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Complete Descriptions of Small Multicut Polytopes

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ABSTRACT. We introduce various new classes of polytopes that are associated with certain cut, equicut, multicut, equimulticut, and balanced multicut problems on graphs. We have computed complete and nonredundant descriptions of these polytopes numerically for the complete graphs of order 4 and 5, and we list and classify in this paper the facet-defining inequalities that we found. Quite a number of new classes of such inequalities arose. We generalize some of the inequalities and prove that they define facets for certain of these polytopes, but many are still waiting for a proper understanding.

1. Introduction

The area of polyhedral investigations of certain cut (or equivalently partitioning) problems has received considerable attention recently. The reason is that many real world problems can be formulated as cut (or partitioning) problems and that cutting plane methods that are based on polyhedral results have shown to be highly efficient algorithms for the solution of these problems.

Without aiming at completeness we mention a few references that corroborate our statements. Long lists of practical problems that can be phrased as certain cut problems can be found, for instance, in [**W, GW1, BJR, BGJR**]. Polyhedral results on various cut, equicut, or multicut polytopes are contained, for example, in [**BMa, DL, CRS, DFL, W, GW2, CR1, CR2**]. Reports about the computational experience with cutting plane algorithms that are based on such polyhedral investigations are, for instance, [**BGJR, BJR, BM, GW1**].

There are many more interesting practical problems that can be modelled as certain cut problems, in particular, a number of questions that are addressed in clustering and qualitative data analysis. The aim of this paper is to introduce some of the cut problems that arise in these applications, to

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define the associated polyhedra, and to present complete and nonredundant descriptions of these polyhedra for small dimensions. These linear descriptions have been calculated by means of a computer program. We classify the facet-defining inequalities that we found. Some of these belong to well-known classes of inequalities, but quite a number are new and are waiting for proper understanding and generalization.

That is, the main purpose of this paper is to provide instructive material (in particular many interesting classes of facets) for those polyhedral combinatorialists who are working in the area of cut polytopes. We believe that starting with these numerically computed classes of inequalities large new classes of facet-defining inequalities may be found and we hope that these inequalities will help to solve practical problems by means of cutting plane methods.

The paper is organized as follows. The polytopes we consider are introduced in §2. In §3 we discuss these polytopes for the complete graph on five nodes and give a complete and nonredundant description of each of these. A list and partial classification of all the classes of facets that we determined can be found in §4 together with a large table summarizing the results.

In §5 we report about some observations we made by analyzing the material of §4 and give a few theoretical explanations. In particular, we generalize some of the inequalities we found and prove that they define facets for certain multicut polytopes. Complete and nonredundant descriptions of the multicut polytopes for the complete graph on four nodes are provided in §6.

2. Multicut and balanced multicut polytopes

Let $G = (V, E)$ be an (undirected) graph with node set V and edge set E. The number n will always denote the cardinality of V. If $S \subseteq V$ we denote the set of edges of E with both endnodes in S by $E(S)$, i.e., $E(S) = \{uv \in E | u, v \in S\}$. It is customary to denote the cut induced by S, i.e., the set $\{uv \in E | u \in S, v \in V \setminus S\}$, by $\delta(S)$. In order to have smooth notation available for more general types of cuts we do not use this symbol and introduce the following variation. Let $S_1, \dots, S_k$ be a partition of V, i.e., $S_i \cap S_j = \emptyset$ for $i \neq j$, $S_i \neq \emptyset$ for $i = 1, \dots, k$, and $\bigcup_{i=1}^k S_i = V$. Then

$$\delta(S_1, \dots, S_k) := \{uv \in E | \exists i, j \text{ with } i \neq j \text{ such that } u \in S_i \text{ and } v \in S_j\}.$$

We call the edge set $\delta(S_1, \dots, S_k)$ a *multicut* and the sets $S_1, \dots, S_k$ the *shores* of the multicut. If we want to stress that the partition of V consists of exactly k sets (shores) we will call $\delta(S_1, \dots, S_k)$ a *k-multicut* or just a *k-cut*. There is only one 1-cut, namely $\delta(V) = \emptyset$, and one n-cut, namely $\delta(\{v_1\}, \dots, \{v_n\}) = E$. The (usual) cuts are our k-cuts with $k \in \{1, 2\}$.

The most intensively studied optimization problem in connection with these problems certainly is the max-cut problem (i.e., given weights c_e for all $e \in E$ find a k-cut with $k \in \{1, 2\}$ of maximum weight; see, e.g.,

[BMa]). Also, the problem of finding a multicut of maximum weight is of particular practical interest. This problem is equivalent to the clique partitioning problem since a clique partitioning (in the complete graph K_n) is nothing but the complement $E \setminus \delta(S_1, \dots, S_k)$ of a multicut (see [GW1, GW2]).

There are further types of multicuts of practical interest.

Definition 1. Let $G = (V, E)$ be a graph and s be an integer with $0 \le s \le |V| - 1$. A multicut $\delta(S_1, \dots, S_k)$, where $k \ge 2$, is called *s-balanced* if

$$||S_i| - |S_j|| \le s \quad \text{for } 1 \le i < j \le k.$$

For two sets S_i, S_j the number $||S_i| - |S_j||$ is called their *discrepancy*.

To turn optimization problems for multicuts into linear programs and questions about multicuts into questions about polyhedra we introduce polyhedra associated with certain sets of multicuts.

Let $G = (V, E)$ be a graph and $\mathbf{R}^E$ the vector space of vectors $x = (x_e)_{e \in E}$, where the components of x are indexed by the edges in E. For every subset F of E, we define its *incidence vector* $\chi^F \in \mathbf{R}^E$ by $\chi^F_e = 1$ if $e \in F$ and $\chi^F_e = 0$ if $e \notin F$. We are particularly interested in incidence vectors $\chi^{\delta(S_1, \dots, S_k)}$ of multicuts $\delta(S_1, \dots, S_k)$. To be able to vary names a bit we will also call $\chi^{\delta(S_1, \dots, S_k)}$ a multicut vector or k-cut vector. We will now introduce the multicut polyhedra considered in this paper.

Definition 2. Let $G = (V, E)$ be a graph with n nodes. Let k and s be integers with $1 \le k \le n$ and $0 \le s \le n - 1$. Set

$$\begin{aligned}
\mathrm{MC}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ a multicut of } G\},\\
\mathrm{MC}_k^{\le}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } h\text{-cut of } G \text{ with } h \le k\},\\
\mathrm{MC}_k^{\ge}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } h\text{-cut of } G \text{ with } h \ge k\},\\
\mathrm{MC}_k^{=}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_k) \text{ a } k\text{-cut of } G\},\\
s\text{-}\mathrm{BMC}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } s\text{-balanced multicut of } G\},\\
s\text{-}\mathrm{BMC}_k^{\le}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } s\text{-balanced } h\text{-cut of } G \text{ with } h \le k\},\\
s\text{-}\mathrm{BMC}_k^{\ge}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } s\text{-balanced } h\text{-cut of } G \text{ with } h \ge k\},\\
s\text{-}\mathrm{BMC}_k^{=}(G) &:= \mathrm{conv}\{\chi^{\delta(S_1, \dots, S_h)} \in \mathbf{R}^E | \delta(S_1, \dots, S_h) \text{ an } s\text{-balanced } k\text{-cut of } G\}.
\end{aligned}$$

In case G is the complete graph K_n we will write $\mathrm{MC}(n)$, etc. instead of $\mathrm{MC}(K_n)$, etc. We call $\mathrm{MC}(G)$ *the multicut polytope* and s-$\mathrm{BMC}(G)$ the *s-balanced multicut polytope* of G. $\mathrm{MC}_k^{\leq}(G)$ is called the *$^{\leq}k$-cut polytope*, $\mathrm{MC}_k^{\geq}(G)$ the *$^{\geq}k$-cut polytope*, $\mathrm{MC}_k^{=}(G)$ the *k-cut polytope*, s-$\mathrm{BMC}_k^{\leq}(G)$ the *s-balanced $^{\leq}k$-cut polytope*, s-$\mathrm{BMC}_k^{\geq}(G)$ the *s-balanced $^{\geq}k$-cut polytope*, and s-$\mathrm{BMC}_k^{=}(G)$ the *s-balanced k-cut polytope* of G. In case it is not necessary to be precise we will simply speak of a multicut polytope $\mathrm{MC}_k^{\leq}(G)$, etc.

Note that the standard *cut polytope* (i.e., the convex hull of all incidence vectors of (usual) cuts) of a graph is our $^{\leq}2$-cut polytope $\mathrm{MC}_2^{\leq}(G)$, and the *equicut polytope* (i.e., the convex hull of all incidence vectors of cuts, where both shores differ in cardinality by at most one) of a graph is our 1-balanced $^{\leq}2$-cut polytope 1-$\mathrm{BMC}_2^{\leq}(G)$.

The polytopes most intensively studied in the literature are

- the cut polytope $\mathrm{MC}_2^{\leq}(G)$ (see, e.g., **[BMa, DL]** and the references therein),
- the multicut polytope $\mathrm{MC}(G)$ (see **[CR1, DGL, GW2, W]**),
- the equicut polytope 1-$\mathrm{BMC}_2^{\leq}(G)$ (see **[CRS, DFL]**),
- the $^{\geq}k$-cut and the $^{\leq}k$-cut polytopes $\mathrm{MC}_k^{\leq}(G)$ and $\mathrm{MC}_k^{\geq}(G)$ (see **[CR1, CR2, DGL]**),

while not much work has been done yet on all the other polytopes. To provide examples and ideas for generalizations we give complete descriptions of all polytopes mentioned above for the complete graphs on 4 and 5 nodes. Let us mention that similar work has been done for small equicut and inequicut cones in **[DFL]**.

3. The multicut polytopes for $n = 5$

We will now discuss the multicut polytopes introduced in Definition 2 for the complete graph $K_5 = (V, E)$ and provide complete and nonredundant descriptions of all polytopes of interest. For the complete graph K_4, similar lists can be found in §6.

By making all possible combinations of the parameters one can see that there are almost 100 cases to be considered for $n = 5$. However, some can be easily derived from others for theoretical reasons and, due to the small dimension, quite a number are trivial or very simple. We will now rule out all trivial cases, discuss the simple cases, and list the interesting ones that will be investigated later.

There is no obvious order of the polytopes since there are many containment relations. We follow here and later the following lexicographic order of the polytopes. We always treat $\mathrm{MC}(n)$ first. Then the multicut polytopes $\mathrm{MC}_k^*(n)$ are studied. Here and further we use the symbol "$*$" to denote one of the symbols "$\leq$", "$\geq$", "$=$". The parameter k specifying a bound on shores is considered in increasing order, "$\leq$" is treated before "$\geq$" and this, in turn, before "=". For the s-balanced multicut polytopes we treat the polytopes with largest s first and, within this class, follow the lexicographic rules for $\mathrm{MC}_k^*(n)$. Then we decrease s by 1 and continue this way.

To be able to give short definitions of all multicut polytopes we now list and number all 52 multicut vectors for $n = 5$. We simply specify the partitions $S_1, \ldots, S_k$ from which the incidence vectors $\chi^{\delta(S_1,\ldots,S_k)} \in \mathbf{R}^E$ can be derived and number them as follows.

1 : 12345	2 : 1, 2345	3 : 2, 1345
4 : 3, 1245	5 : 4, 1235	6 : 5, 1234
7 : 12, 345	8 : 13, 245	9 : 14, 235
10 : 15, 234	11 : 23, 145	12 : 24, 135
13 : 25, 134	14 : 34, 125	15 : 35, 124
16 : 45, 123	17 : 1, 2, 345	18 : 1, 3, 245
19 : 1, 4, 235	20 : 1, 5, 234	21 : 2, 3, 145
22 : 2, 4, 135	23 : 2, 5, 134	24 : 3, 4, 125
25 : 3, 5, 124	26 : 4, 5, 123	27 : 1, 23, 45
28 : 1, 24, 35	29 : 1, 25, 34	30 : 2, 13, 45
31 : 2, 14, 35	32 : 2, 15, 34	33 : 3, 12, 45
34 : 3, 14, 25	35 : 3, 15, 24	36 : 4, 12, 35
37 : 4, 13, 25	38 : 4, 15, 23	39 : 5, 12, 34
40 : 5, 13, 24	41 : 5, 14, 23	42 : 1, 2, 3, 45
43 : 1, 2, 4, 35	44 : 1, 2, 5, 34	45 : 1, 3, 4, 25
46 : 1, 3, 5, 24	47 : 1, 4, 5, 23	48 : 2, 3, 4, 15
49 : 2, 3, 5, 14	50 : 2, 4, 5, 13	51 : 3, 4, 5, 12
52 : 1, 2, 3, 4, 5		

To explain the notation, consider number 29. The partition is $S_1 = \{1\}$, $S_2 = \{2, 5\}$, $S_3 = \{3, 4\}$ which gives the 3-cut vector $\chi^{\delta(S_1, S_2, S_3)} = (\chi_{12}, \chi_{13}, \ldots, \chi_{45}) = (1, 1, 1, 1, 1, 1, 0, 0, 1, 1)$, i.e., this is our vector number 29. To shorten notation, we will now write equations like $\mathrm{MC}_3^{\geq}(5) = \mathrm{conv}\{17, 18, \ldots, 52\}$ which means that in order to get the polytope $\mathrm{MC}_3^{\geq}(5)$ we have to take all convex combinations of the multicut vectors with numbers $17, 18, \ldots, 52$.

After all these preparations we are now going to discuss the multicut polytopes introduced in Definition 2. For each polytope, we give the number v of vertices and the number f of facets and list the classes of facets that are needed to describe the polytope completely and nonredundantly. The classes of facets are numbered (1), (2), ..., (47). These are defined and discussed in §4. Most of these polytopes have dimension $|E| = 10$. We mention whenever a polytope has dimension less than 10 or is particular in some way.

$$\mathrm{MC}(5) = \mathrm{conv}\{1, 2, \ldots, 52\}, \quad v = 52, \quad f = 242,$$

facets : (1), (8), (9), (10), (11), (13), (20), (44), (45), (46).

This polytope, of course, contains all other multicut polytopes. Since the containment relations between the polytopes are quite obvious we do not make any further remarks on this issue. All inequalities valid for MC(n) are also valid for all other multicut polytopes.

Polytopes of the type $\mathrm{MC}_1^*(n)$ are irrelevant. $\mathrm{MC}_1^{\geq}(n) = \mathrm{MC}(n)$ and $\mathrm{MC}_1^{\leq}(n)$, $\mathrm{MC}_1^{=}(n)$ consist of the single point 0.

$$\mathrm{MC}_2^{\leq}(5) = \mathrm{conv}\{1, 2, \dots, 16\}, \quad v = 16, \quad f = 56,$$
$$\text{facets}: (4), (8), (9), (12), (15).$$

This is the cut polytope of K_5. In fact, a complete description of the cut polytope is known for every complete graph up to seven nodes (for details see **[DL]**).

$$\mathrm{MC}_2^{\geq}(5) = \mathrm{conv}\{2, 3, \dots, 52\}, \quad v = 51, \quad f = 264,$$
$$\text{facets}: (1), (8), (9), (10), (11), (13), (17), (20), (27), (44), (45), (46).$$

This polytope is obtained from MC(5) by deleting point 1 (the zero vector). The neighbors of zero are, however, not on a common hyperplane and so $\mathrm{MC}_2^{\geq}(5)$ cannot be derived from MC(5) by adding one "cutting plane". Compared with MC(5) no facet-defining inequality of MC(5) disappears, but the two new classes (17) and (27) appear.

$$\mathrm{MC}_2^{=}(5) = \mathrm{conv}\{2, 3, \dots, 16\}, \quad v = 15, \quad f = 68,$$
$$\text{facets}: (4), (8), (12), (15), (17), (27).$$

$\mathrm{MC}_2^{=}(5)$ arises from the cut polytope $\mathrm{MC}_2^{\leq}(5)$ by removing one vertex, the zero vector. The class (9) of certain hypermetric facets disappears and two new classes of inequalities, (17) and (27), come up. Observe the difference to the transition from MC(5) to $\mathrm{MC}_2^{\geq}(5)$. Here also the zero vector is removed, facets (17) and (27) appear, but (9) remains a class of facets of $\mathrm{MC}_2^{\geq}(5)$.

$$\mathrm{MC}_3^{\leq}(5) = \mathrm{conv}\{1, 2, \dots, 41\}, \quad v = 41, \quad f = 333,$$
$$\text{facets}: (1), (3), (8), (9), (10), (11), (14), (20), (38), (41), (43), (44), (45), (46),$$
$$\mathrm{MC}_3^{\geq}(5) = \mathrm{conv}\{17, 18, \dots, 52\}, \quad v = 36, \quad f = 218,$$
$$\text{facets}: (1), (6), (8), (11), (13), (16), (23), (24), (25), (30), (31), (40),$$
$$\mathrm{MC}_3^{=}(5) = \mathrm{conv}\{17, 18, \dots, 41\}, \quad v = 25, \quad f = 99,$$
$$\text{facets}: (1), (3), (6), (14), (16), (30), (31), (38), (40),$$
$$\mathrm{MC}_4^{\leq}(5) = \mathrm{conv}\{1, 2, \dots, 51\}, \quad v = 51, \quad f = 243,$$
$$\text{facets}: (1), (2), (8), (9), (10), (11), (13), (20), (44), (45), (46).$$

$\mathrm{MC}_4^{\leq}(5)$ arises from MC(5) by deleting vertex 52, i.e., the all-ones vector χ^E. Using the adjacency characterization on the clique partitioning polytope of [W], which, by complementation, gives an adjacency characterization of the vertices of the multicut polytope $\mathrm{MC}(n)$, one can show that the neighbors of χ^E are the 4-cut vectors $42, 43, \ldots, 51$. These points lie on the hyperplane $\{x|x(E) = 9\}$. So it follows that $\mathrm{MC}_4^{\leq}(5) = \mathrm{MC}(5) \cap \{x|x(E) \leq 9\}$. (In fact, this argument holds for general n, see §5.) It may be, though, that some inequalities defining a facet of MC(5) are redundant with respect to $\mathrm{MC}_4^{\leq}(5)$. But it is easy to see that χ^E is only contained in the facets defined by the upper bound constraints (1) $x_e \leq 1$ which also define facets of $\mathrm{MC}_4^{\leq}(5)$ (see §5). So $\mathrm{MC}_4^{\leq}(5)$ has one vertex less and one facet more than MC(5). (For that reason we will not discuss $\mathrm{MC}_4^{\leq}(5)$ further in §4.)

$$\mathrm{MC}_4^{\geq}(5) = \mathrm{conv}\{42, 43, \ldots, 52\}, \quad v = 11, \quad f = 11.$$

$\mathrm{MC}_4^{\geq}(5)$ is a full-dimensional simplex in $\mathbf{R}^E$. The facets are the upper bounds (1) $x_e \leq 1$ and the cardinality constraint $x(E) \geq 9$. $\mathrm{MC}_{n-1}^{\geq}(n)$ is a simplex for any n (see §5). We do not discuss this polytope $\mathrm{MC}_4^{\geq}(5)$ further.

$$\mathrm{MC}_4^{=}(5) = \mathrm{conv}\{42, 43, \ldots, 51\}, \quad v = 10, \quad f = 10.$$

$\mathrm{MC}_4^{=}(5)$ is a 9-dimensional simplex in $\mathbf{R}^E$ arising from $\mathrm{MC}_4^{\geq}(5)$ by turning $x(E) \geq 9$ into an equation. We will not consider $\mathrm{MC}_4^{=}(5)$ in the sequel any more.

Based on this analysis, the multicut polytopes of interest are MC(5), $\mathrm{MC}_2^{\leq}(5)$, $\mathrm{MC}_2^{\geq}(5)$, $\mathrm{MC}_2^{=}(5)$, $\mathrm{MC}_3^{\leq}(5)$, $\mathrm{MC}_3^{\geq}(5)$, $\mathrm{MC}_3^{=}(5)$.

We now turn our attention to the balanced multicut polytopes s-$\mathrm{BMC}_k^*(5)$.

Note that for any k-cut $\delta(S_1, \ldots, S_k)$ in K_5, $k \geq 2$, the discrepancy $||S_i| - |S_j||$ of any two sets $S_i \neq S_j$ is never larger than 3. So, any polytope 4-$\mathrm{BMC}_k^*(5)$ is the same as the polytope 3-$\mathrm{BMC}_k^*(5)$ for $k = 2, \ldots, 5$. Moreover, there is no k-cut $\delta(S_1, \ldots, S_k)$, $k \in \{2, \ldots, 4\}$, in K_5 with discrepancy $||S_i| - |S_j|| = 0$ for all $i \neq j$. Thus the polytopes 0-$\mathrm{BMC}_k^*(5)$, $k = 2, 3, 4$, are empty. The polytope 0-$\mathrm{BMC}_5^*(5)$ consists of the single point χ^E and is also irrelevant. Furthermore, any polytope s-$\mathrm{BMC}(n)$, $s \in \{0, 1, \ldots, n-1\}$, is equal to the polytope s-$\mathrm{BMC}_{n-1}^{\leq}(n)$, so we do not consider the former polytopes in the sequel.

Thus the remaining range of parameters of interest is $s = 3, 2, 1$ and $k = 2, 3, 4, 5$. We will analyze these cases. It will turn out that some of the polytopes are equal to polytopes already considered. In these cases we state this fact without discussing the polytopes further.

$$\begin{aligned}
&3\text{-BMC}_2^{\leq}(5) = \text{MC}_2^{\leq}(5),\\
&3\text{-BMC}_2^{\geq}(5) = \text{MC}_2^{\geq}(5),\\
&3\text{-BMC}_2^{=}(5) = \text{MC}_2^{=}(5),\\
&3\text{-BMC}_3^{\leq}(5) = \text{conv}\{2, 3, \dots, 41\}, \quad v = 40, \quad f = 355,\\
&\qquad =\text{facets}: (1), (3), (8), (9), (10), (11), (14), (17),\\
&\qquad (20), (27), (38), (41), (43), (44), (45), (46).
\end{aligned}$$

This polytope has the second largest number of facets among all polytopes investigated in this paper.

$$\begin{aligned}
&3\text{-BMC}_3^{\geq}(5) = \text{MC}_3^{\geq}(5),\\
&3\text{-BMC}_3^{=}(5) = \text{MC}_3^{=}(5),\\
&3\text{-BMC}_4^{\leq}(5) = \text{conv}\{2, 3, \dots, 51\}, \quad v = 50, \quad f = 265,\\
&\qquad \text{facets}: (1), (2), (8), (9), (10), (11), (13),\\
&\qquad (17), (20), (27), (44), (45), (46).
\end{aligned}$$

$3\text{-BMC}_4^{\leq}(5)$ arises from $\text{MC}(5)$ by deleting the "top vertex" $\chi^E = \mathbf{1}$ and the "bottom vertex" $\chi^{\emptyset} = 0$. Analogously, $3\text{-BMC}_4^{\leq}(5)$ can be derived from $\text{MC}_4^{\leq}(5)$ by deleting the zero vector, or from $\text{MC}_2^{\geq}(5)$ by deleting χ^E. As the computation shows (and one can verify theoretically) $3\text{-BMC}_4^{\leq}(5)$ inherits all facets of $\text{MC}_2^{\geq}(5)$, $\text{MC}_4^{\leq}(5)$, and of $\text{MC}(5)$.

$$\begin{aligned}
&3\text{-BMC}_4^{\geq}(5) = \text{MC}_4^{\geq}(5),\\
&3\text{-BMC}_4^{=}(5) = \text{MC}_4^{=}(5),\\
&2\text{-BMC}_2^{\leq}(5) = 1\text{-BMC}_2^{\leq}(5).
\end{aligned}$$

Note that the 2-cuts $\delta(S, V \setminus S)$ in K_5 have discrepancy $||S| - |V \setminus S||$ equal either to one or to three. For this reason the last equation above holds and thus we discuss this polytope later.

$$\begin{aligned}
&2\text{-BMC}_2^{\geq}(5) = \text{conv}\{7, 8, \dots, 52\}, \quad v = 46, \quad f = 308,\\
&\qquad \text{facets}: (1), (5), (8), (11), (13), (18), (20), (21),\\
&\qquad (23), (29), (32), (35), (36), (39), (42),\\
&2\text{-BMC}_2^{=}(5) = 1\text{-BMC}_2^{=}(5) \quad \text{(see remark after } 2\text{-BMC}_2^{\leq}(5)),\\
&2\text{-BMC}_3^{\leq}(5) = \text{conv}\{7, 8, \dots, 41\}, \quad v = 35, \quad f = 379,\\
&\qquad \text{facets}: (1), (3), (5), (8), (11), (14), (18), (20), (21),\\
&\qquad (23), (29), (32), (35), (36), (38), (39), (42), (43).
\end{aligned}$$

This is the polytope with the largest number of facets considered here. There are 18 different classes of facets.

$$\begin{aligned}
&2\text{-}\mathrm{BMC}_3^{\geq}(5) = \mathrm{MC}_3^{\geq}(5),\\
&2\text{-}\mathrm{BMC}_3^{=}(5) = \mathrm{MC}_3^{=}(5),\\
&2\text{-}\mathrm{BMC}_4^{\leq}(5) = \mathrm{conv}\{7, 8, \dots, 51\}, \quad v = 45, \quad f = 309,\\
&\qquad \text{facets}: (1), (2), (5), (8), (11), (13), (18), (20),\\
&\qquad (21), (23), (29), (32), (35), (36), (39), (42).
\end{aligned}$$

$2\text{-}\mathrm{BMC}_4^{\leq}(5)$ arises from $2\text{-}\mathrm{BMC}_2^{\geq}(5)$ by deleting the vertex χ^E. All facets of $2\text{-}\mathrm{BMC}_2^{\geq}(5)$ are inherited, and the cardinality constraint (2) $x(E) \leq 9$ appears. The behavior can be explained with the same arguments given in the discussion of the polytope $\mathrm{MC}_4^{\leq}(5)$.

$$\begin{aligned}
&2\text{-}\mathrm{BMC}_4^{\geq}(5) = \mathrm{MC}_4^{\geq}(5),\\
&2\text{-}\mathrm{BMC}_4^{=}(5) = 1\text{-}\mathrm{BMC}_4^{=}(5).
\end{aligned}$$

There is no 4-cut in K_5 where two shores have discrepancy 2. So all 2-balanced 4-cuts are in fact 1-balanced, and thus we discuss this polytope later.

$$\begin{aligned}
&1\text{-}\mathrm{BMC}_2^{\leq}(5) = \mathrm{conv}\{7, 8, \dots, 16\}, \quad v = 10, f = 10,\\
&\qquad \text{facets}: (12) \text{ or equivalently } (29), \text{ dimension } = 9.
\end{aligned}$$

This polytope is a 9-dimensional simplex in $\mathbf{R}^E$. It lies on the hyperplane $\{x | x(E) = 6\}$. All inequalities (12) and (29) are facet-defining but the two classes are equivalent in this case. $1\text{-}\mathrm{BMC}_2^{\leq}(5)$ is known in the literature as the *equicut polytope.*

$$\begin{aligned}
&1\text{-}\mathrm{BMC}_2^{\geq}(5) = \mathrm{conv}\{7, 8, \dots, 16, 27, 28, \dots, 52\}, \quad v = 36, \quad f = 113,\\
&\qquad \text{facets}: (1), (5), (8), (20), (26), (33), (34), (37),\\
&1\text{-}\mathrm{BMC}_2^{=}(5) = 1\text{-}\mathrm{BMC}_2^{\leq}(5),\\
&1\text{-}\mathrm{BMC}_3^{\leq}(5) = \mathrm{conv}\{7, 8, \dots, 16, 27, 28, \dots, 41\}, \quad v = 24, \quad f = 169,\\
&\qquad \text{facets}: (1), (3), (5), (8), (15), (20), (26), (28), (34), (37),\\
&\qquad (47),\\
&1\text{-}\mathrm{BMC}_3^{\geq}(5) = \mathrm{conv}\{27, 28, \dots, 52\}, \quad v = 26, \quad f = 26,\\
&\qquad \text{facets}: (1), (7), (19), (22),\\
&1\text{-}\mathrm{BMC}_3^{=}(5) = \mathrm{conv}\{27, 28, \dots, 41\}, \quad v = 15, \quad f = 25,\\
&\qquad \text{facets}: (1), (14) \text{ or equivalently } (22), (19) \text{ or}\\
&\qquad \text{equivalently } (28), \text{ dimension } = 9.
\end{aligned}$$

This polytope lies in the hyperplane $x(E) = 9$, the inequalities (19) are in this case equivalent to the inequalities (28), and the class (14) is equivalent to (22).

$$1\text{-BMC}_4^{\leq}(5) = \text{conv}\{7, 8, \dots, 16, 27, 28, \dots, 51\}, \quad v = 35, \quad f = 114,$$
$$\text{facets}: (1), (2), (5), (8), (20), (26), (33), (34), (37).$$

$1\text{-BMC}_4^{\leq}(5)$ arises from $1\text{-BMC}_2^{\geq}(5)$ by removing vertex $\chi^E = \mathbf{1}$. As in two previous cases (cf. the discussion behind $\text{MC}_4^{\leq}(5)$) $1\text{-BMC}_4^{\leq}(5)$ keeps all facets of $1\text{-BMC}_2^{\geq}(5)$ and only one new facet, (2) $x(E) \leq 9$, appears.

$$1\text{-BMC}_4^{\geq}(5) = \text{MC}_4^{\geq}(5),$$
$$1\text{-BMC}_4^{=}(5) = \text{MC}_4^{=}(5).$$

This finishes our discussion of the multicut polytopes for $n = 5$. For convenient reference we list in Table 1 which of the polytopes are equal and which are of further interest.

TABLE 1

1.	$\text{MC}(5)$
2.	$\text{MC}_2^{\leq}(5) = 3\text{-BMC}_2^{\leq}(5)$
3.	$\text{MC}_2^{\geq}(5) = 3\text{-BMC}_2^{\geq}(5)$
4.	$\text{MC}_2^{=}(5) = 3\text{-BMC}_2^{=}(5)$
5.	$\text{MC}_3^{\leq}(5)$
6.	$\text{MC}_3^{\geq}(5) = 3\text{-BMC}_3^{\geq}(5) = 2\text{-BMC}_3^{\geq}(5)$
7.	$\text{MC}_3^{=}(5) = 3\text{-BMC}_3^{=}(5) = 2\text{-BMC}_3^{=}(5)$
8.	$3\text{-BMC}_3^{\leq}(5)$
9.	$3\text{-BMC}_4^{\leq}(5)$
10.	$2\text{-BMC}_2^{\geq}(5)$
11.	$2\text{-BMC}_3^{\leq}(5)$
12.	$2\text{-BMC}_4^{\leq}(5)$
13.	$1\text{-BMC}_2^{\leq}(5) = 1\text{-BMC}_2^{=}(5) = 2\text{-BMC}_2^{\leq}(5) = 2\text{-BMC}_2^{=}(5)$
14.	$1\text{-BMC}_2^{\geq}(5)$
15.	$1\text{-BMC}_3^{\leq}(5)$
16.	$1\text{-BMC}_3^{\geq}(5)$
17.	$1\text{-BMC}_3^{=}(5)$
18.	$1\text{-BMC}_4^{\leq}(5)$

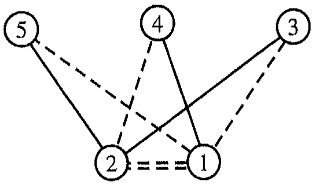


FIGURE 1

The following polytopes will not be considered further.

$$\mathrm{MC}_4^{\leq}(5)$$
$$\mathrm{MC}_4^{\geq}(5) = 3\text{-}\mathrm{BMC}_4^{\geq}(5) = 2\text{-}\mathrm{BMC}_4^{\geq}(5) = 1\text{-}\mathrm{BMC}_4^{\geq}(5),$$
$$\mathrm{MC}_4^{=}(5) = 3\text{-}\mathrm{BMC}_4^{=}(5) = 2\text{-}\mathrm{BMC}_4^{=}(5) = 1\text{-}\mathrm{BMC}_4^{=}(5).$$

4. A dictionary of the facets of the multicut polytopes for $n = 5$

We will now list all classes of inequalities that came up in our numerical calculations and that define facets of at least one of our multicut polytopes listed in Table 1. As before, $K_5 = (V, E)$ denotes the complete graph on five nodes and $V = \{1, 2, \ldots, 5\}$. For a convenient description of the inequalities it seems more appropriate to use numbers as indices rather than letters. Instead of saying that the inequality $x_{ij} \leq 1$ defines a facet for every edge $ij \in E$, we will simply state that $x_{12} \leq 1$ defines a facet. This is justified since the property of an inequality to define a facet is preserved under permutation of indices. In other words, writing that $x_{12}+x_{23}+x_{13} \leq 2$ defines a facet means that the inequalities $x_{ij}+x_{jk}+x_{ik} \leq 2$ define facets for all $i, j, k \in V$, $i \neq k \neq j \neq i$. Whenever we list a representative of a class of inequalities we list in brackets the number of permutations of $\{1, \ldots, 5\}$ that lead to different facet-defining inequalities.

We will also present some drawings of the (weighted) support graphs of the inequalities. The convention is that a solid line represents a $+1$-coefficient, two parallel solid lines a $+2$-coefficient, etc., while a broken line represents a -1-coefficient and two parallel broken lines represent a -2-coefficient, etc. For example, the inequality

$$-2x_{12} - x_{13} + x_{14} - x_{15} + x_{23} - x_{24} + x_{25} \leq 0$$

is depicted by the graph shown in Figure 1.

With each class of inequalities we provide a list of those polytopes for which this class defines facets. We restrict reference to the 18 (interesting) polytopes of Table 1.

In the sequel we will drop the "(5)" in the names of the multicut polytopes to avoid unnecessary symbols, i.e., instead of "$\mathrm{MC}_2^{\geq}(5)$" we will simply write "$\mathrm{MC}_2^{\geq}$" etc.

(4.1) Upper bounds. Since all multicut polytopes are 0/1-polytopes, the upper bound inequalities

(1) $$x_{12} \le 1 \qquad \text{(10 permutations)}$$

are valid for these polytopes. In fact, they define facets for all the 18 polytopes except $\mathrm{MC}_2^{\le}$, $\mathrm{MC}_2^{=}$, and 1-$\mathrm{BMC}_2^{\le}$. It is interesting to note that no lower bound inequality $x_{12} \ge 0$ ever defines a facet.

(4.2) Cardinality constraints. We call constraints of the type $x(E) \le u$, where u is some positive integer, an upper cardinality constraint, and similarly we call $x(E) \ge l$ a lower cardinality constraint. Such constraints appear as facet-defining inequalities with different right-hand sides as follows:

(2) $$x(E) \le 9 \qquad \text{(1 permutation)}$$

is facet-defining for 3-$\mathrm{BMC}_4^{\le}$, 2-$\mathrm{BMC}_4^{\le}$, 1-$\mathrm{BMC}_4^{\le}$;

(3) $$x(E) \le 8 \qquad \text{(1 permutation)}$$

defines a facet of $\mathrm{MC}_3^{\le}$, $\mathrm{MC}_3^{=}$, 3-$\mathrm{BMC}_3^{\le}$, 2-$\mathrm{BMC}_3^{\le}$, 1-$\mathrm{BMC}_3^{\le}$;

(4) $$x(E) \le 6 \qquad \text{(1 permutation)}$$

defines a facet of $\mathrm{MC}_2^{\le}$, $\mathrm{MC}_2^{=}$. The lower bound

(5) $$x(E) \ge 6 \qquad \text{(1 permutation)}$$

defines a facet of 2-$\mathrm{BMC}_2^{\ge}$, 2-$\mathrm{BMC}_3^{\le}$, 2-$\mathrm{BMC}_4^{\le}$, 1-$\mathrm{BMC}_2^{\ge}$, 1-$\mathrm{BMC}_3^{\le}$, 1-$\mathrm{BMC}_4^{\le}$;

(6) $$x(E) \ge 7 \qquad \text{(1 permutation)}$$

is facet-defining for $\mathrm{MC}_3^{\ge}$ and $\mathrm{MC}_3^{=}$; and

(7) $$x(E) \ge 8 \qquad \text{(1 permutation)}$$

defines a facet of 1-$\mathrm{BMC}_3^{\ge}$.

(4.3) Hypermetric inequalities. We first define some symbols to make notation easier. If $b = (b_1, \dots, b_5)$ is a vector of integers and $x \in \mathbf{R}^E$ then we set

$$Q(b) \cdot x := \sum_{ij \in E} b_i b_j x_{ij}.$$

Moreover, if $\sigma := \sum_{i=1}^{5} b_i$ then the inequality

$$Q(b) \cdot x \le \tfrac{1}{2}\sigma(\sigma - 1)$$

is called the *hypermetric inequality* associated with b. For any integral vector b, the hypermetric inequality associated with b is valid for MC and thus for all multicut polytopes considered here (see **[DGL]**). We now list the hypermetric inequalities that come up for $n = 5$:

(8) $$Q(1, 1, -1, 0, 0) \cdot x \le 0 \qquad \text{(30 permutations)}$$

(8) can be written (in a less sophisticated way) as $x_{12} - x_{13} - x_{23} \leq 0$; this inequality is also known under the name *triangle inequality*. It is facet-defining for all multicut polytopes except $MC_3^=$, $1\text{-}BMC_2^{\leq}$, $1\text{-}BMC_3^{\geq}$, $1\text{-}BMC_3^=$.

(9) $\qquad Q(1, 1, 1, -1, -1) \cdot x \leq 0 \qquad$ (10 permutations)

is facet-defining for MC, $MC_2^{\leq}$, $MC_2^{\geq}$, $MC_3^{\leq}$, $3\text{-}BMC_3^{\leq}$, $3\text{-}BMC_4^{\leq}$;

(10) $\qquad Q(1, 1, 1, 1, -2) \cdot x \leq 1 \qquad$ (5 permutations)

is facet-defining for MC, $MC_2^{\geq}$, $MC_3^{\leq}$, $3\text{-}BMC_3^{\leq}$, $3\text{-}BMC_4^{\leq}$;

(11) $\qquad Q(1, 1, 1, -1, 0) \cdot x \leq 1 \qquad$ (20 permutations)

is facet-defining for MC, $MC_2^{\geq}$, $MC_3^{\leq}$, $MC_3^{\geq}$, $3\text{-}BMC_3^{\leq}$, $3\text{-}BMC_4^{\leq}$, $2\text{-}BMC_2^{\geq}$, $2\text{-}BMC_3^{\leq}$, $2\text{-}BMC_4^{\leq}$;

(12) $\qquad Q(1, 1, 1, 0, 0) \cdot x \leq 2 \qquad$ (10 permutations)

is facet-defining for $MC_2^{\leq}$, $MC_2^=$, $1\text{-}BMC_2^{\leq}$;

(13) $\qquad Q(1, 1, 1, 1, -1) \cdot x \leq 3 \qquad$ (5 permutations)

is facet-defining for MC, $MC_2^{\geq}$, $MC_3^{\geq}$, $3\text{-}BMC_4^{\leq}$, $2\text{-}BMC_2^{\geq}$, $2\text{-}BMC_4^{\leq}$.

(4.4) *Q*-inequalities. It turns out that the left-hand sides of some inequalities we found can be written in the form $Q(b) \cdot x$ but that the right-hand sides differ from the value $\frac{1}{2}\sigma(\sigma - 1)$ of the hypermetric inequalities. We will call these inequalities *Q-inequalities.*

(14) $\qquad Q(1, 1, 1, 1, 0) \cdot x \leq 5 \qquad$ (5 permutations)

can also be viewed as an upper bound on any complete subgraph K_4 of K_5. It is facet-defining for $MC_3^{\leq}$, $MC_3^=$, $3\text{-}BMC_3^{\leq}$, $2\text{-}BMC_3^{\leq}$, $1\text{-}BMC_3^=$.

(15) $\qquad Q(1, 1, 1, 1, -1) \cdot x \leq 2 \qquad$ (5 permutations)

is facet-defining for $MC_2^{\leq}$, $MC_2^=$, $1\text{-}BMC_3^{\leq}$.

(4.5) Lower bounds on cycles. The next inequalities are of the form $x(C) \geq l$, where C is a 3-, 4-, or 5-cycle in K_5 and l is some positive integer.

(16) $\qquad x_{12} + x_{23} + x_{34} + x_{45} + x_{15} \geq 3 \qquad$ (12 permutations)

specifies a lower bound of 3 on any pentagon in K_5. This inequality defines a facet of $MC_3^{\geq}$, $MC_3^=$. Similarly,

(17) $\qquad x_{12} + x_{23} + x_{34} + x_{45} + x_{15} \geq 2 \qquad$ (12 permutations)

specifies a lower bound of 2 on any 5-cycle in K_5. It is facet-defining for $MC_2^{\geq}$, $MC_2^=$, $3\text{-}BMC_3^{\leq}$, $3\text{-}BMC_4^{\leq}$.

(18) $\qquad x_{12} + x_{23} + x_{34} + x_{14} \geq 2 \qquad$ (15 permutations)

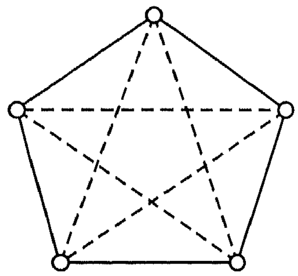


FIGURE 2

requires that each feasible multicut contains at least two edges of every 4-cycle. (There is only one such 4-cycle inequality.) (18) is facet-defining for 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, 2-$\mathrm{BMC}_4^{\leq}$.

(19) $$x_{12} + x_{23} + x_{13} \geq 2 \qquad \text{(10 permutations)}$$

specifies a lower bound of 2 on any triangle. (19) defines a facet of 1-$\mathrm{BMC}_3^{\geq}$, 1-$\mathrm{BMC}_3^{=}$.

(4.6) 2-chorded odd cycle inequalities. The next inequality can be viewed as a combination of two 5-cycles. It comes from the general class of 2-chorded odd cycle inequalities introduced in [**W, GW2**]. Its form, in our special case, is as follows. Let C be any 5-cycle in K_5 and $\overline{C}$ its complement (which is also a 5-cycle). Then

(20) $$x(C) - x(\overline{C}) \leq 2 \qquad \text{(12 permutations)}$$

is valid for MC and thus for all multicut polytopes. (20) defines a facet of MC, $\mathrm{MC}_2^{\geq}$, $\mathrm{MC}_3^{\leq}$, 3-$\mathrm{BMC}_3^{\leq}$, 3-$\mathrm{BMC}_4^{\leq}$, 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, 2-$\mathrm{BMC}_4^{\leq}$, 1-$\mathrm{BMC}_2^{\geq}$, 1-$\mathrm{BMC}_3^{\leq}$, 1-$\mathrm{BMC}_4^{\leq}$. A picture of a 2-chorded 5-cycle inequality is shown in Figure 2.

(4.7) Degree conditions. Some polytopes have degree constraints requiring that among all edges incident to any node at least a certain number have to be in a feasible multicut. The first lower bound on a star is of the form

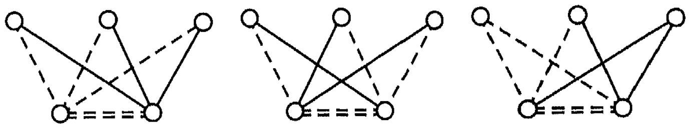
(21) $$x_{12} + x_{13} + x_{14} + x_{15} \geq 2 \qquad \text{(5 permutations)}$$

and is facet-defining for 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, and 2-$\mathrm{BMC}_4^{\leq}$. The second degree bound reads

(22) $$x_{12} + x_{13} + x_{14} + x_{15} \geq 3 \qquad \text{(5 permutations)}$$

and defines a facet of 1-$\mathrm{BMC}_3^{\geq}$ and 1-$\mathrm{BMC}_3^{=}$.

(4.8) Crowns. A *crown* in K_5 is an edge set $C = E \setminus T$, where T is a triangle. (Figures 3 and 4 make it clear why such edge sets are called crowns.) Crowns are supports of seven different classes of facet-defining inequalities



(a) (b) (c)

FIGURE 3

for multicut polytopes. The first three classes are characterized by the fact that the two nodes of the "basis" of the crown are linked by an edge with weight -2 and that exactly three of the edges linking these two nodes to the other three (nonbasis) nodes have weight $+1$ and -1, respectively.

The first class, shown in Figure 3(a), is of the form

(23) $-2x_{12} - x_{13} - x_{14} - x_{15} + x_{23} + x_{24} + x_{25} \leq 0$ (20 permutations),

i.e., all edges linking one node of the basis to the three nonbasis nodes have weight -1 while the edges leaving the other basis node have weight $+1$. These inequalities define facets of $\mathrm{MC}_3^{\geq}$, 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, and 2-$\mathrm{BMC}_4^{\leq}$.

In the second class of crown inequalities, one of the basis nodes is linked to exactly one nonbasis node by an edge with weight -1 while the other basis node is linked to the other two nonbasis nodes by edges with weight -1 (see Figure 3(b)), i.e., these inequalities are of the form

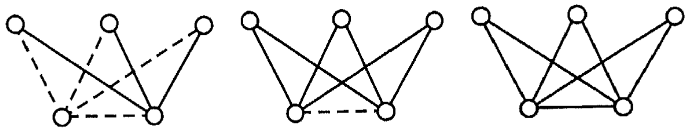
(24) $-2x_{12} - x_{13} + x_{14} + x_{15} + x_{23} - x_{24} - x_{25} \leq 0$ (30 permutations).

These inequalities induce facets of $\mathrm{MC}_3^{\geq}$.

In the third class of inequalities (see Figure 3(c)) one basis node is linked by two edges of weight -1 to two nonbasis nodes, the other basis node is linked by one edge of weight -1 to one of the nonbasis nodes that is already incident to a negative edge, so one nonbasis node is always incident to two positive edges. Thus, these inequalities are of the form

(25) $-2x_{12} - x_{13} - x_{14} + x_{15} - x_{23} + x_{24} + x_{25} \leq 0$ (30 permutations).

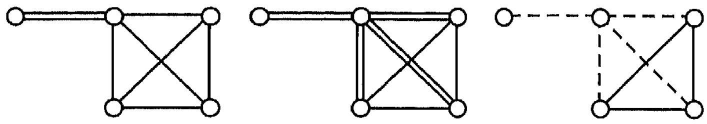
They define facets of $\mathrm{MC}_3^{\geq}$.



(a) (b) (c)

FIGURE 4

The other four classes of crown inequalities have nonzero coefficients $+1$ and -1. In the first of these, the two basis nodes are linked by a -1-edge, all



(a) (b) (c)

FIGURE 5

other edges leaving one of the basis nodes have weight -1, the other edges leaving the other basis node have weight $+1$ (see Figure 4(a)). Thus, they can be written as

$$(26)\quad -x_{12} - x_{13} - x_{14} - x_{15} + x_{23} + x_{24} + x_{25} \leq 0 \qquad \text{(20 permutations)}.$$

They define facets of 1-$\mathrm{BMC}_2^{\geq}$, 1-$\mathrm{BMC}_3^{\leq}$, 1-$\mathrm{BMC}_4^{\leq}$.

In the second version of this type of crown inequalities, the two basis nodes are linked by a -1-edge, all other crown edges have weight $+1$ (see Figure 4(b)), i.e., we have

$$(27)\quad -x_{12} + x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25} \geq 2 \qquad \text{(10 permutations)}.$$

Inequalities (27) define facets of $\mathrm{MC}_2^{\geq}$, $\mathrm{MC}_2^{=}$, 3-$\mathrm{BMC}_3^{\leq}$, 3-$\mathrm{BMC}_4^{\leq}$.

Finally, there are two types of inequalities that specify an upper bound and a lower bound, respectively, on the edges in a crown C shown in Figure 4(c). The first cardinality constraint reads

$$(28)\qquad x(C) \leq 6 \qquad \text{(10 permutations)}$$

and defines a facet of 1-$\mathrm{BMC}_3^{\leq}$ and 1-$\mathrm{BMC}_3^{=}$; the lower bound constraint requires

$$(29)\qquad x(C) \geq 4 \qquad \text{(10 permutations)}$$

and is facet-defining for 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, 2-$\mathrm{BMC}_4^{\leq}$, and 1-$\mathrm{BMC}_2^{\leq}$.

(4.9) Bipartite inequalities. A further class of inequalities specifies a lower bound on the edge set of any complete bipartite graph $K_{2,3}$ contained in K_5. It is of the form

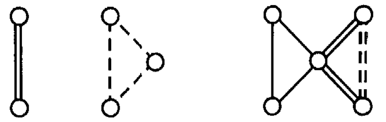
$$(30)\quad x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25} \geq 4 \qquad \text{(10 permutations)}$$

and defines a facet of $\mathrm{MC}_3^{\geq}$ and $\mathrm{MC}_3^{=}$. It is a special case of the class of bipartite inequalities introduced in **[CR1]** for $\mathrm{MC}_k^{\geq}(n)$.

(4.10) Casserole inequalities. We call the next type of inequalities casserole inequalities since their supports resemble a casserole (see Figure 5).

They come in three classes. The first one, shown in Figure 5(a), specifies a lower bound on the edges of a casserole, but where the edges of the "handle" are counted twice, i.e.,

$$(31)\quad 2x_{12} + x_{23} + x_{24} + x_{25} + x_{34} + x_{35} + x_{45} \geq 5 \qquad \text{(20 permutations)}.$$



(a) (b)

FIGURE 6

These inequalities define facets of $\mathrm{MC}_3^{\geq}$ and $\mathrm{MC}_3^{=}$. The second one (see Figure 5(b)) is of the form

(32) $2x_{12} + 2x_{23} + 2x_{24} + 2x_{25} + x_{34} + x_{35} + x_{45} \geq 6$ (20 permutations)

and is facet-defining for 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, and 2-$\mathrm{BMC}_4^{\leq}$.

The third class of casserole inequalities—the support is shown in Figure 5(c)—is of the form

(33) $-x_{12} - x_{23} - x_{24} - x_{25} + x_{34} + x_{35} + x_{45} \leq 0$ (20 permutations)

and is facet-defining for 1-$\mathrm{BMC}_2^{\geq}$ and 1-$\mathrm{BMC}_4^{\leq}$.

Observe that the left-hand side of (31) can be written as $2x_{12} + x(E(\{2, 3, 4, 5\}))$, while the left-hand sides of (32) and (33) can be viewed as $2x(\delta(2))+x(T)$ and $x(T)-x(\delta(2))$, respectively, where T is the triangle on the nodes 3, 4, 5.

(4.11) Inequalities with low-connectivity support. One usually expects the support graph of a facet-defining inequality to be highly (at least two) connected. The casserole inequalities already provide a support graph that is not 2-connected. There are two more classes of inequalities with such low-connectivity support graph. In the first case (see Figure 6(a)), the support graph is disconnected and can be viewed as the sum of twice an edge minus a triangle, i.e.,

(34) $2x_{12} - x_{34} - x_{35} - x_{45} \leq 0$ (10 permutations).

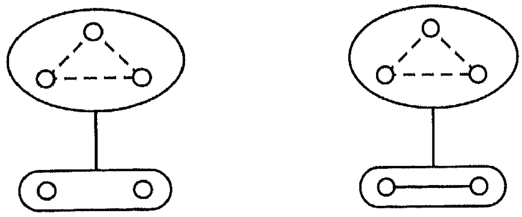
These inequalities define facets of 1-$\mathrm{BMC}_2^{\geq}$, 1-$\mathrm{BMC}_3^{\leq}$, and 1-$\mathrm{BMC}_4^{\leq}$.

The second class has a 1- but not 2-connected support graph (see Figure 6(b)), and reads

(35) $x_{12} + x_{13} + x_{23} + 2x_{34} + 2x_{35} - 2x_{45} \geq 2$ (30 permutations).

These inequalities are facet-defining for 2-$\mathrm{BMC}_2^{\geq}$, 2-$\mathrm{BMC}_3^{\leq}$, and 2-$\mathrm{BMC}_4^{\leq}$.

(4.12) Further inequalities. From now on we will make no attempt towards classifying the remaining inequalities further. Their common characteristic is that their support graph is the complete graph with at most two edges missing. With two exceptions all inequalities have coefficients that are different from $0, \pm 1$. In order to be able to draw some of the support graphs a little more nicely and to display structure we will sometimes group the nodes into "supernodes" (large ellipses) and link ellipses by edges. If two ellipses are



(a) (b)

FIGURE 7

linked by an edge, then all nodes of one ellipse are linked to all nodes of the other ellipse by an edge (of the same weight). Nodes inside an ellipse are not necessarily connected by edges.

We begin with the two classes of pure inequalities (all coefficients are $0, \pm 1$). The first one, displayed in Figure 7(a), reads

$$\begin{aligned} &-x_{12} - x_{13} - x_{23} + x_{14} + x_{15} + x_{24} \\ (36) \qquad &\qquad + x_{25} + x_{34} + x_{35} \geq 1 \qquad \text{(10 permutations)} \end{aligned}$$

and is facet-defining for $2\text{-BMC}_2^{\geq}$, $2\text{-BMC}_3^{\leq}$, and $2\text{-BMC}_4^{\leq}$.

The second pure class is almost identical. Just one more positive coefficient, the last missing nonzero, appears. It is of the form (see Figure 7(b))

$$\begin{aligned} &-x_{12} - x_{13} - x_{23} + x_{14} + x_{15} + x_{24} + x_{25} \\ (37) \qquad &\qquad + x_{34} + x_{35} + x_{45} \geq 2 \qquad \text{(10 permutations)} \end{aligned}$$

and is facet-defining for $1\text{-BMC}_2^{\geq}$, $1\text{-BMC}_3^{\leq}$, and $1\text{-BMC}_4^{\leq}$. If we denote the triangle on nodes $1, 2, 3$ by T, the edges of $K_{2,3}$ (with $1, 2, 3$ on one side) by F, and the edge between 4 and 5 by e, we can write (36) and (37), respectively, as

$$-x(T) + x(F) \geq 1,$$
$$-x(T) + x(F) + x_e \geq 2.$$

Another way to view (36) and (37) in our Q-notation is

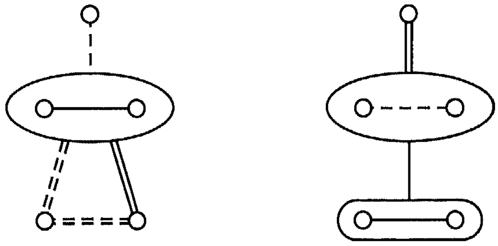
$$-Q(1, 1, 1, -1, -1) \cdot x + x_e \geq 1,$$
$$-Q(1, 1, 1, -1, -1) \cdot x + 2x_e \geq 2.$$

All further inequalities have some coefficients whose absolute value is at least 2. The largest coefficient appearing has absolute value 4.

The support of the next class of inequalities is the complete graph K_5. All coefficients have value +2, except for one triangle, where all edges have value 1. In Q-notation, it reads as follows:

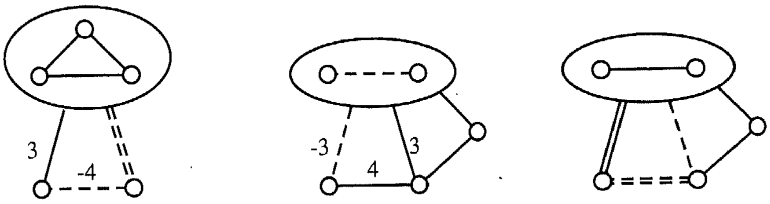
(38) $2Q(1, 1, 1, 1, 1) \cdot x - Q(1, 1, 1, 0, 0) \cdot x \leq 14$ (10 permutations)

and is facet-defining of $\text{MC}_3^{\leq}$, $\text{MC}_3^{=}$, $3\text{-BMC}_3^{\leq}$, and $2\text{-BMC}_3^{\leq}$.



(a) (b)

FIGURE 8



(a) (b) (c)

FIGURE 9

The support graph of inequality

$$-x_{12} - x_{13} + x_{23} + 2x_{24} - 2x_{25} + 2x_{34} \tag{39}$$
$$- 2x_{35} - 2x_{45} \leq 0 \qquad \text{(60 permutations)}$$

is shown in Figure 8(a). Inequalities of type (39) define facets of 2-$BMC_2^{\geq}$, 2-$BMC_3^{\leq}$, and 2-$BMC_4^{\leq}$.

The inequality depicted in Figure 8(b) reads

$$2x_{12} + 2x_{13} - x_{23} + x_{24} + x_{25} + x_{34} \tag{40}$$
$$+ x_{35} + x_{45} \geq 5 \qquad \text{(30 permutations)}$$

and defines a facet of $MC_3^{\geq}$ and $MC_3^{=}$.

The support graphs of the next three classes of inequalities are shown in Figure 9.

The inequality depicted in Figure 9(a) reads

$$-4x_{12} + 3x_{13} + 3x_{14} + 3x_{15} - 2x_{23} - 2x_{24} \tag{41}$$
$$- 2x_{25} + x_{34} + x_{35} + x_{45} \leq 5 \qquad \text{(20 permutations)}$$

and defines a facet of $MC_3^{\leq}$ and 3-$BMC_3^{\leq}$. In Q-notation we can write it as

$$Q(3, -2, 1, 1, 1) \cdot x + 2x_{12} \leq 5.$$

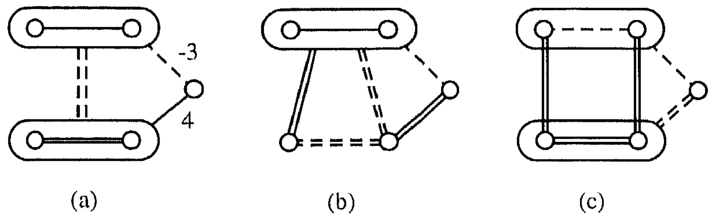


FIGURE 10

The inequality shown in Figure 9(b) is

$$4x_{12} - 3x_{13} - 3x_{14} + 3x_{23} + 3x_{24} + x_{25} - x_{34} + x_{35} + x_{45} \geq 1 \qquad \text{(60 permutations)} \tag{42}$$

and is facet-defining for 2-BMC$_2^{\geq}$, 2-BMC$_3^{\leq}$, and 2-BMC$_4^{\leq}$.

The inequality belonging to Figure 9(c) can be written as

$$-2x_{12} + 2x_{13} + 2x_{14} - x_{23} - x_{24} + x_{25} + x_{34} + x_{35} + x_{45} \leq 5 \qquad \text{(60 permutations)}. \tag{43}$$

In Q-notation, we can view it as

$$Q(2, -1, 1, 1, 1) \cdot x - 2x_{15} + 2x_{25} \leq 5.$$

It is facet-defining for MC$_3^{\leq}$, 3-BMC$_3^{\leq}$, and 2-BMC$_3^{\leq}$.

Figure 10 shows three further weighted support graphs. The inequality belonging to Figure 10(a) reads

$$4x_{12} + 4x_{13} - 3x_{14} - 3x_{15} + 2x_{23} - 2x_{24} - 2x_{25} - 2x_{34} - 2x_{35} + x_{45} \leq 2 \qquad \text{(30 permutations)} \tag{44}$$

and can be written in Q-notation as

$$Q(3, 2, 2, -1, -1) \cdot x - 2Q(1, 1, 1, 0, 0) \cdot x \leq 2.$$

It is facet-defining for MC, MC$_2^{\geq}$, MC$_3^{\leq}$, 3-BMC$_3^{\leq}$, and 3-BMC$_4^{\leq}$ and is a special case of the generalized cycle inequalities introduced in **[CR2]** for MC$_k^{\leq}(n)$.

The next inequality

$$2x_{12} - x_{14} - x_{15} - 2x_{23} - 2x_{24} - 2x_{25} + 2x_{34} + 2x_{35} + x_{45} \leq 2 \qquad \text{(60 permutations)} \tag{45}$$

is depicted in Figure 10(b) and can be written in Q-notation as

$$Q(-1, -2, 2, 1, 1) \cdot x + 2(x_{13} + x_{23}) \leq 2.$$

It is facet-defining for MC, MC$_2^{\geq}$, MC$_3^{\leq}$, 3-BMC$_3^{\leq}$, and 3-BMC$_4^{\leq}$. The inequality shown in Figure 10(c) has the form

$$-2x_{12} - 2x_{13} - x_{14} - x_{15} + 2x_{23} + 2x_{25} + 2x_{34} - x_{45} \leq 2 \qquad \text{(60 permutations)} \tag{46}$$

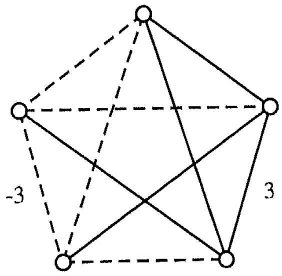


FIGURE 11

and is facet-defining for MC, $MC_2^{\geq}$, $MC_3^{\leq}$, 3-$BMC_3^{\leq}$, and 3-$BMC_4^{\leq}$. So the classes (44), (45), and (46) are facet-defining for the same multicut polytopes.

Finally, the class of inequalities represented by

$$\begin{aligned} &x_{12} + x_{13} - x_{14} - x_{15} + 3x_{23} + x_{24} - x_{25} \\ &\quad - x_{34} + x_{35} - 3x_{45} \leq 2 \qquad \text{(60 permutations)} \end{aligned} \tag{47}$$

and shown in Figure 11 is facet-defining for 1-$BMC_3^{\leq}$.

This finishes our list of inequalities. In Table 2 we give a comprehensive overview over all multicut polytopes of K_5 and their facets. The polytopes of Table 1 index the first 18 rows of the table. The column labeled v contains the number of vertices, the column labeled f the number of facets, and the column labeled d the dimension of each polytope. The columns labeled (1), (2), (3), ..., (47) correspond to the 47 classes of facets listed above. An entry "x" in the table denotes the fact that all inequalities of the class of facets of its column define a facet of the polytope of its row. The last three rows specify whether inequality is of "$\leq$" or "$\geq$" type, the value of the right-hand side of each inequality in the class, and the number of different inequalities in the class, respectively.

Without going into too much detail we mention a few artifacts that can be read from Table 2. The range of right-hand sides of the inequalities (assuming that they are in "$\leq$" form with integral and coprime coefficients) is -8 to 14. Only few facets appear frequently, namely the upper bounds (1), the triangle inequalities (8), and the 2-chorded 5-cycle inequalities (20). Four classes are facet-defining for just one polytope: (7), (24), (25), and (47). And there are clusters of facets that always appear together. For instance,

classes (18), (21), (29), (32), (35), (36), (39), (42)

define facets of 2-$BMC_2^{\geq}$, 2-$BMC_3^{\leq}$, and 2-$BMC_4^{\leq}$ but not of any other polytope (disregarding the lower-dimensional polytope 1-$BMC_2^{\leq}$). These polytopes, in fact, have 303 common facets in total. Moreover,

classes (6), (16), (30), (31), (40)

TABLE 2

polytopes	v	f	d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
MC	52	242	10	x							x	x	x	x		x							x				
$MC_2^{\le}$	16	56	10				x				x	x			x			x									
$MC_2^{\ge}$	51	264	10	x							x	x	x	x		x				x			x				
$MC_2^{=}$	15	68	10				x				x				x			x		x							
$MC_3^{\le}$	41	333	10	x		x					x	x	x	x			x						x				
$MC_3^{\ge}$	36	218	10	x					x		x			x		x			x							x	x
$MC_3^{=}$	25	99	10	x		x			x								x		x								
3-$BMC_3^{\le}$	40	355	10	x		x					x	x	x	x			x			x			x				
3-$BMC_4^{\le}$	50	265	10	x	x						x	x	x	x		x				x			x				
2-$BMC_2^{\ge}$	46	308	10	x				x			x			x		x					x		x	x		x	
2-$BMC_3^{\le}$	35	379	10	x		x		x			x			x			x				x		x	x		x	
2-$BMC_4^{\le}$	45	309	10	x	x			x			x			x		x					x		x	x		x	
1-$BMC_2^{\le}$	10	10	9												x												
1-$BMC_2^{\ge}$	36	113	10	x				x			x												x				
1-$BMC_3^{\le}$	24	169	10	x		x		x			x							x					x				
1-$BMC_3^{\ge}$	26	26	10	x						x												x			x		
1-$BMC_3^{=}$	15	25	9	x													x					x			x		
1-$BMC_4^{\le}$	35	114	10	x	x			x			x												x				
inequality type				≤	≤	≤	≤	≥	≥	≥	≤	≤	≤	≤	≤	≤	≤	≤	≥	≥	≥	≥	≤	≥	≥	≤	≤
right-hand side				1	9	8	6	6	7	8	0	0	1	1	2	3	5	2	3	2	2	2	2	2	3	0	0
no. of inequalities				10	1	1	1	1	1	1	30	10	5	20	10	5	5	5	12	12	15	10	12	5	5	20	30

polytopes	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
MC																				x	x	x	
$MC_2^{\le}$																							
$MC_2^{\ge}$			x																	x	x	x	
$MC_2^{=}$			x																				
$MC_3^{\le}$														x			x		x	x	x	x	
$MC_3^{\ge}$	x					x	x									x							
$MC_3^{=}$						x	x							x		x							
3-$BMC_3^{\le}$			x											x			x		x	x	x	x	
3-$BMC_4^{\le}$			x																	x	x	x	
2-$BMC_2^{\ge}$					x			x			x	x			x			x					
2-$BMC_3^{\le}$					x			x			x	x		x	x			x	x				
2-$BMC_4^{\le}$					x			x			x	x			x			x					
1-$BMC_2^{\le}$					x																		
1-$BMC_2^{\ge}$		x							x	x			x										
1-$BMC_3^{\le}$		x		x						x			x										x
1-$BMC_3^{\ge}$																							
1-$BMC_3^{=}$				x																			
1-$BMC_4^{\le}$		x							x	x			x										
inequality type	≤	≤	≥	≤	≥	≥	≥	≥	≤	≤	≥	≥	≥	≤	≤	≥	≤	≥	≤	≤	≤	≤	≤
right-hand side	0	0	2	6	4	4	5	6	0	0	2	1	2	14	0	5	5	1	5	2	2	2	2
no. of inequalities	30	20	10	10	10	10	20	20	20	10	30	10	10	10	60	30	20	60	60	30	60	60	60

are only facet-defining for $\mathrm{MC}_3^{\geq}$ and $\mathrm{MC}_3^{=}$,

classes (10), (44), (45), (46)

are only facet-defining for MC, $\mathrm{MC}_2^{\geq}$, $\mathrm{MC}_3^{\leq}$, 3-$\mathrm{BMC}_3^{\leq}$, and 3-$\mathrm{BMC}_4^{\leq}$ and do not appear anywhere else. Similarly,

classes (7), (19), (22)

are only facets of 1-$\mathrm{BMC}_3^{\geq}$ and do not appear anywhere else (disregarding the lower-dimensional polytope 1-$\mathrm{BMC}_3^{=}$),

classes (14), (38)

are only facet-defining for $\mathrm{MC}_3^{\geq}$, $\mathrm{MC}_3^{=}$, 3-$\mathrm{BMC}_3^{\leq}$, and 2-$\mathrm{BMC}_3^{\leq}$, and finally

classes (24), (25)

only define facets of the polytope $\mathrm{MC}_3^{\geq}$ and of no other multicut polytope.

5. Some observations

(5.1) On $(n-1)$-cut polytopes. We observed that the multicut polytopes $\mathrm{MC}_4^{=}(5)$ and $\mathrm{MC}_4^{\geq}(5)$ are both simplices in $\mathbf{R}^{10}$ of dimension 9 and 10, respectively; also, that the polytope $\mathrm{MC}_4^{\leq}(5)$ can be deduced from the polytope MC(5) simply by adding the following cardinality constraint: $\sum_{ij\in E} x_{ij} \leq 9$. In fact, these observations remain valid more generally for the multicut polytopes $\mathrm{MC}(n)$, $\mathrm{MC}_{n-1}^{=}(n)$, $\mathrm{MC}_{n-1}^{\geq}(n)$, and $\mathrm{MC}_{n-1}^{\leq}(n)$ defined, respectively, as the convex hull of all incidence vectors of multicuts, all $(n-1)$-cuts, all k-cuts with $k \geq n-1$, and all k-cuts with $k \leq n-1$, of the complete graph $K_n = (V, E)$ on n nodes.

The polytope $\mathrm{MC}_{n-1}^{\geq}(n)$ has $\binom{n}{2}+1$ vertices, all linearly independent, so $\mathrm{MC}_{n-1}^{\geq}(n)$ is the $\binom{n}{2}$-simplex in $\mathbf{R}^E$ defined completely and nonredundantly by

$$(48)\quad \mathrm{MC}_{n-1}^{\geq}(n) = \left\{ x \in \mathbf{R}^E : x_{ij} \leq 1 \text{ for all } ij \in E,\ \sum_{ij\in E} x_{ij} \geq \binom{n}{2} - 1 \right\}.$$

The polytope $\mathrm{MC}_{n-1}^{=}(n)$ has precisely one vertex less than $\mathrm{MC}_{n-1}^{\geq}(n)$, namely the incidence vector of the n-cut, i.e., the vector χ^E whose coordinates are all equal to 1, is a vertex of $\mathrm{MC}_{n-1}^{\geq}(n)$ but not of $\mathrm{MC}_{n-1}^{=}(n)$. So, $\mathrm{MC}_{n-1}^{=}(n)$ is the $(\binom{n}{2}-1)$-simplex defined completely and nonredundantly by

$$(49)\quad \mathrm{MC}_{n-1}^{=}(n) = \left\{ x \in \mathbf{R}^E : x_{ij} \leq 1 \text{ for all } ij \in E,\ \sum_{ij\in E} x_{ij} = \binom{n}{2} - 1 \right\}.$$

Moreover, the polytope $\mathrm{MC}_{n-1}^{\leq}(n)$ has precisely one less vertex than $\mathrm{MC}(n)$; namely, the only multicut vector that is not a vertex of $\mathrm{MC}_{n-1}^{\leq}(n)$

is the n-cut vector χ^E. The cardinality condition $\sum_{e\in E} x_e \le \binom{n}{2} - 1$ clearly defines a facet of $\mathrm{MC}^{\le}_{n-1}(n)$, so the following inclusion holds: $\mathrm{MC}^{\le}_{n-1}(n) \subseteq \mathrm{MC}(n) \cap \{x \in \mathbf{R}^E \mid \sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1\}$. Using the fact [W] that the vertices of $\mathrm{MC}(n)$ which are adjacent to χ^E on $\mathrm{MC}(n)$ are precisely the $(n-1)$-cut vectors, i.e., the multicut vectors lying on the hyperplane defined by the equation $\sum_{ij\in E} x_{ij} = \binom{n}{2} - 1$, we deduce that $\mathrm{MC}^{\le}_{n-1}(n)$ indeed coincides with $\mathrm{MC}(n) \cap \{x \in \mathbf{R}^E \mid \sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1\}$. Furthermore, every inequality defining a facet of $\mathrm{MC}(n)$ also defines a facet of $\mathrm{MC}^{\le}_{n-1}(n)$. This is clear for facets that do not contain χ^E. Note that the only facets of $\mathrm{MC}(n)$ containing χ^E are those defined by the upper bound conditions (1) $x_{ij} \le 1$. To see this, take a facet F of $\mathrm{MC}(n)$ containing χ^E. Then the vertices of F adjacent to χ^E together with χ^E form a set of rank $\binom{n}{2} - 1$. Since, as mentioned above, the multicut vectors adjacent to χ^E are the $(n-1)$-cut vectors, we deduce that F contains a set of rank $\binom{n}{2} - 1$ consisting of χ^E and $(n-1)$-cut vectors and hence F is indeed supported by some upper bound condition (1). Summarizing these observations we obtain

PROPOSITION 1. *For* $n \ge 3$, *the facets of* $\mathrm{MC}^{\le}_{n-1}(n)$ *are those of* $\mathrm{MC}(n)$ *together with the cardinality condition* $\sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1$. *Moreover, the following equality holds*:

$$\mathrm{MC}^{=}_{n-1}(n) = \mathrm{MC}^{\ge}_{n-1}(n) \cap \mathrm{MC}^{\le}_{n-1}(n).$$

One can observe, by looking at the table of facets (Table 2), that the following equality holds:

$$s\text{-}\mathrm{BMC}^{\le}_{4}(5) = s\text{-}\mathrm{BMC}^{\ge}_{2}(5) \cap \left\{ x \in \mathbf{R}^E \,\middle|\, \sum_{ij\in E} x_{ij} \le 9 \right\} \quad \text{for } s = 1, 2, 3.$$

Generally, the polytope $s\text{-}\mathrm{BMC}^{\le}_{n-1}(n)$ has the same vertices as the polytope $s\text{-}\mathrm{BMC}^{\ge}_{2}(n)$ except the n-cut vector χ^E which is cut off by the cardinality condition $\sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1$. It seems that the equality

$$s\text{-}\mathrm{BMC}^{\le}_{n-1}(n) = s\text{-}\mathrm{BMC}^{\ge}_{2}(n) \cap \left\{ x \in \mathbf{R}^E \,\middle|\, \sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1 \right\}$$

holds for any $n \ge 5$ and that the facets of $s\text{-}\mathrm{BMC}^{\le}_{n-1}(n)$ are those of $s\text{-}\mathrm{BMC}^{\ge}_{2}(n)$ together with the cardinality condition $\sum_{ij\in E} x_{ij} \le \binom{n}{2} - 1$.

(5.2) Connectivity of the facet support graphs. One can observe that, among the inequalities listed in §4, those defining a facet of some multicut polytope $\mathrm{MC}^{\le}_{k}(5)$ for some $k \in \{2, 3, 4, 5\}$ have a support graph that is 2-connected. More generally, one has the following result.

PROPOSITION 2. *The support graph of an inequality defining a facet of the polytope* $\mathrm{MC}_k^{\leq}(n)$, *for some* $2 \leq k \leq n$, *is 2-connected, i.e., the smallest number of nodes whose removal disconnects the graph is greater than or equal to* 2.

PROOF. The proof is a direct extension of the proof given in **[D]** for the case $k = 2$, i.e., for the cut polytope case. Take an inequality $v^T x \leq \alpha$ which defines a facet of $\mathrm{MC}_k^{\leq}(n)$ and let us assume that its support graph $G_v = (V, E(v))$ is not 2-connected. Suppose first that G_v admits an articulation node i_0. So $V = V_1 \cup V_2$, $E(v) = E_1 \cup E_2$ with $V_1 \cap V_2 = \{i_0\}$, and E_i is the edge set of G_v induced by V_i for $i = 1, 2$. Denote by v_i the vector of $\mathbf{R}^E$ whose projection on $\mathbf{R}^{E_i}$ coincides with the projection of v and whose projection on $\mathbf{R}^{E \setminus E_i}$ is the zero vector, for $i = 1, 2$. Thus, $v = v_1 + v_2$ holds.

Set $\alpha_i := \max\{v_i^T x | x \in \mathrm{MC}_k^{\leq}(n)\}$. Then the inequality $v_i^T x \leq \alpha_i$ is valid for $\mathrm{MC}_k^{\leq}(n)$ for $i = 1, 2$. We prove that $\alpha = \alpha_1 + \alpha_2$ holds. First, inequality $\alpha \leq \alpha_1 + \alpha_2$ holds trivially. To prove the converse inequality, take a multicut vector x_i of $\mathrm{MC}_k^{\leq}(n)$ satisfying $v_i^T x_i = \alpha_i$, for $i = 1, 2$. So x_1 is the incidence vector of an h-cut of the form $\delta(S_1, S_2, \ldots, S_h)$ with $h \leq k$, and x_2 is the incidence vector of an l-cut $\delta(T_1, \ldots, T_l)$ with $l \leq k$. We may assume that $h \leq l$ and that node i_0 belongs to $S_1 \cap T_1$. Consider the multicut vector x whose shores are the sets $(S_j \cap V_1) \cup (T_j \cap V_2)$ for $1 \leq j \leq h$, and $T_j \cap V_2$ for $h+1 \leq j \leq l$. So x is the incidence vector of a multicut having at most $l \leq k$ shores and satisfying $v_i^T x = v_i^T x_i = \alpha_i$ and, thus, $v^T x = \alpha_1 + \alpha_2 \leq \alpha$. Therefore, the inequality $v^T x \leq \alpha$ can be written as the sum of two valid inequalities, contradicting the fact that it is facet inducing. The proof is analogous in the case when the support graph is disconnected. □

This result does not extend to the case of the other multicut polytopes $\mathrm{MC}_k^{\geq}(n)$, $\mathrm{MC}_k^{=}(n)$ or the s-balanced multicut polytopes. For instance, the casserole inequalities (31), (32), (33) are connected, but not 2-connected. (See also the example of the facets from Proposition 4.) Moreover, the support graph of inequality (34) is disconnected.

(5.3) Cycle inequalities. We introduce a class of inequalities containing inequalities (17), (18), (19) as special cases. We call them *cycle inequalities* since their supporting graph is a cycle C and they are of the form:

$$\sum_{ij \in C} x_{ij} \geq 2. \tag{50}$$

Note that a multicut vector violates inequality (50) if and only if one of its shores contains all nodes of the cycle C. This implies easily that, for $2 \leq p \leq n-1$, if C is a cycle of length $p+1$, then the cycle inequality (50) is valid for the polytopes $\mathrm{MC}_{n-p+1}^{\geq}(n)$ and $(p-1)$-$\mathrm{BMC}_{n-p}^{\geq}$. Denote by $1, 2, \dots, p+1$ the nodes of C and by $p+2, \dots, n$ the remaining nodes. One checks easily that the roots of inequality (50) in $\mathrm{MC}^{\geq}_{n-p}(n)$, i.e., the multicut vectors x of $\mathrm{MC}^{\geq}_{n-p}(n)$ that satisfy inequality (50) at equality, are, setting $I_u := \{1, 2, \dots, u-1, u\}$ for $1 \leq u \leq p$, as follows.

(50a) x is a multicut vector with shores I_u, $\{1, \dots, p+1\}\backslash I_u$, and the singletons $\{i\}$ for $p+2 \leq i \leq n$. So x is the incidence vector of an $(n-p+1)$-cut that is $(p-1)$-balanced.

(50b) x is a multicut vector with shores $I_u \cup \{j\}$, $\{1, \dots, p+1\}\backslash I_u$, and the singletons $\{i\}$ for $p+2 \leq i \leq n$, $i \neq j$ with $p+2 \leq j \leq n$. So x is the incidence vector of an $(n-p)$-cut and is $(p-1)$-balanced if $u \neq p$.

Hence, if $p < n-1$, inequality (50) does not define a facet of $\mathrm{MC}^{\geq}_{n-p+1}(n)$, since the only multicut vectors in $\mathrm{MC}^{\geq}_{n-p+1}(n)$ satisfying inequality (50) with equality are those described in (50a). But these also satisfy the equation $x_{1n} = 1$. The next result indicates that (50) is indeed facet-defining in the remaining cases.

PROPOSITION 3. 1. *Let C be a cycle of length n. Then the cycle inequality* (50) *defines a facet of the polytopes* $\mathrm{MC}^{=}_{2}(n)$ *and* $\mathrm{MC}^{\geq}_{2}(n)$.

2. *Let C be a cycle of length $p+1$, $2 \leq p \leq n-1$. Then the cycle inequality* (50) *defines a facet of the polytope* $(p-1)$-$\mathrm{BMC}^{\geq}_{n-p}(n)$.

PROOF. We shall use the description given in (50a), (50b) of the roots of inequality (50).

1. Take an inequality $v^T x \geq \alpha$ that is valid for $\mathrm{MC}^{=}_{2}(n)$ and is satisfied with equality by all the roots of inequality (50) in $\mathrm{MC}^{=}_{2}(n)$. Set $S := \{2, \dots, u\}$ for $2 \leq u \leq p-1$; then we have

$$v^T \chi^{\delta(S)} = v^T \chi^{\delta(S\cup\{1\})} = v^T \chi^{\delta(S\cup\{u+1\})} = v^T \chi^{\delta(S\cup\{1, u+1\})} = \alpha,$$

from which one easily deduces that $v_{1, u+1} = 0$. Hence, by symmetry, $v_{ij} = 0$ for every edge ij that is not an edge of C. Using the fact that $v^T \chi^{\delta(\{1\})} = v^T \chi^{\delta(\{2\})} = v^T \chi^{\delta(\{1,2\})} = \alpha$, one deduces the $v_{12} = v_{23} = \alpha/2$. Therefore, we obtain that inequality $v^T x \geq \alpha$ is a multiple of inequality (50), implying that inequality (50) indeed defines a facet of $\mathrm{MC}^{=}_{2}(n)$ and thus of $\mathrm{MC}^{\geq}_{2}(n)$ too.

2. Take again an inequality $v^T x \geq \alpha$ that is valid for $(p-1)$-$\mathrm{BMC}^{\geq}_{n-p}(n)$ and admits the same roots in $(p-1)$-$\mathrm{BMC}^{\geq}_{n-p}(n)$ as inequality (50). Set $S := \{2, \dots, u\}$ for $2 \leq u \leq p-1$ and $S' := \{u+2, \dots, p+1\}$. Let δ_1 (respectively, δ_2, δ_3, δ_4) denote the multicut whose shores are the sets S and $S' \cup \{1, u+1\}$ (respectively, $S \cup \{1\}$ and $S' \cup \{u+1\}$, $S \cup \{1, u+1\}$ and S', $S \cup \{u+1\}$ and $S' \cup \{1\}$) together with the singletons $\{i\}$ for

$p+2 \leq i \leq n$. So $\delta_1, \delta_2, \delta_3, \delta_4$ are of type (50a). Using the fact that $v^T\chi^{\delta_1} - v^T\chi^{\delta_2} = v^T\chi^{\delta_3} - v^T\chi^{\delta_4} = 0$, one deduces easily that $v_{1u+1} = 0$. Take $j \in \{p+2, \dots, n\}$. Then both multicuts with shores $\{1\}, \{2, \dots, p+1\}$ and singletons $\{i\}$ for $p+2 \leq i \leq n$ and with shores $\{1, j\}, \{2, \dots, p+1\}$ and singletons $\{i\}$ for $p+2 \leq i \leq n$, $i \neq j$, are roots (of type (50b)) yielding that $v_{1j} = 0$. One now deduces easily that $v^T x \geq \alpha$ is a multiple of inequality (50), which indeed shows that (50) defines a facet of $(p-1)$-$\mathrm{BMC}^{\geq}_{n-p}(n)$. □

Finally note that, if C is a cycle of length n, then the inequality $\sum_{ij \in C} x_{ij} \geq k$ is trivially valid on $\mathrm{MC}^{\geq}_k(n)$.

(5.4) A class of "casserole" facets. We saw that the casserole inequality (31) defines a facet of the polytope $\mathrm{MC}^{\geq}_3(5)$. The following inequality can be seen as an extension of inequality (31):

$$\sum_{1 \leq i < j \leq n-1} x_{ij} + 2x_{1n} \geq \binom{n-1}{2} - 1 \tag{51}$$

Proposition 4. *Inequality* (51) *defines a facet of the polytopes* $\mathrm{MC}^{\geq}_{n-2}(n)$ *and* $\mathrm{MC}^{=}_{n-2}(n)$ *for* $n \geq 5$.

Proof. Validity of inequality (51) for $\mathrm{MC}^{\geq}_{n-2}(n)$ is easy to check. Its roots are the following multicut vectors x:

(1) x is the incidence vector of an $(n-2)$-cut whose shores are singletons except two shores that are pairs $\{1, n\}, \{i, j\}$ for $2 \leq i < j \leq n-1$.
(2) x is the incidence vector of an $(n-2)$-cut whose shores are singletons except one shore that is a triple which is either $\{1, i, j\}$ with $2 \leq i < j \leq n-1$, or $\{1, n, i\}$ with $2 \leq i \leq n-1$, or $\{i, j, k\}$ with $2 \leq i < j < k \leq n-1$.

Using the above description of the roots, it is not difficult to show that any inequality $v^T x \geq \alpha$ which is valid for $\mathrm{MC}^{=}_{n-2}(n)$ and is satisfied at equality by the multicut vectors defined in (1), (2), (3) above is indeed a multiple of inequality (51). □

(5.5) Some classes of "crown" facets. We introduce two classes of inequalities (52), (53), which contain as special cases the "crown" inequalities (23), (24), and (27), respectively.

Proposition 5. *Consider a partition of* $\{1, 2, \dots, n\}$ *into subsets* A, A', *and* $\{1, n\}$. *Then the inequality*

$$-(n-3)x_{1n} + \sum_{i \in A}(x_{1i} - x_{in}) + \sum_{i \in A'}(x_{in} - x_{1i}) \leq 0 \tag{52}$$

defines a facet of $\mathrm{MC}^{\geq}_3(n)$ *for* $n \geq 5$.

PROOF. One can easily check that inequality (52) is valid for $MC_3^{\geq}(n)$ and that its roots in $MC_3^{\geq}(n)$ are the incidence vectors x of the multicuts for which either

- both nodes 1, n belong to the same shore of the multicut, or
- the multicut is of the form $\delta(A' \cup \{1\}, A \setminus \{i\} \cup \{n\}, \{i\})$ for some $i \in A$, or
- the multicut is of the form $\delta(A' \setminus \{i\} \cup \{1\}, A \cup \{n\}, \{i\})$ for some $i \in A'$.

Then one can verify that the set of roots of (52) has rank $\binom{n}{2} - 1$ implying that inequality (52) defines a facet for $MC_3^{\geq}(n)$. □

Note that inequality (52) is not valid for $MC_2^{=}(n)$. It is, in fact, violated by the incidence vector of the 2-cut $\delta(A \cup \{n\})$ but by no other 2-cut vector. Note also that, for $n = 5$, inequality (52) coincides with (23) if $|A| = 3$ and with (24) if $|A| = 1$ or 2.

PROPOSITION 6. *The inequality*

$$(53) \qquad (n-4)x_{1n} - \sum_{2 \leq i \leq n-2} (x_{1i} + x_{in}) \leq -2$$

defines a facet of the polytopes $MC_2^{=}(n)$, $MC_2^{\geq}(n)$.

PROOF. It is easy. We simply mention that the roots of inequality (53) in $MC_2^{=}(n)$ are the incidence vectors of the 2-cuts $\delta(\{i\})$ for $2 \leq i \leq n-1$, and $\delta(S \cup \{n\})$ for any subset S of $\{2, 3, \ldots, n-1\}$. □

Note that inequality (53) for $n = 5$ coincides with inequality (27).

(5.6) On the polytopes $1\text{-}BMC_{n-2}^{\geq}(n)$, $MC_{n-2}^{=}(n)$. The balanced multicut polytope $1\text{-}BMC_3^{\geq}(5)$ has 26 vertices and 26 facets which are of one of the following four types:

- upper bound condition (1) $x_{12} \leq 1$,
- cardinality condition (7) $\sum_{ij \in E} x_{ij} \geq 8$,
- lower bound on 3-cycle condition (19) $x_{12} + x_{13} + x_{23} \geq 2$,
- degree condition (22) $\sum_{i=2}^{5} x_{1i} \geq 3$.

All of them extend to the general balanced multicut polytope $1\text{-}BMC_{n-2}^{\geq}(n)$. The polytope $1\text{-}BMC_{n-2}^{\geq}(n)$ has $\frac{1}{2}\binom{n}{2}\binom{n-1}{2} + \binom{n}{2} + 1$ vertices that are incidence vectors of the n-cut, the $(n-1)$-cuts, and the $(n-2)$-cuts whose shores are singletons except two shores which are pairs. First, the upper bound condition (1) clearly defines a facet of $1\text{-}BMC_{n-2}^{\geq}(n)$. The cardinality condition for general n is as follows:

$$(54) \qquad \sum_{1 \leq i < j \leq n} x_{ij} \geq \binom{n}{2} - 2.$$

One can easily check that inequality (54) indeed defines a facet of 1-$\mathrm{BMC}_{n-2}^{\geq}(n)$. We already proved in Proposition 3 that the lower bound on a 3-cycle defines a facet of 1-$\mathrm{BMC}_{n-2}^{\geq}(n)$. Finally, the degree condition for general n is as follows:

$$\sum_{i=2}^{n} x_{1i} \geq n-2, \tag{55}$$

and it is easily verified that inequality (55) defines a facet of 1-$\mathrm{BMC}_{n-2}^{\geq}(n)$.

We finally mention another multicut polytope for which all its facets admit an easy generalization. This is the case for the polytope $\mathrm{MC}_3^{=}(5)$ that has nine types of facets, namely those defined by inequalities (1), (3), (6), (14), (16), (30), (31), (38), and (40). All of them extend to the polytope $\mathrm{MC}_{n-2}^{=}(n)$ for any $n \geq 5$. The polytope $\mathrm{MC}_{n-2}^{=}(n)$ has as vertices the incidence vectors of the $(n-2)$-cuts, i.e., the multicuts whose shores are all singletons except two pairs, or all singletons except one triple. For each of the facet-defining inequalities of $\mathrm{MC}_3^{=}(5)$, we can state a general inequality that is valid for $\mathrm{MC}_{n-2}^{=}(n)$ as follows:

- as an extension of the cardinality constraints (3), (6), the inequalities

  $$\binom{n}{2} - 3 \leq \sum_{1 \leq i < j \leq n} x_{ij} \leq \binom{n}{2} - 2$$

  are valid for $\mathrm{MC}_{n-2}^{=}(n)$,
- as an extension of (14), inequality

  $$\sum_{1 \leq i < j \leq n-1} x_{ij} \leq \binom{n-1}{2} - 1$$

  is valid for $\mathrm{MC}_{n-2}^{\leq}(n)$,
- as an extension of (16), inequality

  $$\sum_{ij \in C} x_{ij} \geq n-2,$$

  where C is a cycle of length n, is valid for $\mathrm{MC}_{n-2}^{\geq}(n)$,
- as an extension of (30), inequality

  $$\sum_{1 \leq i < j \leq n} x_{ij} - (x_{12} + x_{13} + x_{23} + x_{45}) \geq \binom{n}{2} - 6$$

  is valid for $\mathrm{MC}_{n-2}^{=}(n)$,
- as an extension of (31), we have the casserole inequality (51) which is facet-defining for $\mathrm{MC}_{n-2}^{=}(n)$ (see Proposition 4),

- as an extension of (38), inequality

$$2 \sum_{1 \le i < j \le n} x_{ij} - (x_{12} + x_{13} + x_{23}) \le 2\binom{n}{2} - 6$$

is valid for $\mathrm{MC}^{\le}_{n-2}(n)$,

- as an extension of (40), inequality

$$\sum_{1 \le i < j \le n-1} x_{ij} + 2(x_{1n} + x_{2n} - x_{12}) \ge \binom{n-1}{2} - 1$$

is valid for $\mathrm{MC}^{\ge}_{n-2}(n)$.

6. Complete descriptions of the multicut polytopes for $n = 4$

For completeness we now list also all multicut polytopes introduced in Definition 2 for the complete graph $K_4 = (V, E)$. Our numerical computations did not produce any new and interesting class of inequalities. All classes of inequalities that appear as facet-defining inequalities for at least one of the polytopes belong to some general type.

As in §3 we number the 15 different multicut vectors to be able to give short definitions of the polytopes of interest. We do this as follows:

1 : 1234	2 : 1, 234	3 : 2, 134
4 : 3, 124	5 : 4, 123	6 : 12, 34
7 : 13, 24	8 : 14, 23	9 : 1, 2, 34
10 : 1, 3, 24	11 : 1, 4, 23	12 : 2, 3, 14
13 : 2, 4, 13	14 : 3, 4, 12	15 : 1, 2, 3, 4

So if we sort the edges ij of E as $12, 13, 14, 23, 24, 34$, then the 4th multicut vector, for instance, is the vector $\chi^{(\delta\{3\}, \{1,2,4\})} = (0, 1, 0, 1, 0, 1)$.

We now list all polytopes that arise for $n = 4$, $k = 2, 3$ and $s = 0, 1, 2, 3$ leaving out trivial cases of polytopes that are empty, single points, or otherwise trivial or those that are equal to some other polytope listed. The facets mentioned will be introduced in the sequel. The ordering follows the same lexicographic scheme as explained for $n = 5$.

$\mathrm{MC}(4) = \mathrm{conv}\{1, 2, \dots, 15\}, \quad v = 15, \quad f = 22,$
facets: (56), (64), (65),

$\mathrm{MC}_2^{\leq}(4) = \mathrm{conv}\{1, 2, \dots, 8\}, \quad v = 8, \quad f = 16,$
facets: (59), (64),

$\mathrm{MC}_2^{\geq}(4) = \mathrm{conv}\{2, 3, \dots, 15\}, \quad v = 14, \quad f = 25,$
facets: (57), (61), (64), (65),

$\mathrm{MC}_2^{=}(4) = \mathrm{conv}\{2, 3, \dots, 8\}, \quad v = 7, \quad f = 7,$
facets: (59), (61),

$\mathrm{MC}_3^{\leq}(4) = \mathrm{conv}\{1, 2, \dots, 14\}, \quad v = 14, \quad f = 23,$
facets: (57), (58), (64), (65),

$\mathrm{MC}_3^{\geq}(4) = \mathrm{conv}\{9, 10, \dots, 15\}, \quad v = 7, \quad f = 7,$
facets: (57), (60),

$\mathrm{MC}_3^{=}(4) = \mathrm{conv}\{9, 10, \dots, 14\}, \quad v = 6, \quad f = 6,$ dimension $= 5$,
facets: (57), equation: (56),

$2\text{-}\mathrm{BMC}_3^{\leq}(4) = \mathrm{conv}\{2, 3, \dots, 14\}, \quad v = 13, \quad f = 26,$
facets: (57), (58), (61), (64), (65),

$1\text{-}\mathrm{BMC}_2^{\geq}(4) = \mathrm{conv}\{6, 7, \dots, 15\}, \quad v = 10, \quad f = 14,$
facets: (57), (62), (63),

$1\text{-}\mathrm{BMC}_3^{\leq}(4) = \mathrm{conv}\{6, 7, \dots, 14\}, \quad v = 9, \quad f = 14,$
facets: (57), (58), (62), (63).

The polytopes $\mathrm{MC}_2^{=}(4)$, $\mathrm{MC}_3^{\geq}(4)$, and $\mathrm{MC}_3^{=}(4)$ are simplices.

The following equation defines an affine space that is spanned by one of the polytopes:

(56) $$x(E) = 5.$$

Upper bounds on edge sets

(57) $x_{12} \leq 1$ (6 permutations),

(58) $x(E) \leq 1$ (1 permutation),

(59) $x(T) \leq 2$ (T a triangle) (4 permutations).

Lower bounds on edge sets

(60) $x(E) \geq 5$ (1 permutation),

(61) $x(C) \geq 2$ (C a 4-cycle) (3 permutations),

(62) $x(T) \geq 2$ (T a triangle) (4 permutations),

(63) $x(\delta(i)) \geq 2$ (4 permutations).

Hypermetric inequalities

(64) $Q(1, 1, -1, 0) \cdot x \leq 0$ (triangle inequalities) (12 permutations),

(65) $Q(1, 1, 1, -1) \cdot x \leq 1$ (4 permutations).

This finishes our list of facet-defining inequalities for multicut polytopes of K_4.

References

[BJR] F. Barahona, M. Jünger, and G. Reinelt, *Experiments in quadratic 0-1 programming*, Math. Programming **44** (1989), 127–137.

[BGJR] F. Barahona, M. Grötschel, M. Jünger, and G. Reinelt, *An application of combinatorial optimization to statistical physics and circuit layout design*, Oper. Res. **36** (1988), 493–513.

[BM] F. Barahona and E. Maccioni, *On the exact ground states of three-dimensional Ising spin glasses*, J. Phys. A **15** (1982), L611–L615.

[BMa] F. Barahona and A. R. Mahjoub, *On the cut polytope*, Math. Programming **36** (1986), 157–173.

[CR1] S. Chopra and M. R. Rao, *The partition problem* (submitted).

[CR2] ——, *Facets of the k-partition polytope* (submitted).

[CRS] M. Conforti, M. R. Rao, and A. Sassano, *The equipartition polytope*. I, II, Math. Programming **49** (1990/91), 49–70 and 71–90.

[D] C. De Simone, *Lifting facets of the cut polytope*, Operations Research Letters **9** (1990), 341–344.

[DFL] M. Deza, K. Fukuda and M. Laurent, *The inequicut cone*, Research report No. 89-04, The University of Tsukuba, Tokyo. Japan, 1989.

[DGL] M. Deza, M. Grötschel and M. Laurent, *Clique-Web facets for multicut polytopes*, Report No. 186, Schwerpunktprogramm der Deutschen Forschungsgemeinschaft, Universität Augsburg, Germany, 1989.

[DL] M. Deza and M. Laurent, *Facets for the cut cone*. I, Research Memorandum RMI 88-13, University of Tokyo (1988); Math. Programming (to appear).

[GW1] M. Grötschel and Y. Wakabayashi, *A cutting plane algorithm for a clustering problem*, Math. Programming B **45** (1989), 59–96.

[GW2] ——, *Facets of the clique partitioning polytope*, Math. Programming **47** (1990), 367–387.

[W] Y. Wakabayashi, *Aggregation of binary relations: algorithmic and polyhedral investigations*, Ph.D. Thesis, Universität Augsburg, West Germany, 1986.

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