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On the Cycle Polytope of a Binary Matroid

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The convex hull of the incidence vectors of the cycles of a binary matroid is studied. We prove that a description of the facets of this polytope can be obtained from a description of the facets that contain any given vertex. The facet-inducing inequalities are given for matroids with no F_7^*, R_{10}, or $M(K_5)^*$ minor. We also characterize adjacency on this polytope. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let M be an $m \times n$ matrix with zero-one coefficients and b a vector in $\{0, 1\}^m$. In this paper we study the polytope

$$P(M, b) := \text{conv}\{x \in \{0, 1\}^n \mid Mx \equiv b (\text{mod } 2)\},$$

i.e., the convex hull of the set of zero-one solutions of $Mx \equiv b(\text{mod } 2)$. A related polyhedron, called the *binary group polyhedron*, has been investigated by Gastou and Johnson [6]. It is defined as follows

$$P(mC_2, M, b) := \text{conv}\{x \in \{0, 1, 2, ...\}^n \mid Mx \equiv b(\text{mod } 2)\}.$$

$P(mC_2, M, b)$ is the *dominant* of $P(M, b)$, that is,

$$P(mC_2, M, b) = P(M, b) + \mathbb{R}^n_+.$$

It is easy to see that for $c \in \mathbb{R}^n$,

$$\min\{cx \mid x \in P(mC_2, M, b)\} = \min\{cx \mid x \in P(M, b)\}$$

if $c > 0$. But this relation does not hold if we wish to maximize the linear function cx. This, however, is the problem in which we are interested and which led us to study $P(M, b)$ in more detail.

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Namely, we are currently trying to develop practically efficient cutting plane algorithms for a number of real-world problems (e.g., determining the ground state of spin glasses at 0°K, or solving certain quadratic 0/1-problems) which can be phrased as $\max\{cx \mid x \in P(M, b)\}$, where M and b have particular properties. After investigation of these special cases it turned out that most of our results for these cases could be stated in the more general framework to be studied here. Moreover, almost all of our proofs became shorter and more elegant. So we decided to present our theoretical investigations in this framework.

Let us mention first a few similarities of and differences between $P(M, b)$ and $P(mC_2, M, b)$. Clearly, every vertex of $P(mC_2, M, b)$ is a vertex of $P(M, b)$, but not vice versa. If $ax \leqslant a_0$ defines a bounded facet of $P(mC_2, M, b)$ then it also defines a facet (in fact, the same) of $P(M, b)$, but $P(M, b)$ has other bounded facets.

We shall prove that $P(M, b)$ has a nice property that $P(mC_2, M, b)$ does not have: to characterize the facet defining inequalities of $P(M, b)$ it is enough to characterize the facets that contain a given vertex. Roughly speaking, the cones associated to each vertex are all the same.

This property enables us to use a result of Seymour to characterize the matrices M such that $P(M, b)$ is defined by the so-called *cocircuit inequalities.*

We shall characterize adjacency on $P(M, b)$, and we shall prove the Hirsch Conjecture for $P(M, b)$ in the case that M does not contain a certain minor. We shall assume familiarity with matroid theory. For an introduction to it see Welsh [11]. Given a set $F \subseteq E$ the *incidence vector* x^F of F is defined by

$$x_e^F = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{if } e \in E \setminus F. \end{cases}$$

The symmetric difference between F and G, $(F \setminus G) \cup (G \setminus F)$, will be denoted by $F \triangle G$.

Let us also recall some notions of the theory of polyhedra. If P is a polyhedron, the inequality $ax \leqslant \alpha$ is *valid* for P if every $x \in P$ satisfies it. The face induced by the valid inequality $ax \leqslant \alpha$ is $\{x \in P \mid ax = \alpha\}$. A face $F \neq \emptyset$ of P is called a *facet* of P if the dimension of F is equal to the dimension of P minus one.

If P is a polyhedron and

$$P = \{x \mid Ax = b, Cx \leqslant d\},$$

then $Ax = b$, $Cx \leqslant d$ is a *minimal linear system* defining P if and only if

(i) $\{x \mid Ax = b\}$ is the affine hull of P and the rows of A are linearly independent,

(ii) each inequality of $Cx \leqslant d$ induces a facet of P, and no two inequalities induce the same facet.

2. Basic Properties of $P(M, b)$ and the Relation to Binary Matroids

First, we shall prove that the polytope $P(M, b)$ can be obtained from $P(M, 0)$ by a simple transformation. For $x \in \mathbb{R}^n$ with $0 \leqslant x \leqslant 1$, and $y \in \{0, 1\}^n$, the vector $x \oplus y \in \mathbb{R}^n$ is defined by

$$(x \oplus y)_i := \begin{cases} 1 - x_i & \text{if } y_i = 1, \\ x_i & \text{if } y_i = 0. \end{cases}$$

Note that for 0/1-vectors x, y, the operation $x \oplus y$ is just componentwise additon modulo 2.

(2.1) Lemma. *Let $y \in \{0, 1\}^n$ such that $My \equiv b$ (mod 2).*

(a) $P(M, b) = \{x \oplus y \in \mathbb{R}^n \mid x \in P(M, 0)\}$,

(b) *$x \in \mathbb{R}^n$ is a vertex of $P(M, 0)$ iff $x \oplus y$ is a vertex of $P(M, b)$.*

(c) *$ax \leqslant \alpha$ is valid for (resp. defines a facet of) $P(M, 0)$ iff $a(x \oplus y) \leqslant \alpha$ is valid for (resp. defines a facet of) $P(M, b)$.*

Proof. (b) If x is a vertex of $P(M, 0)$ then $x \in \{0, 1\}^n$, and hence, $x \oplus y \in \{0, 1\}^n$. Moreover, it is easy to see that $x \oplus y \in P(M, b)$. This implies that $x \oplus y$ is a vertex of $P(M, b)$. Similarly, $x \oplus y$ is a vertex of $P(M, b)$ implies that x is a vertex of $P(M, 0)$.

(a) If $x \in P(M, 0)$ then $x = \sum \lambda_j x^j$, with $\lambda_j \geqslant 0$, x^j a vertex of $P(M, 0)$, for all j, and $\sum \lambda_j = 1$. This implies that

$$x \oplus y = \sum \lambda_j (x^j \oplus y).$$

Since $x^j \oplus y$ is a vertex of $P(M, b)$ for all j, by (b) we conclude $x \oplus y \in P(M, b)$.

Conversely, let us suppose that $z \in P(M, b)$, $z = \sum \lambda_j z^j$, with $\lambda_j \geqslant 0$, z^j a vertex of $P(M, b)$, for all j, and $\sum \lambda_j = 1$. By (b) $z^j \oplus y$ is a vertex of $P(M, 0)$, for all j. Set $x := \sum \lambda_j (z^j \oplus y)$; then $x \in P(M, 0)$ and $z = x \oplus y$.

(c) The equivalence of the validity of the two inequalities is clear. To prove that one of the inequalities defines a facet iff the other does, observe that the vectors $x_1, \ldots, x_k$ are affinely independent vertices of $P(M, 0)$ satisfying $ax = \alpha$ iff the vectors $x_1 \oplus y, \ldots, x_k \oplus y$ are affinely independent vertices of $P(M, b)$ satisfying $a(x \oplus y) = \alpha$. ▮

Lemma (2.1)(c) in particular implies that if $y \in P(M, b) \cap \{0, 1\}^n$ and if

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant \alpha_i \qquad \text{for} \quad i = 1, \ldots, k$$

is a system of inequalities for $P(M, 0)$ which is valid (nonredundant, complete) then the system of inequalities

$$\sum_{j, y_j = 0} a_{ij} x_j - \sum_{j, y_j = 1} a_{ij} x_j \leqslant \alpha_i - \sum a_{ij} y_j \qquad \text{for} \quad i = 1, \ldots, k$$

is valid (nonredundant, complete) for $P(M, b)$.

Lemma (2.1) shows that the polyhedra $P(M, 0)$ are the essential objects for the investigation of the polyhedra $P(M, b)$. Therefore we shall study only the polyhedra $P(M, 0)$ in the sequel.

The zero-one matrix M defines a matroid as follows. Let us denote the column index set of M by E. Consider E as the ground set of a matroid $\hat{M}$ in which a set $S \subseteq E$ is dependent if and only if the columns of M corresponding to S are linearly dependent in the m-dimensional vector space over $GF(2)$. This matroid $\hat{M}$ is binary.

A set $C \subseteq E$ in a binary matroid $\hat{M}$ is called a *cycle* if either $C = \emptyset$ or C is the disjoint union of circuits. It follows immediately from our definitions that the 0/1-solutions of $Mx \equiv 0 \pmod 2$ are the incidence vectors of the cycles of $\hat{M}$. Thus $P(M, 0)$ can be viewed as the convex hull of the incidence vectors of the cycles of $\hat{M}$. In fact, many different matrices may lead to one and the same binary matroid $\hat{M}$ and the same polyhedron $P(M, 0)$. All the results we state in the sequel are independent of the particular matrix M chosen to define $P(M, 0)$. All characterizations of $P(M, 0)$ can be stated (much more nicely) in terms of the associated binary matroid $\hat{M}$.

Therefore, from now on we take the matroid viewpoint. In the sequel M (instead of the notationally inconvenient $\hat{M}$) denotes a binary matroid with ground set E, and $P(M)$ (instead of the longer $P(M, 0)$) denotes the convex hull of the incidence vectors of the cycles of M.

Before going on let us remark that there is a third way to look at the subject we address. If A is a zero-one matrix, the set $\{x \in \{0, 1\}^n \mid Ax \equiv 0 \pmod 2\}$ is a linear subspace of $(GF(2))^n$. Of course, every subspace of $(GF(2))^n$ can be represented as the $GF(2)$-solutions of an equation $Ax = 0$, i.e., as the kernel of a linear mapping. So the problem we aim at is to describe the convex hull (considered in $\mathbb{R}^n$) of the linear (resp. affine) subspaces of $(GF(2))^n$.

Note that the set of cycles of M is a family of subsets of the ground set E closed under symmetric difference. So, if x^C and x^D are incidence vectors of cycles C and D of M, then $x^C \oplus x^D$ is the incidence vector of the cycle $C \triangle D$.

If M is graphic (M can e.g. be defined as described above by a matrix whose rows are the incidence vectors of all cuts of a graph G), then $P(M)$ is the convex hull of the incidence vectors of Eulerian subgraphs of G. In this case $P(M)$ can be obtained from matching theory, cf. [3–5, 9]. If M is cographic (the rows of a matrix defining M are, e.g., the incidence vectors of all cycles of a graph), then $P(M)$ is the cut polytope which has been studied in [2].

Let M^* denote the *dual matroid* of M; a *cocycle* of M is a cycle of M^*.

When $Z \subseteq E$, $M\backslash Z$ denotes the matroid obtained by *deleting* Z, M/Z is the matroid obtained by *contracting* Z. We will write $M\backslash e$ instead of $M\backslash\{e\}$. If C is a cycle and D a cocycle of M then $|C \cap D|$ is an even number (since M is binary).

3. Faces of $P(M)$ and the "Sum of Circuits Property"

In this section we show that the polytopes $P(M)$ are—in a sense to be made precise—highly symmetric; we introduce some valid inequalities and derive from a deep theorem of Seymour [10] that these inequalities suffice to describe $P(M)$ if M has the "sum of circuits property." We begin with a transformation theorem for faces of $P(M)$.

(3.1) Theorem. *If $ax \leqslant \alpha$ defines a face of $P(M)$ of dimension d, and C is a cycle of M, then the inequality $\bar{a}x \leqslant \bar{\alpha}$ also defines a face of $P(M)$ of dimension d, where*

$$\bar{a}_e := \begin{cases} a_e & \text{if } e \notin C, \\ -a_e & \text{if } e \in C, \end{cases}$$

and $\bar{\alpha} := \alpha - ax^C$.

Proof. First, we shall prove that $\bar{a}x \leqslant \bar{\alpha}$ is valid for $P(M)$. Let us suppose that B is a cycle of M such that $\bar{a}x^B > \bar{\alpha}$. This implies

$$\begin{aligned} ax^{C \triangle B} &= ax^{B \setminus C} + ax^{C \setminus B} = ax^{B \setminus C} + ax^C - ax^{B \cap C} \\ &= \bar{a}x^B + ax^C > \bar{\alpha} + ax^C = \alpha, \end{aligned}$$

which contradicts the validity of $ax \leqslant \alpha$.

By assumption, there are $d+1$ cycles $D_0, \ldots, D_d$ such that $ax^{D_i} = \alpha$, for $i = 0, \ldots, d$, and the vectors x^{D_i}, $i = 0, \ldots, d$, are affinely independent. Set

$$F_i := D_i \triangle C, \qquad i = 0, \ldots, d.$$

Then $\bar{a}x^{F_i} = \bar{a}x^{D_i \setminus C} + \bar{a}x^{C \setminus D_i} = ax^{D_i \setminus C} - ax^{C \setminus D_i} = \alpha - ax^{D_i \cap C} - (ax^C - ax^{D_i \cap C}) = \alpha - ax^C = \bar{\alpha}$.

Suppose the vectors x^{F_i} are affinely dependent. We may assume that $x^{F_0} = \sum_{i=1}^d \lambda_i x^{F_i}$ and $\sum_{i=1}^d \lambda_i = 1$. Then by Lemma (2.1) $x^{F_0} \oplus x^C = \sum_{i=1}^d \lambda_i (x^{F_i} \oplus x^C)$, and this implies $x^{D_0} = \sum_{i=1}^d \lambda_i x^{D_i}$, a contradiction.

The arguments above show that $\bar{a}x \leqslant \bar{\alpha}$ defines a face of dimension at least d. If the dimension was greater than d we could apply the same transformation to the inequality $\bar{a}x \leqslant \bar{\alpha}$, and this would imply that $ax \leqslant \alpha$ would define a face of dimension greater than d. ∎

From this we can derive a somewhat surpirsing symmetry property of vertices of $P(M)$.

(3.2) COROLLARY. *Let v, w be two vertices of $P(M)$, and let $\mathscr{F}(v, d)$ and $\mathscr{F}(w, d)$ be the sets of faces of dimension d that contain v and w, respectively. Then there exists a bijective mapping*

$$f: \mathscr{F}(v, d) \to \mathscr{F}(w, d).$$

Proof. The vertices v and w are incidence vectors of cycles of M, say V and W. Let F be a d-dimensional face containing v defined by $ax \leqslant \alpha$. Apply the transformation of Theorem (3.1) using $C := V \triangle W$. Then the set $\bar{F} := \{x \in P(M) \mid \bar{a}x = \bar{\alpha}\}$ is a face of $P(M)$ of dimension d containing w. It is easy to see that $F \mapsto \bar{F}$ is the desired bijection. ∎

This corollary shows that in order to describe $P(M)$ completely it is enough to know all the facets of $P(M)$ containing a given vertex. Since 0 is a vertex, it is sufficient to describe all facets of $P(M)$ containing 0, i.e., the "homogeneous" facets of $P(M)$. We will see that this property will help us to describe $P(M)$ completely for some binary matroids M.

Now let us look for inequalities which are valid with respect to $P(M)$. As $P(M)$ is contained in the unit hypercube we know that the *trivial inequalities*

$$0 \leqslant x_e \leqslant 1 \qquad \text{for all} \quad e \in E \tag{3.3}$$

are valid. Moreover, since M is binary we know that the cardinality of the intersection of a cycle and a cocycle is even. This implies that the *cocircuit inequalities*

$$x(F) - x(C \setminus F) \leqslant |F| - 1 \qquad \text{for all cocircuits } C \subseteq E \text{ and all } F \subseteq C,\ |F| \text{ odd} \tag{3.4}$$

are valid with respect to $P(M)$. (As usual we abbreviate the sum $\sum_{e \in F} x_e$ by $x(F)$.) A natural question to ask is: when do the inequalities (3.3) and

(3.4) suffice to describe $P(M)$? By Corollary (3.2) this is equivalent asking when the homogeneous among these inequalities contain all the facets containing 0, i.e., when is

$$\mathrm{CONE}(P(M)) := \{ y \in \mathbb{R}^E \mid y = \lambda x, \lambda \geqslant 0, x \in P(M) \}$$

defined by

$$x_e - x(C \setminus \{e\}) \leqslant 0 \qquad \text{for all cocircuits } C \subseteq E \text{ and all } e \in C,$$

$$-x_e \leqslant 0 \qquad \text{for all } e \in E.$$

This is called the "sum of circuits property" by Seymour [10]. Actually Seymour proved that M has this property if and only if M has no F_7^*, R_{10} or $M(K_5)^*$ minor. ($M(K_5)^*$ is the cographic matroid of the complete graph K_5, F_7^* is the dual Fano matroid, see Section 4, and R_{10} is the binary matroid associated with the 5×10 matrix whose columns are the ten 0/1-vectors with 3 ones and 2 zeros. The matroid R_{10} is regular but neither graphic nor cographic.) Thus, we can state

(3.5) THEOREM. *The cycle polytope $P(M)$ of a binary matroid M is equal to the polytope defined by the inequality systems* (3.3) *and* (3.4) *if and only if M has no F_7^*, R_{10}, $M(K_5)^*$ minor.*

Note that this theorem provides a complete characterization of the Eulerian subgraph polytope of any graph and of the polytope of cuts of a graph not contractible to K_5, see (4.23) and (4.24) for a more concise description.

4. DIMENSION AND FACETS OF $P(M)$

In this section we shall study the problem of characterizing linear inequalities which, for a given binary matroid M, define facets of $P(M)$. To be able to do this we have to know the dimension of $P(M)$.

Let us first give one more definition. If $\{e, f\}$ is a cocircuit of M we say that e and f are *coparallel* (e.g., if M is graphic, two elements are coparallel if they form a cut of size two in the corresponding graph; if M is cographic, then coparallel elements correspond to parallel edges). Recall that a *coloop* of M is a loop of M^* (e.g., if M is graphic then a coloop is a bridge in the corresponding graph, if M is cographic a coolop is an ordinary loop in the graph). A *coparallel class* of M is a maximal subset $F \subseteq E$ which contains no coloops, so that every two distinct members of F are coparallel.

(4.1) THEOREM. *The dimension of $P(M)$ is equal to the number of coparallel classes of M.*

Proof. First, observe that every vector $x \in P(M)$ satisfies the following systems of equations

$$x_e = 0 \qquad \text{for each coloop } e \in E, \tag{4.2}$$

$$x_{e_0} - x_{e_i} = 0 \qquad \text{for each coparallel class, } F = \{e_0, e_1, ..., e_k\} \text{ with } k \geq 1 \text{ and } i = 1, ..., k. \tag{4.3}$$

Clearly Eqs. (4.2) and (4.3) are linearly independent which implies that the dimension of $P(M)$ is at most the number of coparallel classes of M.

Now suppose that $ax = \alpha$ is an equation satisfied by all $x \in P(M)$. Since $0 \in P(M)$ we must have $\alpha = 0$, and moreover, by adding appropriate linear combinations of Eqs. (4.2) and (4.3) we may assume that

$$a_e = 0 \qquad \text{if } e \text{ is a coloop and}$$

$$a_{e_i} = 0 \qquad \text{for } i = 1, ..., k \text{ if } F = \{e_0, e_1, ..., e_k\} \text{ is a coparallel class.}$$

This implies that we can restrict our attention to the matroid $\bar{M}$ obtained from M by:

(i) deleting coloops,

(ii) contracting $\{e_1, ..., e_k\}$ for each coparallel class $F = \{e_0, e_1, ..., e_k\}$.

$\bar{M}$ has no coloops and no two elements which are coparallel.

Seymour [10, Theorem (3.2)] proved that if $\bar{M}$ has no coloops then there is a number $r > 0$ and a list of circuits L such that every element of $\bar{M}$ is in precisely r circuits.

By assumption we have that $ax^C = 0$ for each $C \in L$. If we sum up these $|L|$ equations we obtain

$$r \sum_{e \in E(\bar{M})} a_e = 0,$$

and hence

$$\sum_{e \in E(\bar{M})} a_e = 0. \tag{4.4}$$

Now, pick any $f \in E(\bar{M})$. $\bar{M} \backslash f$ has no coloop (because $\bar{M}$ has no coparallel elements). For the same reasons as above we conclude that

$$\sum_{e \in E(\bar{M} \backslash f)} a_e = 0. \tag{4.5}$$

Equations (4.4) and (4.5) imply that $a_f = 0$, i.e., a is the zero vector. Hence the dimension of $P(M)$ is equal to the number of elements of $\bar{M}$. This proves our claim. ∎

In fact, Theorem (4.1) also follows from the proof of Corollary (3.3) in [10]. The proof of the preceding theorem also shows

(4.6) COROLLARY. *If $P(M) \subseteq \{x \mid bx = 0\}$ then for each coparallel class $F = \{e_0, e_1, ..., e_k\}$ of M we have*

$$\sum_{i=0}^{k} b_{e_i} = 0.$$

It follows from the proof of Theorem (4.1) that the equation system (4.2) and (4.3) is a minimal system of equations defining the affine hull of $P(M)$. In what follows we will exhibit some classes of facet-defining inequalities of $P(M)$. If

$$ax \leqslant \alpha \tag{4.7}$$

defines a face F of $P(M)$ and $\bar{a}x \leqslant \bar{\alpha}$ is obtained from (4.7) by adding a linear combination of Eqs. (4.2) and (4.3), then $\bar{a}x \leqslant \bar{\alpha}$ induces the same face F. Thus we can assume that $\bar{a}_e = 0$ for each coloop e of M, and for each coparallel class $F = \{e_0, e_1, ..., e_k\}$ we have $\bar{a}_{e_i} = 0$ for $i = 1, ..., k$. Then we can restrict our attention to facet-defining inequalities of $P(\bar{M})$, where $\bar{M}$ is the matroid defined in the proof of Theorem (4.1). The same inequalities will define facets of $P(M)$. Let us first study the trivial inequalities (3.3).

(4.8) THEOREM. *Let $\bar{M}$ be a binary matroid without coloops and without coparallel elements. If $f \in E$ does not belong to a cotriangle (a cocircuit with three elements), then the inequality*

$$x_f \geqslant 0$$

defines a facet of $P(\bar{M})$.

Proof. Suppose $f \in E$ does not belong to a cotriangle of $\bar{M}$. Clearly, the set $F := \{x \in P(\bar{M}) \mid x_f = 0\}$ and its projection $P(\bar{M} \setminus f)$ have the same dimension. Since f does not belong to a cotriangle, $\bar{M} \setminus f$ has no coloops and no coparallel elements. Thus, by Theorem (4.1) $P(\bar{M} \setminus f)$ has dimension $|E| - 1$, which proves our claim. ∎

(4.9) COROLLARY. *Let f be defined as in Theorem* (3.2); *then the inequality*

$$x_f \leqslant 1$$

defines a facet of $P(\bar{M})$.

Proof. Pick a cycle that contains f (such a cycle exists, since $\bar{M}$ has no coloops) and apply Theorem (3.1) to the inequality $x_f \geqslant 0$. ∎

Now, we shall study the inequalities (3.4) associated with cocircuits. For a cotriangle $T = \{e, f, g\}$, formula (3.4) gives the following four inequalities valid for $P(M)$.

$$x_e - x_f - x_g \leqslant 0, \tag{4.10}$$

$$-x_e + x_f - x_g \leqslant 0, \tag{4.11}$$

$$-x_e - x_f + x_g \leqslant 0, \tag{4.12}$$

$$x_e + x_f + x_g \leqslant 2. \tag{4.13}$$

Let us note that (4.11) and (4.12) imply $x_e \geqslant 0$, and (4.10) and (4.13) imply $x_e \leqslant 1$, so if e is in a cotriangle the inequalities $0 \leqslant x_e \leqslant 1$ do not define facets of $P(M)$.

But the inequalities (4.10)–(4.13) also do not always define facets. For instance, the binary matroid associated with the following matrix

$$A := \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is the dual Fano Matroid F_7^*. The incidence vectors of the cycles of F_7^* are the columns of

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The polytope $P(F_7^*)$ is full-dimensional by Theorem (4.1) because the dual of F_7^*, the Fano Matroid F_7, contains no loop and no circuit of cardinality two. $P(F_7^*)$ has eight vertices which form an affinely independent set. Thus $P(F_7^*)$ is a 7-dimensional simplex in $\mathbb{R}^7$.

The set $\{2, 3, 4\}$ is a cotriangle of F_7^*, but as one can easily see, there are only 6 cycles in F_7^* whose incidence vectors satisfy

$$x_2 - x_3 - x_4 \leqslant 0$$

with equality. Hence this cotriangle inequality does not define a facet of $P(F_7^*)$. Note that $P(F_7^*)$ has only one facet which does not contain the origin. This facet is defined by

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7 \leqslant 4. \tag{4.14}$$

All the other facet-defining inequalities of $P(F_7^*)$ are obtained by applying Theorem (3.1) to the inequality (4.14). Thus, a minimal system that defines $P(F_7^*)$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \leqslant \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We shall prove that if M has no F_7^* minor then the cotriangle inequalities (4.10)–(4.13) define facets of $P(M)$.

To shorten our proofs the following notation will be convenient. If M is a binary matroid without coloops, $S \subseteq E$, and $h \in E \setminus S$ then $C(h, S)$ denotes the coparallel class of h in $M \setminus S$. The following lemmas will be used.

(4.15) LEMMA. *Let $\bar{M}$ be a binary matroid without coloops and without coparallel elements.*

(a) $|C(h, \{f\})| \leqslant 2$ *for all* $f, h \in E$.

(b) $C(h, \{f\}) \cap C(h, \{g\}) = \{h\}$ *for all different* $f, g, h \in E$.

(c) *If, in addition,* $\bar{M}$ *has no* F_7^* *minor,* $T = \{e, f, g\}$ *is a cotriangle, and* $|C(h, \{f\})| = |C(h, \{g\})| = 2$ *then*

$$C(h, T) = C(h, \{f\}) \cup C(h, \{g\}) \qquad \textit{for all} \quad h \in E \setminus T.$$

Proof. (a) Suppose $i, j, h \in C(h, \{f\})$, then $T_1 := \{i, h, f\}$ and $T_2 := \{j, h, f\}$ are cotriangles of $\bar{M}$, and so $T_1 \triangle T_2 = \{i, j\}$ is a cocycle, i.e., the elements i and j are coparallel or coloops, which contradicts our assumption.

(b) follows in the same way.

(c) Suppose $C(h, \{f\}) = \{h, i\}$, $C(h, \{g\}) = \{h, j\}$ and $k \in C(h, T)$ with i, j, k different. Then $T_1 := \{i, h, f\}$, $T_2 := \{j, h, g\}$ and $T = \{e, f, g\}$

are cotriangles of $\bar{M}$. Since $k \in C(h, T)$ there must be a cocircuit of $\bar{M}$ containing h, k and some members of T. We have to discuss several cases.

Assume first that $S = \{e, h, k\}$ is a cocircuit of $\bar{M}$. Then consider the 4×7 matrix whose rows are formed by the incidence vectors of T, T_1, T_2 and $S \triangle T$. This matrix is the matrix A (the sequence of column indices is h, g, f, e, i, j, k) defined above which gives the matroid F_7^*, i.e., $\bar{M}$ contains a F_7^* minor, which is a contradiction.

Second, assume that $S' = \{e, f, h, k\}$ is a cocircuit of $\bar{M}$. Then $T \triangle S' \triangle T_2 = \{j, k\}$, i.e., $\{j, k\}$ is a cocycle which implies that j, k are either coparallel or coloops, a contradiction. The other two cases follow similarly. ∎

(4.16) Lemma. *Let M be a binary matroid. If $C = \{e_1, e_2, \ldots, e_k\}$ is a cocircuit of M, there exist cycles $D_2, \ldots, D_k$ of M such that*

$$C \cap D_i = \{e_1, e_i\} \qquad \textit{for} \quad i = 2, \ldots, k.$$

Proof. If $k = 2$, the assertion is true, otherwise e_1 or e_2 would be a coloop contained in C. Let us proceed by induction and suppose that the statement is true for $2 \leqslant k \leqslant p$. If the cocircuit C of M has $p + 1$ elements, $C \setminus \{e_{p+1}\}$ is a cocircuit of $M \setminus \{e_{p+1}\}$. Then by the induction hypothesis there exist cycles $D_2, \ldots, D_p$ of $M \setminus \{e_{p+1}\}$ and thus of M such that $C \cap D_i = \{e_1, e_i\}$.

Since e_{p+1} is not a coloop of M there is a circuit F of M such that $e_{p+1} \in F$ and $|F \cap C|$ is a positive even number. If $F \cap C = \{e_1, e_2, \ldots, e_{2l+1}, e_{p+1}\}$, set

$$F' := F \triangle D_2 \triangle \cdots \triangle D_{2l+1}.$$

If $F \cap C = \{e_2, e_3, \ldots, e_{2l}, e_{p+1}\}$, set

$$F' := F \triangle D_2 \triangle \cdots \triangle D_{2l}.$$

In both cases F' is a cycle of M satisfying

$$F' \cap C = \{e_1, e_{p+1}\}. \quad ∎$$

(4.17) Lemma. *If $C = \{e_1, e_2, \ldots, e_k\}$ is a cocircuit of M, then there exists a cycle D such that $D \cap C = \{e_1, e_2, \ldots, e_{2l}\}$, for every l with $1 \leqslant l \leqslant k/2$.*

Proof. By Lemma (4.16) there exist cycles D_i such that $D_i \cap C = \{e_1, e_i\}$, $2 \leqslant i \leqslant 2l$. Set

$$D = D_2 \triangle D_3 \triangle \cdots \triangle D_{2l}. \quad ∎$$

(4.18) THEOREM. *Let $\bar{M}$ be a binary matroid without coloops, without coparallel elements and without F_7^* minor. If $T=\{e, f, g\}$ is a cocircuit of $\bar{M}$ then inequality* (4.10)

$$x_e - x_f - x_g \leqslant 0$$

defines a facet of $P(\bar{M})$.

Proof. Let us denote inequality (4.10) by $ax \leqslant 0$, and let us assume that

$$\{x \in P(\bar{M}) \mid ax = 0\} \subseteq \{x \in P(\bar{M}) \mid bx = \beta\}$$

for a facet-defining inequality $bx \leqslant \beta$. Since $0 \in \{x \in P(\bar{M}) \mid ax = 0\}$, we have $\beta = 0$. Pick any $h \in E \backslash T$. By Lemma (4.15)(a), the coparallel classes of h in $\bar{M} \backslash f$ and $\bar{M} \backslash g$ have size at most two. We have to discuss several cases. Let us first suppose that both coparallel classes have size two, say

$$C(h, \{f\}) = \{h, i\},$$
$$C(h, \{g\}) = \{h, j\}.$$

Then Lemma (4.15)(c) implies that

$$C(h, T) = \{h, i, j\}.$$

Let a' (resp. b') denote the vectors in $\mathbb{R}^{E \backslash f}$ which arise from a (resp. b) by deleting the component corresponding to f. Clearly, the inequalities $a'x \leqslant 0$ and $b'x \leqslant 0$ are valid for $P(\bar{M} \backslash f)$. Moreover, $\{x \in P(\bar{M} \backslash f) \mid a'x = 0\} \subseteq \{x \in P(\bar{M} \backslash f) \mid b'x = 0\}$. But e and g form a coparallel class in $\bar{M} \backslash f$, and so $a'x = 0$ and thus $b'x = 0$ are satisfied by all $x \in P(\bar{M} \backslash f)$ by Theorem (4.1).

Corollary (4.6) now implies

$$b_h + b_i = 0.$$

And using the same arguments we can conclude

$$b_h + b_j = 0,$$
$$b_h + b_i + b_j = 0.$$

The only solution of these three equations is 0, hence $b_h = 0$.

If one of the two coparallel classes $C(h, \{f\})$ and $C(h, \{g\})$ has size one, $b_h = 0$ follows immediately. Thus we can conclude

$$b_h = 0 \qquad \text{for all} \quad h \in E \backslash T.$$

By Lemma (4.16) there are a cycle C_1 with $T \cap C_1 = \{e, f\}$ and a cycle C_2 with $T \cap C_2 = \{e, g\}$, so

$$bx^{C_1} = b_e + b_f = 0,$$
$$bx^{C_2} = b_e + b_g = 0,$$

and thus $b_e = -b_f = -b_g$ which implies $b = \alpha a$. Since $bx \leqslant 0$ is valid, $\alpha > 0$. This completes the proof. ▮

(4.19) COROLLARY. *Let T be defined as in Theorem* (4.18); *then the inequality*

$$x(T) \leqslant 2$$

defines a facet of $P(\bar{M})$.

Proof. Apply Theorem (3.1) to the inequality $x_e - x_f - x_g \leqslant 0$, with a cycle that contains $\{f, g\}$. Clearly such a cycle exists, since both f and g are contained in circuits of $\bar{M}$. ▮

Given a cocircuit C then $h \in E \backslash C$ is called a *chord* of C if there exist two cocircuits D and F such that $D \cap F = \{h\}$, and $D \triangle F = C$.

(4.20) THEOREM. *If the binary matroid M has no F_7^* minor, and $C = \{e_1, e_2, ..., e_k\}$, $k \geqslant 3$, is a cocircuit without chord, then the inequality*

$$x_{e_1} - x(\{e_2, ..., e_k\}) \leqslant 0$$

defines a facet of $P(M)$.

Proof. As for Theorem (4.1) it suffices to prove the theorem for the matroid $\bar{M}$ which has no coloops and no coparallel elements. We denote the ground set of $\bar{M}$ by E. Let us use induction. By Theorem (4.18) the statement is true if C has three elements. We assume that the theorem is proved when C has $p \geqslant 3$ elements, and we will study the case when C has $p+1$ elements. Let us denote the inequality $x_{e_1} - x(\{e_2, ..., e_{p+1}\}) \leqslant 0$ by $ax \leqslant 0$. As in the preceding theorems we suppose that

$$\{x \in P(\bar{M}) \mid ax = 0\} \subseteq \{x \in P(\bar{M}) \mid bx = 0\},$$

for a facet defining inequality $bx \leqslant 0$. First, we shall prove that $b_h = 0$ for all $h \in E \backslash C$.

Since C has no chord, the cocircuits $C \backslash \{e_{p+1}\}$ and $C \backslash \{e_p\}$ have no chord in $\bar{M} \backslash e_{p+1}$ and in $\bar{M} \backslash e_p$, respectively. Now consider the coparallel classes of h in these matroids. Suppose $\{h, i\} = C(h, \{e_p\})$, and $\{h, j\} =$

$C(h, \{e_{p+1}\})$. Since h is no chord of C, we have $i, j \in E \setminus C$. By the induction hypothesis

$$a'x := x_{e_1} - x(C \setminus \{e_p\}) \leqslant 0$$

and

$$a''x := x_{e_1} - x(C \setminus \{e_{p+1}\}) \leqslant 0,$$

define facets F_p and F_{p+1} of $P(\bar{M} \setminus e_p)$ and of $P(\bar{M} \setminus e_{p+1})$, respectively. Our assumption on b implies $F_s \subseteq \{x \in P(\bar{M} \setminus e_s) \mid \sum_{e \neq e_s} b_e x_e = 0\}$, $s = p, p+1$.

Let b' (resp. b'') denote the vectors obtained from b by deleting component e_p (resp. e_{p+1}). Since $a'x \leqslant 0$ defines a facet of $P(\bar{M} \setminus e_p)$ we can conclude that $b' = \rho a' + \sum \lambda_q c_q$ with $\rho \geqslant 0$, $\lambda_q \in \mathbb{R}$ where c_q are the left-hand sides of Eqs. (4.3). (Note that $\bar{M} \setminus e_p$ has no coloops.) Since $\{h, i\}$ is a coparallel class in $\bar{M} \setminus e_p$ and $a_h = a_i = 0$ we obtain

$$b_h = b'_h = a_h + \lambda,$$

$$b_i = b'_i = a_i - \lambda,$$

for some λ, and hence

$$b_h + b_i = 0.$$

Similarly, we get

$$b_h + b_j = 0.$$

If the coparallel class of j in $M \setminus e_p$ or the coparallel class of i in $M \setminus e_{p+1}$ has size one we immediately obtain from the arguments given above $b_j = 0$ or $b_i = 0$, and hence $b_h = 0$. Now suppose $C(j, \{e_p\}) = \{j, k\}$ and $C(i, \{e_{p+1}\}) = \{i, k'\}$. This implies that $T_1 = \{e_p, h, i\}$, $T_2 = \{e_p, j, k\}$, $T_3 = \{e_{p+1}, h, j\}$ and $T_4 = \{e_{p+1}, i, k'\}$ are cotriangles, and hence $T_1 \triangle T_2 \triangle T_3 \triangle T_4 = \{k, k'\}$ is a cocycle which is impossible, i.e., at least one of the sets $C(j, \{e_p\})$, $C(i, \{e_{p+1}\})$ must have size one. Therefore, we can conclude

$$b_h = 0 \qquad \text{for all} \quad h \in E \setminus C.$$

By Lemma (4.16) there are cycles $D_2, ..., D_p$ such that $D_i \cap C = \{e_1, e_i\}$. From $bx^{D_i} = 0$, for $i = 2, ..., p$, we conclude

$$b = \alpha a.$$

Clearly $\alpha > 0$, because $bx \leqslant 0$ is valid. Our proof is complete. ∎

On the other hand, if C is a cocircuit, and if there exist two cocircuits D and F such that $D \cap F = \{h\}$ and $D \triangle F = C$, then—assuming $e \in D \setminus \{h\}$—the inequality

$$x_e - x(C \setminus \{e\}) \leqslant 0$$

is the sum of

$$x_e - x(D \setminus \{e\}) \leqslant 0$$

and

$$x_h - x(F \setminus \{h\}) \leqslant 0.$$

So $x_e - x(C \setminus \{e\}) \leqslant 0$ does not define a facet if the cocircuit C has a chord.

(4.21) COROLLARY. *Under the hypotheses of Theorem* (4.20) *the inequality*

$$x(F) - x(C \setminus F) \leqslant |F| - 1, \; F \subseteq C, \; |F| \text{ odd},$$

defines a facet of $P(M)$.

Proof. Apply Theorem (3.1) with a cycle D such that $D \cap C = F \setminus \{e\}$, to the inequality

$$x_e - x(C \setminus \{e\}) \leqslant 0. \quad \blacksquare$$

With the results about facets of $P(M)$ shown above we can strengthen Theorem (3.5) as

(4.22) THEOREM. *The system*

(a) $x_e = 0$ *for each coloop* $e \in E$,

(b) $x_{e_0} - x_{e_i} = 0$ *for each coparallel class* $F = \{e_0, e_1, ..., e_k\}$ *and each* $i \in \{1, ..., k\}$,

(c) $0 \leqslant x_e \leqslant 1$ *for each* $e \in E(\bar{M})$ *such that* e *does not belong to a cotriangle*,

(d) $x(F) - x(C \setminus F) \leqslant |F| - 1$ *for each cocircuit* C *of* $\bar{M}$ *with no chord and each* $F \subseteq C$, $|F|$ *odd*,

is a minimal system that defines $P(M)$ *if and only if* M *has no* F_7^*, R_{10}, $M(K_5)^*$ *minor.*

From Theorem (4.22) one can derive the following known special cases, see for instance Schrijver [9], Barahona and Mahjoub [2]. (Recall that an

Eulerian subgraph of a graph is a subgraph (not necessarily connected) in which each node has even degree.)

(4.23) COROLLARY. *Let $G=(V, E)$ be a graph, let E_3 be the edges not contained in a cut of size at most 3 and let E' be a maximal set of edges not containing bridges or cuts of size two. Then the convex hull of the incidence vectors of the edge sets of the Eulerian subgraphs of G is given by*

(a) $x_e = 0$ *for each bridge* $e \in E$,

(b) $x_e - x_f = 0$ *for each minimal cut* $\{e, f\}$ *of size two*,

(c) $0 \leqslant x_e \leqslant 1$ *for each* $e \in E \setminus E_3$,

(d) $x(F) - x(C \setminus F) \leqslant |F| - 1$ *for each minimal cut* $C \subseteq E'$ *with no chord and each* $F \subseteq C$, $|F|$ *odd.*

Moreover, the system above is nonredundant.

(4.24) COROLLARY. *Let $G=(V, E)$ be a graph not contractible to K_5. Let E_3 be the edges not contained in a cycle of size at most 3, and let E' be a maximal subset of E which does neither contain loops nor parallel edges. The convex hull of the incidence vectors of the cuts of G is given by*

(a) $x_e = 0$ *for each loop* $e \in E$,

(b) $x_e - x_f = 0$ *for each pair* $\{e, f\}$ *of parallel edges*,

(c) $0 \leqslant x_e \leqslant 1$ *for each* $e \in E \setminus E_3$,

(d) $x(F) - x(C \setminus F) \leqslant |F| - 1$ *for each cycle* $C \subseteq E'$ *with no chord and each* $F \subseteq C$, $|F|$ *odd.*

Moreover, the system above is nonredundant.

5. ADJACENCY AND THE HIRSCH CONJECTURE

We shall now study adjacency on $P(M)$, give an upper bound on the diameter of $P(M)$, and verify the Hirsch conjecture of $P(M)$ for binary matroids without F_7^* minor.

Giles [7] has given a characterization of adjacency of vertices of the Chinese Postman Polyhedron, which is a special case of the binary group polyhedron $P(mC_2, M, b)$. This characterization applies also to the latter, as was shown by Gastou and Johnson [6]. Moreover, the same criterion also describes adjacency of vertices of $P(M)$, see also [2].

(5.1) THEOREM. *Two different vertices of $P(M)$ are adjacent on $P(M)$ if and only if the symmetric difference of their supports is a circuit of M.*

Proof. Let x^A and x^B be two vertices of $P(M)$, i.e., A and B are cycles of M. Suppose first that $A \triangle B$ is not a circuit. Since $A \triangle B \neq \emptyset$ is dependent in M, $A \triangle B$ contains a circuit, say C. By elementary calculation we get

$$\tfrac{1}{2}x^A + \tfrac{1}{2}x^B = \tfrac{1}{2}x^{A \triangle C} + \tfrac{1}{2}x^{B \triangle C}.$$

Since $C \neq A \triangle B$, the incidence vectors of $A \triangle C$ and $B \triangle C$ are different from those of A and B. Thus x^A and x^B are not adjacent.

Now suppose that $A \triangle B$ is a circuit. We shall construct an objective function which is maximized by x^A and x^B but by no other vertex of $P(M)$. Define $c \in \mathbb{R}^E$ by

$$c_e := \begin{cases} 1 & \text{if } e \in A \cap B \\ 0 & \text{if } e \in A \triangle B \\ -1 & \text{if } e \notin A \cup B \end{cases}$$

for all $e \in E$. Clearly

$$\max\{cx \mid x \in P(M)\} = |A \cap B|,$$

and the maximum is attained at x^A and x^B. Let x^D be any vertex which gives this maximum. Then clearly $A \cap B \subseteq D \subseteq A \cup B$. Since $D \triangle A$ is a cycle, $D \triangle A \subseteq A \triangle B$, and $A \triangle B$ is a circuit we must have that $D \triangle A = B \triangle A$ or $D \triangle A = \emptyset$, and thus $D = B$ or $D = A$ holds. ▌

Given a polytope P, we can associate a graph $G(P)$ with P such that every vertex of P corresponds to a node of $G(P)$, and between two nodes of $G(P)$ we put an edge if the corresponding vertices are adjacent on P. The *distance* between two vertices of P is the cardinality of the shortest path between the corresponding nodes in $G(P)$.

(5.2) THEOREM. *Let A and B be two cycles of M, then the distance from x^A to x^B on $P(M)$ is bounded from above by the number of circuits contained in $A \triangle B$.*

Proof. By induction on the number k of circuits contained in $A \triangle B$. By the theorem above our statement is true for $k = 0, 1$. Suppose it is true for $k \geqslant 1$ and let $A \triangle B = C_1 \cup \cdots \cup C_{k+1}$ be the disjoint union of $k+1$ circuits. $A \triangle C_1$ is adjacent to A, and we have $(A \triangle C_1) \triangle B = C_2 \cup \cdots \cup C_{k+1}$. Then the distance between A and $A \triangle C_1$ is 1, and $A \triangle C_1$ has a distance of at most k to B; thus the distance between x^A and x^B is at most $k+1$. ▌

(5.3) COROLLARY. *The diameter of $P(M)$ is at most $|E|$.*

This bound can be achieved. If M is defined by a matrix consisting of one row with only zeros (i.e., all singletons are loops of M) then $P(M)$ is a hypercube in $\mathbb{R}^E$, and thus $P(M)$ has diameter $|E|$.

It is of course more natural to assume that the matrix from which M is derived has no zero column (i.e., that M has no loops). In such a case every circuit has at least two elements, so the diameter is at most $|E|/2$. Again, this bound can be achieved. Consider the matrix $[I, I]$, (two identity matrices); the distance from the origin to the all-ones vector in the corresponding polytope $P(M)$ is $|E|/2$.

The well-known *Hirsch conjecture* which is related to the diameter (and thus to lower bounds for the number of iterations of edge-following *LP*-algorithms like the Simplex method) states that every d-dimensional polyhedron P with k facets has diameter at most $k-d$. Let us say that P has the *Hirsch property* if the Hirsch conjecture is true for P.

(5.4) THEOREM. *If the binary matroid M has no F_7^* minor then $P(M)$ has the Hirsch property.*

Proof. Let us work again with the matroid $\bar{M}$ defined in the proof of (4.1). $P(M)$ and $P(\bar{M})$ have the same diameter and the same number of facets. $E(\bar{M})$ can be partitioned into E_1 and E_2, where E_1 consists of the elements that belong to a cotriangle. Let $T_1, \ldots, T_p$ be the cotriangles of $\bar{M}$. $P(\bar{M})$ has at least

$$4p + 2\,|E_2|$$

facets (cotriangle and trivial inequalities). Since $p \geqslant |E_1|/3$,

$$4p + 2\,|E_2| - \dim(P(\bar{M})) \geqslant |E_1|/3 + |E_2|.$$

On the other hand, by Theorem (5.2) a bound for the diameter of $P(\bar{M})$ is the maximum number of circuits that can be contained in a cycle. This number is at most

$$|E_1|/3 + |E_2|.$$

That finishes our proof. ∎

In fact, $P(F_7^*)$ also has the Hirsch property, but we do not know sufficiently many facets for the cycle polytopes of binary matroids with F_7^* minor to draw the same conclusion.

6. RELATIONS BETWEEN $P(M, b)$ AND $P(mC_2, M, b)$

In this section let M denote again an $m \times n$-matrix with zero-one coefficients, b a vector in $\{0, 1\}^m$, and let E denote the set of column indices of

M. Since $P(mC_2, M, b) = P(M, b) + \mathbb{R}^n_+$, it is natural to see whether an inequality inducing a facet of one of these polytopes also induces a facet of the other. Gomory [8] has shown that all facet-defining inequalities of $P(mC_2, M, b)$ can be written as nonnegativity constraints

$$x_e \geqslant 0, \qquad e \in E,$$

or as

$$ax \geqslant 1$$

with $a \geqslant 0$. Since $P(mC_2, M, b)$ is always full dimensional, these representations of facets are unique up to multiplication by positive constants. There are various difficulties in exploring the facet relation between the two polyhedra. $P(M, b)$ may have any dimension between -1 and $|E|$, while $P(mC_2, M, b)$ is always empty or full dimensional. Since $P(M, b) \subseteq P(mC_2, M, b)$ it is clear that every inequality valid for $P(mC_2, M, b)$ is also valid for $P(M, b)$. Of course, this does not hold the other way around. But it is easy to see that every valid inequality $ax \geqslant \alpha$ for $P(M, b)$ with $a \geqslant 0$ is valid for $P(mC_2, M, b)$. Now, we will explore the relation between the faces induced by these inequalities.

(6.1) THEOREM. (a) *Let $ax \geqslant 1$ define a facet of $P(mC_2, M, b)$; then $F = \{x \in P(M, b) \mid ax = 1\}$ is a face of $P(M, b)$ with dimension at least $|E| - 1$ minus the number of* 0*-coefficients of a.*

(b) *If $ax \geqslant 1$, $a \geqslant 0$, defines a facet of $P(M, b)$ then $G := \{x \in P(mC_2, M, b) \mid ax = 1\}$ is a face of $P(mC_2, M, b)$ of dimension at least* dim $P(M, b) - 1$.

Proof. (a) $P(mC_2, M, b)$ is full dimensional; thus there are $n = |E|$ affinely independent points in $P(mC_2, M, b)$ spanning the facet defined by $ax \geqslant 1$. Since $P(mC_2, M, b)$ is pointed, we may choose such a set of points in the following form

$$v_1, ..., v_k, v_1 + e_{k+1}, ..., v_1 + e_n,$$

where $v_1, ..., v_k$, $k \geqslant 1$, are vertices of $P(mC_2, M, b)$ and $e_{k+1}, ..., e_n$ are in the recession cone (i.e., in $\mathbb{R}^n_+$). Moreover the number k should be as large as possible. We clearly have in this case

$$av_i = 1, \qquad i = 1, ..., k,$$

$$ae_i = 0, \qquad i = k + 1, ..., n.$$

Since every vertex of $P(mC_2, M, b)$ is also a vertex of $P(M, b)$, the dimension of F is therefore at least $k - 1$. Since $a \geqslant 0$, $e_i \geqslant 0$, and $ae_i = 0$, we necessarily have that the support of a and the support of e_i have an empty

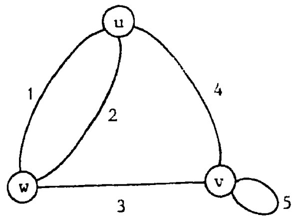

FIGURE 1

intersection. Thus the vectors e_i span a space that is contained in $\{x \in \mathbb{R}^n \mid x_e = 0 \text{ for all } e \in E \text{ with } a_e = 0\}$. Therefore the assertion follows.

(b) As remarked above $ax \geqslant 1$ is valid for $P(mC_2, M, b)$ and each vertex of $P(M, b)$ contained in the face of $P(M, b)$ defined by this inequality is contained in G. This implies the statement. ∎

The statements made in (a) and (b) are in a sense best possible. We know of examples where the dimensions of the faces F resp. G attain exactly the lower bound.

Finally, we would like to give an example showing how one can apply the results described before and also showing some differences between $P(M, b)$ and $P(mC_2, M, b)$.

Let G be the graph with 3 nodes and 5 edges shown in Fig. 1. Consider the polytope

$$\begin{aligned} P(M, b) := \operatorname{conv}\{x \in 0, 1\}^5 \mid x_1 + x_2 + x_3 \quad &\equiv 0 \pmod 2, \\ x_1 + x_2 \quad + x_4 &\equiv 1 \pmod 2, \\ x_3 + x_4 &\equiv 1 \pmod 2\}. \end{aligned}$$

$P(M, \mathrm{b})$ has the following 8 vertices

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	0	0	1	1	0	0	1	1
2	0	0	0	0	1	1	1	1
3	0	0	1	1	1	1	0	0
4	1	1	0	0	0	0	1	1
5	0	1	0	1	0	1	0	1

In graph theoretic terms, these eight vertices correspond to the T-joins of G for $T = \{u, v\}$. Now transform $P(M, b)$ into $P(M, 0)$ as described in

Lemma (2.1). The vertices of $P(M, 0)$ are the points $v_i \oplus v_1$, $i = 1, \ldots, 8$. These points correspond to the edge sets of the Eulerian subgraphs of G. By Corollary (4.23) $P(M, 0)$ is defined nonredundantly by the system

$$x_3 - x_4 = 0, \tag{6.2}$$

$$0 \leqslant x_5 \leqslant 1, \tag{6.3}$$

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\leqslant 2, \\ x_1 - x_2 - x_3 &\leqslant 0, \\ -x_1 + x_2 - x_3 &\leqslant 0, \\ -x_1 - x_2 + x_3 &\leqslant 0. \end{aligned}\right\} \tag{6.4}$$

Now we have to transform the inequalities above to get a description of $P(M, b)$. If $ax \leqslant \alpha$ is valid (a facet) for $P(M, 0)$ then $a(x \oplus v_1) \leqslant \alpha$ is valid (a facet) for $P(M, b)$. So, a nonredundant system defining $P(M, b)$ is given by

$$x_3 + x_4 = 1 \tag{6.5}$$

and the inequalities (6.3), (6.4) above. Clearly, the dimension of $P(M, 0)$ and $P(M, b)$ is four.

The dimension of $P(mC_2, M, b)$ is five, and the vertices of $P(mC_2, M, b)$ are just the points v_1, v_3 and v_5. It follows from [6] that $P(mC_2, M, b)$ is described nonredundantly by the following system

$$x_i \geqslant 0, \qquad i = 1, \ldots, 5, \tag{6.6}$$

$$\left.\begin{aligned} x_3 + x_4 &\geqslant 1, \\ x_1 + x_2 + x_4 &\geqslant 1. \end{aligned}\right\} \tag{6.7}$$

So $P(mC_2, M, b)$ has 6 facets, but no bounded facet, while $P(M, b)$ has 7 facets (which are all bounded). Apparently the systems describing $P(M, b)$ and $P(mC_2, M, b)$ look quite unrelated. Moreover, each vertex of $P(M, b)$ is contained in 4 facets (i.e., $P(M, b)$ is nondegenerate) while vertex v_1 is contained in 6 facets of $P(mC_2, M, b)$ and v_3, v_5 are contained in 5 facets of $P(mC_2, M, b)$. In particular, v_1 is degenerate.

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