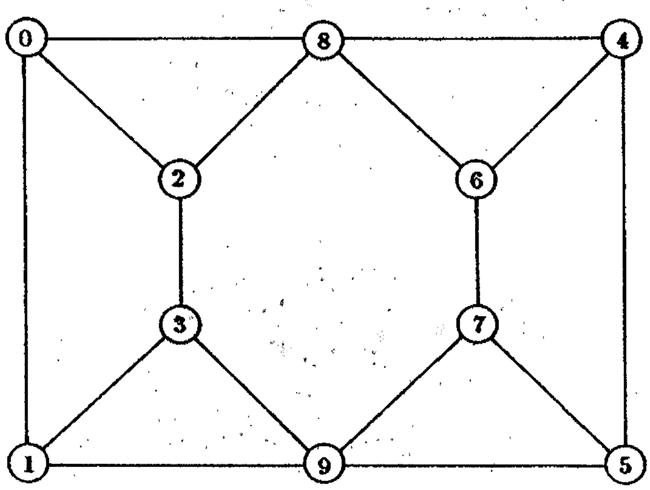

Fig. 4

By Theorem (4.6) and Proposition (4.3) $Q_T(G)$ is given by

$$
\begin{aligned}
0 \le x_e \le 1 \quad & \text{for all } e \in E \\
x(\delta(i)) = 2 \quad & \text{for } i = 0, 1, \dots, 7 \\
x_{08} + x_{28} = 1 & \\
x_{48} + x_{68} = 1 & \\
x_{19} + x_{39} = 1 & \\
x_{59} + x_{79} = 1 & \\
x_{01} + x_{23} = 1 & \\
x_{45} + x_{67} = 1 &
\end{aligned}
$$

By Theorem (4.7) the system describing $Q_T(G)$ is

$$
\begin{aligned}
0 \le x_e \le 1 \quad & \text{for all } e \in E \\
x(\delta(i)) = 2 \quad & \text{for all } e \in V \\
x(\delta(W)) = 2 \quad & \text{for all } W \in \{\{0,2,8\}, \{4,6,8\}, \{0,1,2,3\}, \{0,1,2,3,83\}
\end{aligned}
$$

It is somewhat surprising that the results described above are not implied by the more complicated constructions of CORNUÉJOLS, NADDEF & PULLEYBLANK (1983, 1985). For example, the graph G of Figure 4 does not belong to any of their classes, and so the linear description of $Q_T(G)$ given above cannot be derived from their results. Moreover, the well-studied system (4.1), (4.2), plus subtour elimination constraints, plus 2-matching constraints (see GRÖTSCHEL & PADBERG (1985)) does not provide a complete linear system for the polytopes $Q_T(G)$, $G \in \mathcal{K}$. For instance, the vector $x \in \mathbb{R}^E$ — graphically displayed in Figure 5 — does not belong to $Q_T(G)$ (G is the graph of Figure 4),but we can show that it is a vertex of the system (4.1), (4.2), plus subtour elimination constraints, plus 2-matching constraints. However, x violates some comb constraints (see GRÖTSCHEL & PADBERG (1985)). We do not know whether the system (4.1), (4.2), plus subtour elimination constraints, plus comb constraints (and possibly plus clique tree constraints) provides a complete description of $Q_T(G)$, $G \in \mathcal{K}$.

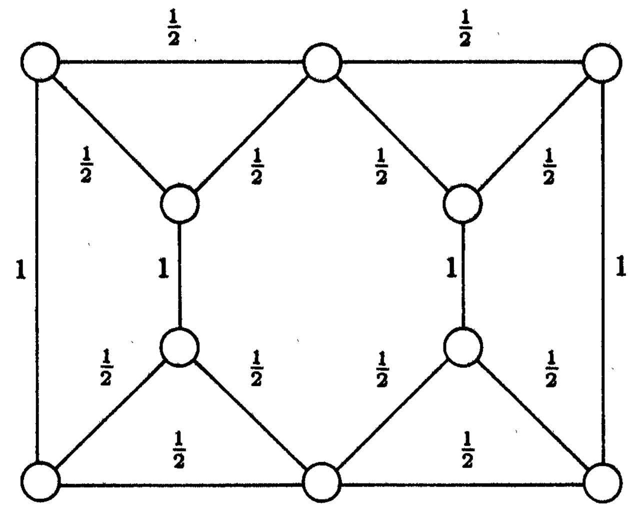

Fig. 5

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