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GRAPHS WITH CYCLES CONTAINING GIVEN PATHS

M. GRÖTSCHEL
Institut für Ökonometrie und Operations Research, D–53 Bonn, Nassestraße 2, F.R.G.

In this note we establish a sufficient condition for the following property of a graph: given any path of length r there is a cycle of length at least $m \geq r+3$ containing this path. The theorem implies the well-known theorem of Chvátal [4] on hamiltonian graphs and the theorem of Pósa [7] which gives sufficient conditions for a graph to contain cycles of a certain length. It is shown that the theorem is neither stronger nor weaker than the theorem of Bondy [3] and the still unsettled conjecture of Woodall [8].

1. Notation

The graphs $G=(V,E)$ considered are undirected, loopless, and without multiple edges. The degree $d(v)$ of a vertex $v \in V$ is the number of edges $e \in E$ containing v. A non-decreasing sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers will be called a *degree sequence* if there is a graph G with n vertices $v_1, \ldots, v_n$ such that $d(v_i) = d_i$, $i = 1, \ldots, n$. A sequence $t_1, \ldots, t_n$ *majorizes* a sequence $d_1, \ldots, d_n$ if $t_i \geq d_i$, $i = 1, \ldots, n$. A sequence $P = (v_1, \ldots, v_p)$ of distinct vertices of V is called a *path* if $\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, p-1$. The *length* of the path is $p-1$. $\bar{P} = (v_p, v_{p-1}, \ldots, v_1)$ is also a path and will be called the *reverse* of P. If furthermore $\{v_1, v_p\} \in E$, P is a *cycle* of length p and will be denoted by $[v_1, \ldots, v_p]$. Sometimes we will write $[v_1, \ldots, v_p, v_1]$ instead of $[v_1, \ldots, v_p]$ for clarity. A path from v_1 to v_q, $q \leq p$, along P will be denoted by (v_1, P, v_q). If two paths $P' = (v'_1, \ldots, v'_p)$ and $P'' = (v''_1, \ldots, v''_q)$ have exactly one vertex $v'_r = v''_s$ in common then $P = (v'_1, P', v'_r, P'', v''_q)$ is a well-defined path from v'_1 to v''_q. By $N(v)$ we denote the set of neighbours of v, i.e. the set of vertices $w \in V$ such that $\{v, w\} \in E$. $|M|$ is the cardinality of a set M. $\lfloor x \rfloor$ is the greatest integer k with $k \leq x$, $\lceil x \rceil$ is the smallest integer k with $k \geq x$.

2. Properties of h-connected graphs

As a tool for further proofs we cite and prove some results concerning h-connected graphs, i.e. graphs which remain connected after the deletion of any $h-1$ vertices.

The first theorem is due to Bondy, see [1, p. 173].

Proposition 1 (Bondy). *Let G be a graph with degree sequence $d_1, \ldots, d_n$ such that for some integer $h < n$ the following holds:*

$$d_k \geq k + h - 1 \quad \text{for all} \quad 1 \leq k \leq n - d_{n-h+1} - 1. \tag{1}$$

Then G is h-connected. □

A well-known property of h-connected graphs is the following, cf. [1, p. 168]:

Proposition 2. *If G is h-connected then the induced subgraph obtained by removing one vertex is $(h-1)$-connected.* □

The next two theorems can also be found in [1, p. 169]:

Proposition 3. *Let $G = (V, E)$ be h-connected. Let $W = \{w_1, \ldots, w_h\}$ be a set of vertices, $|W| = h$. If $v \in V - W$, there exist h vertex-disjoint paths $(v, \ldots, w_i)$, $i = 1, \ldots, h$, joining v and W.* □

Proposition 4. *Let G be a h-connected graph, $h \geq 2$. Then there is a cycle passing through an arbitrary set of two edges and $h-2$ vertices.* □

A frequently used theorem is the following, see [2, p. 192]:

Proposition 5 (Menger–Dirac). *Let $P = (a_0, a_1, \ldots, a_p)$ be a path. If G is 2-connected then there exist two paths P' and P'' with the following properties:*

(a) *the endpoints of P' and P'' are a_0 and a_p,*

(b) *P' and P'' have no other points in common,*

(c) *if P' (or P'') contains vertices of P, then they appear in P' (or P'') in the same order as they do in P.* □

We now give an extension of Proposition 3 which will be of interest later.

Proposition 6. *Let G be a 3-connected graph and $P = (a_0, \ldots, a_p)$ be a path, let $\{a_s, a_{s+1}\}$ be an edge of this path. Then there exists a pair of paths P', P'' with the following properties:*

(a) *The endpoints of P' and P'' are a_0 and a_p,*

(b) *P' and P'' have no other points in common,*

(c) *if P' (or P'') contains vertices of P, then they appear in P' (or P'') in the same order as they do in P,*

(d) *P' contains $\{a_s, a_{s+1}\}$.*

Proof. By induction.

(1) Let $P = (a_0, a_1)$, i.e. P is an edge. Then necessarily $s = 0$. As G is 2-connected, there is another path P'' from a_0 to a_1. Take $P' = P$.

(2) $P = (a_0, a_1, a_2)$, $s = 1$. By Proposition 3 there are two vertex-disjoint paths $P_1 = (a_0, \ldots, a_1)$ and $P_2 = (a_0, \ldots, a_2)$. Define $P' = (a_0, P_1, a_1, a_2)$, $P'' = P_2$. The case $s = 0$ is similar.

(3) $P = (a_0, a_1, a_2, a_3)$, $s = 1$. By Proposition 3 there are three vertex-disjoint paths (G is 3-connected): $P_1 = (a_0, \ldots, a_1)$, $P_2 = (a_0, \ldots, a_2)$, $P_3 = (a_0, \ldots, a_3)$. Define $P' = (a_0, P_1, a_1, a_2, a_3)$ and $P'' = P_3$. All other cases are similar.

Now suppose the theorem is true for paths of length k. We prove that it is true for paths of length $k + 1$.

Let $P = (a_0, a_1, \ldots, a_{k+1})$, $P_1 = (a_0, P, a_k)$.

We may assume that $s < k - 1$, otherwise we take the reverse $\bar{P}$ of P. By assumption there exist paths P_1' and P_1'' connecting a_0 and a_k having the desired properties with respect to P_1. From G we now remove the vertex a_k and add the edge $\{a_0, a_{k+1}\}$, if it does not already exist. By Proposition 2 the new graph G' is 2-connected. By Proposition 4 there is a cycle in G' containing the edges $\{a_s, a_{s+1}\}$ and $\{a_0, a_{k+1}\}$. Thus there is a path $Q = (a_0, a_1', \ldots, a_q', a_{k+1})$ in G connecting a_0 and a_{k+1}, which contains the edge $\{a_s, a_{s+1}\}$ and does not contain the vertex a_k.

Let x be the vertex of path Q which is as close as possible to a_{k+1} and is contained in the union of the vertex sets P_1, P_1', and P_1''. Clearly x lies between a_{s+1} and a_{k+1} on the path Q as a_{s+1} is in Q and in P_1'. If x is in P_1' then x lies between a_{s+1} and a_k in P_1'. We now have to investigate several cases.

(i) $x = a_{k+1}$	(a) $x \in P_1'$	$P' = (a_0, P_1', x)$, $P'' = (a_0, P_1'', a_k, a_{k+1})$,
	(b) $x \in P_1''$	$P' = (a_0, P_1', a_k, a_{k+1})$, $P'' = (a_0, P_1'', x)$.
(ii) x not in P	(a) $x \in P_1'$	$P' = (a_0, P_1', x, Q, a_{k+1})$, $P'' = (a_0, P_1'', a_k, a_{k+1})$,
	(b) $x \in P_1''$	$P' = (a_0, P_1', a_k, a_{k+1})$, $P'' = (a_0, P_1'', x, Q, a_{k+1})$.

(iii) x in P but $x \neq a_{k+1}$, say $x = a_r$, $r \geq s + 1$. Let $p \leq r$ be the largest index such that a_p is contained in the union of the vertex sets of P_1' and P_1''.

(a) $a_p \in P_1'$	$P' = (a_0, P_1', a_p, P, a_r, Q, a_{k+1})$, $P'' = (a_0, P_1'', a_k, a_{k+1})$,
(b) $a_p \in P_1''$	$P' = (a_0, P_1', a_k, a_{k+1})$, $P'' = (a_0, P_1'', a_p, P, a_r, Q, a_{k+1})$.

These are all the cases which have to be considered and hence we are done. □

Corollary 7. *Let G be $(r + 2)$-connected and $P = (a_0, \ldots, a_p)$ be a path, $r \leq p$, let $Q = (a_s, \ldots, a_{s+r})$ be a path of length r contained in P. Then there exists a pair of paths P', P'' with the following properties:*

(a) *the endpoints of P' and P'' are a_0 and a_p,*

(b) *P' and P'' have no other points in common,*

(c) *if P' (or P'') contains vertices of P, then they appear in P' (P'') in the same order as they do in P,*

(d) *P' contains the path Q.*

Proof. $r = 0$: Then by definition Q is an empty path and Corollary 7 reduces to Proposition 5.

$r = 1$: This is Proposition 6.

$r > 1$: Remove the $r-1$ vertices $a_{s+1}, a_{s+2}, \ldots, a_{s+r-1}$ and add the edge $\{a_s, a_{s+r}\}$. The resulting graph G' is 3-connected by Proposition 2. The path $P_1 = (a_0, \ldots, a_s, a_{s+r}, \ldots, a_p)$, contains the edge $\{a_s, a_{s+r}\}$. Application of Proposition 6 gives two paths P_1' and P_1'', and P_1' contains $\{a_s, a_{s+r}\}$. The path $P' = (a_1, P_1', a_s, Q, a_{s+r}, P_1', a_p)$ is well defined in G. Define $P'' = P_1''$, then the pair P', P'' has the desired properties. □

3. The theorem and its corollaries

The following theorem establishes a sufficient condition — in terms of the degree sequence — for the following property of a graph: given any path of a specified length, there exists a cycle containing this path and having a certain minimum length. Formally the theorem is very like a theorem of Berge [1, p. 204], which is an extension of a theorem of Chvátal [4] on hamiltonian graphs. The proof of case (i) below is a slight variation of their proof which — in spirit — is due to Nash-Williams [6]. Case (ii) of the proof was motivated by Pósa's proof of his own theorem [7] which is also included in the following:

Theorem 8. *Let $d_1, \ldots, d_n$ be the degree sequence of a graph $G = (V, E)$. Let $n \geq 3$, $m \leq n$, $0 \leq r \leq m-3$, and let the following condition be satisfied:*

$$d_k \leq k + r \implies d_{n-k-r} \geq n - k \quad \text{for all} \quad 0 < k < \tfrac{1}{2}(m-r). \tag{2}$$

Furthermore, let G be $(r+2)$-connected if $\frac{1}{2}(m-r) \leq n - d_{n-r-1} - 1$ holds and $d_k > k + r$ holds for all $0 < k < \frac{1}{2}(m-r)$. Then for each path Q of length r there exists a cycle in G of length at least m which contains Q.

Proof. (1) We prove: G is $(r+2)$-connected. Let $h = r + 2 < n$, then (2) is equivalent to

$$d_k \leq k + h - 2 \implies d_{n-h+2-k} \geq n - k \quad \text{for all } 0 < k < \tfrac{1}{2}(m - h + 2). \tag{2'}$$

(a) Suppose there exists a j such that $0 < j < \frac{1}{2}(m-h+2)$ and $d_j \leq j + h - 2$. Condition (2') implies $d_{n-h+2-j} \geq n - j$. As $d_{n-h+1} \geq d_{n-h+2-j}$, we obtain $j > n - (n-j) - 1 \geq n - d_{n-h+1} - 1$. Thus if $d_k < k + h - 1$, then $k > n - d_{n-h+1} - 1$. Therefore the conditions of Proposition 1 are satisfied and G is h-connected.

(b) Suppose $d_k \geq k + h - 1$ for all $0 < k < \frac{1}{2}(m - h + 2)$, then G is h-connected by Proposition 1 if $\frac{1}{2}(m-h+2) > n - d_{n-h+1} - 1$. Otherwise h-connectedness follows from the assumption. We note for the following that $(r+2)$-connectedness implies $d_1 \geqslant r+2$.

(2) It is an easy exercise to see that a graph G' obtained from G by adding any new edge to G also satisfies (2) and the other conditions of the theorem.

(3) Suppose now that G is a graph satisfying the required conditions but which contains a path Q of length r such that Q is not contained in a cycle of length $\geqslant m$. By adding new edges to G we construct a "maximal" graph (also called G) which satisfies all the conditions of the theorem, contains a path Q of length r, has no cycle of length $\geqslant m$ containing Q, and has the property that the addition of any new edge to G creates a cycle of length $\geqslant m$ which contains Q. In the following we shall deal with this maximal graph G.

(4) Let $u, v \in V$ be two nonadjacent vertices of G. The addition of the edge $\{u, v\}$ will create a cycle with the desired properties. Thus there exists a path

$$P := (u_1, \ldots, u_p),\ u_1 = u,\ u_p = v,\ p \geqslant m$$

of length $\geqslant m-1$ connecting u and v, and which contains

$$Q := (u_s, \ldots, u_{s+r}), \quad \text{where } s \in \{1, \ldots, p-r\}.$$

Let

$$S := \{i \in \{1, \ldots, p\} : \{u_1, u_{i+1}\} \in E\} \cap (\{1, \ldots, s-1\} \cup \{s+r, \ldots, p\})$$

$$T := \{i \in \{1, \ldots, p\} : \{u_p, u_i\} \in E\}.$$

(a) We prove: $S \cap T = \phi$. Suppose $i \in S \cap T$, then $[u_1, u_{i+1}, P, u_p, u_i, \bar{P}, u_1]$ is a cycle with the desired properties. Contradiction!

(b) $|S| + |T| \leqslant |P| - 1$ because $p \notin S \cup T$.

(5) The degree sequence of G necessarily has exactly one of the following properties:

Case (i) there is a k_0, $0 < k_0 < \frac{1}{2}(m-r)$, such that $d_{k_0} \leqslant k_0 + r$,

Case (ii) $d_k > k + r$ for all $0 < k < \frac{1}{2}(m-r)$.

These cases will be handled separately.

Case (i).

(6) As $d_1 \geqslant r+2$ and as the degree sequence $d_1, \ldots, d_n$ is increasing there is a $j \leqslant k_0$ such that $d_j = j + r$. (2) implies $d_{n-j-r} \geqslant n - j$, i.e. there are $j+r+1$ vertices of V having degree at least $n-j$. The vertex having degree $j+r$ cannot be adjacent to all of these. Thus there exist two nonadjacent vertices $a, b \in V$ such that $d(a) + d(b) \geqslant n + r$.

(7) Among all nonadjacent vertices of G choose u, v such that $d(u) + d(v)$ is as large as possible. Define P, S, T, Q as in (4). We calculate $d(u) + d(v)$. Obviously

$$d(v) = |T| + \alpha \text{ where } \alpha \leqslant |V - P|$$

and

$$d(u) \leqslant |S| + r + \beta \text{ where } \beta \leqslant |V - P|.$$

Suppose there is a $w \in V-P$ which is adjacent to both u and v. Then $[u_1, u_2, \ldots, u_p, w]$ would be a desired cycle. Therefore $\alpha+\beta \leqslant |V-P|$, which leads, using (4) (a) and (b), to

$$\begin{aligned} d(u)+d(v) &\leqslant |T|+\alpha+|S|+r+\beta \\ &\leqslant |P|-1+\alpha+\beta+r \\ &\leqslant |P|+|V-P|+r-1 \\ &\leqslant n+r-1. \end{aligned}$$

By (6) $d(u)+d(v)$ cannot be maximal. Contradiction!

Case (ii).

(8) Among all longest paths in G containing Q choose a path such that the sum of the degrees of the endpoints is as large as possible. As G is maximal, the length of this path is at least $m-1$, and the endpoints are not joined by an edge. Let this path be $P=(u_1, \ldots, u_p)$ and Q, T, S be defined as in (4).

(9) We prove: $d(u_1) \geqslant \frac{1}{2}(m+r)$, $d(u_p) \geqslant \frac{1}{2}(m+r)$. Suppose $d(u_1) < \frac{1}{2}(m+r)$. All neighbours of u_1 and u_p are contained in P, otherwise P would not have maximal length. As $d_1 \geqslant r+2$, we have $d(u_1) > r+1$ and therefore $|S| \geqslant d(u_1)-r > 1$. All vertices u_i, $i \in S$, have degree at most $d(u_1)$, otherwise $(u_i, u_{i-1}, \ldots, u_1, u_{i+1}, u_{i+2}, \ldots, u_p)$ would be a path of the same length as P and $d(u_i)+d(u_p) > d(u_1)+d(u_p)$, contradicting the maximality assumption on the endpoints of P. Let $j_0 := d(u_1)$, then there are $|S| \geqslant j_0 - r$ vertices of degree at most j_0. As we are in case (ii), $d_k > k+r$ holds for all $0 < k < \frac{1}{2}(m-r)$, which is equivalent to $d_{j-r} > j$ for all $r < j < \frac{1}{2}(m+r)$. Therefore $j_0 \geqslant \frac{1}{2}(m+r)$. By similar arguments $d(u_p) \geqslant \frac{1}{2}(m+r)$.

(10) From (9) it follows that

$$|S|+r+|T| \geqslant d(u_1)+d(u_p) \geqslant m+r.$$

Thus $|S|+|T| \geqslant m$, and from (4) (b) we have $|P| \geqslant m+1$. Therefore if $m=n$ we have $n=|P|>n$ which is a contradiction, and in this case we are done.

(11) Let $N := N(u_1) \cup N(u_p) \cup \{u_s, \ldots, u_{s+r}\} \cup \{u_1, u_p\}$. We prove: $|N| \geqslant m$. As $r \leqslant m-3$, $|\{u_s, \ldots, u_{s+r}\} \cap \{u_1, u_p\}| \leqslant 1$.

(a) Suppose $\max\{i \in S\} < \min\{j \in T'\}$, where $T' := T - \{s, \ldots, s+r\}$. This means that the index of a neighbour of u_1 which is not among $u_s, \ldots, u_{s+r}$ is less than or equal to the smallest of the indices of the neighbours of u_p not among $u_s, \ldots, u_{s+r}$. Thus $|(N(u_1) \cap N(u_p)) - \{u_s, \ldots, u_{s+r}\}| \leqslant 1$. Obviously

$$\begin{aligned} |N| \geqslant\ & |N(u_1)-\{u_s, \ldots, u_{s+r}\}|+|N(u_p)-\{u_s, \ldots, u_{s+r}\}| \\ & +|\{u_s, \ldots, u_{r+s}\}|+|\{u_1, u_p\}|-|(N(u_1) \cap N(u_p))-\{u_s, \ldots, u_{r+s}\}| \\ & -|\{u_s, \ldots, u_{r+s}\} \cap \{u_1, u_p\}| \\ \geqslant\ & |S|-1+|T'|+(r+1)+2-1-1 \\ \geqslant\ & |S|+|T| \geqslant m. \end{aligned}$$

(b) Suppose $\max\{i \in S\} \geqslant \min\{j \in T'\}$. Let

$$d := \min\{(i+1) - j : i \in S,\ j \in T' \text{ such that } i \geqslant j\},$$

then we have $d > 0$. Now let $i_0 + 1 - j_0 = d$.

(b$_1$) $i_0 + 1 \leqslant s$. By definition $j_0 < s$ and no vertex of the path P between u_{j_0} and u_{i_0+1} is linked to u_1 or u_p by an edge. Thus

$$[u_1, u_{i_0+1}, u_{i_0+2}, \ldots, u_p, u_{j_0}, u_{j_0-1}, \ldots, u_1]$$

is a cycle containing the path Q, all vertices u_i, $i \in S$, with the possible exception of $i = i_0$, and all vertices u_j, $j \in T'$. It also contains u_1 and u_p. Thus the length of this cycle is at least:

$$(r+2) + |S| - 1 + |T'| \geqslant |S| + |T| \geqslant m$$

which is impossible by assumption.

(b$_2$) $r + s \leqslant j_0$. Define the same cycle as in (b$_1$) and by the same arguments we obtain a contradiction.

(b$_3$) $j_0 < s$, $i_0 > r + s$. Define

$$j_1 := \min\{j \in T'\} \leqslant j_0, \qquad i_1 := \max\{i + 1 : i \in S\} \geqslant r + s + 1.$$

The conditions of case (b$_3$) imply the following:

$$u_1 \neq u_s,\ u_p \neq u_{s+r},$$

none of the vertices u_i, $j_1 < i \leqslant s$, can be linked to u_1 by an edge, none of the vertices u_i, $i_1 < i \leqslant p$, is a neighbour of u_1, thus

$$N(u_1) \subset \{u_2, \ldots, u_{j_1}\} \cup \{u_{s+1}, \ldots, u_{i_1}\},$$

none of the vertices u_i, $1 \leqslant i < j_1$, is a neighbour of u_p, none of the vertices u_i, $s + r < i < i_1$, is a neighbour of u_p, thus

$$N(u_p) \subset \{u_{j_1}, \ldots, u_{s+r}\} \cup \{u_{i_1}, \ldots, u_{p-1}\}.$$

Furthermore

$$|N(u_1) - \{u_s, \ldots, u_{s+r}\}| = |S|,$$

$$|N(u_p) - \{u_s, \ldots, u_{s+r}\}| = |T'|.$$

The only vertices which might be neighbours of both u_1 and u_p are u_{j_1}, u_{i_1} and $u_{s+1}, \ldots, u_{s+r}$. This implies

$$|(N(u_1) \cap N(u_p)) - \{u_s, \ldots, u_{s+r}\}| \leqslant 2.$$

Therefore

$$\begin{aligned} |N| &\geqslant |N(u_1) - \{u_s, \ldots, u_{s+r}\}| + |N(u_p) - \{u_s, \ldots, u_{s+r}\}| + (r+1) + 2 - 2 - 1 \\ &\geqslant |S| + |T'| + r \\ &\geqslant |S| + |T| \geqslant m. \end{aligned}$$

These are all the cases which can occur, therefore $|N| \geq m$ is proved.

(12) Among all pairs of paths satisfying Corollary 7 with respect to P and Q choose a pair P', P'' such that the cycle $K = [u_1, P', u_p, \bar{P}'', u_1]$ contains as many vertices of P as possible.

(13) To show that K has length $\geq m$, we will prove: K contains all vertices of N. Suppose there is a vertex of N which is not contained in K. Trivially the vertex is either in $N(u_1) - \{u_s, \ldots, u_{s+r}\}$ or in $N(u_p) - \{u_s, \ldots, u_{s+r}\}$. Without loss of generality we assume that the vertex $u_k \in N(u_1) - \{u_s, \ldots, u_{s+r}\}$ is not contained in K. Let

$$i_0 = \max\{i \mid u_i \in N \cap K, i < k\}, \qquad j_0 = \min\{i \mid u_i \in N \cap K, i > k\}.$$

(a) Suppose u_{i_0}, $u_{j_0} \in P'$, then

$$P_1' = (u_1, P', u_{i_0}, P, u_{j_0}, P', u_p),$$
$$P_1'' = P'',$$

is a pair of paths satisfying Corollary 7, and $K_1 = [u_1, P_1', u_p, \bar{P}_1'', u_1]$ contains more vertices of P then K does. Contradiction! If u_{i_0}, $u_{j_0} \in P''$ the contradiction follows similarly.

(b) Suppose $u_{i_0} \in P'$, $u_{j_0} \in P''$. Let

$$P_1' = (u_1, P, u_{i_0}, P', u_p),$$
$$P_1'' = (u_1, u_k, P, u_{j_0}, P'', u_p).$$

If $i_0 \leq s$, then Q is contained in (u_{i_0}, P', u_p), otherwise Q is contained in (u_1, P, u_{i_0}). Therefore P_1' and P_1'' satisfy the conditions of Corollary 7, and K_1 contains more vertices of P then K does. Contradiction!

(c) Suppose $u_{i_0} \in P''$, $u_{j_0} \in P'$.

(c_1) $i_0 \leq s$: this implies $j_0 \leq s$.

Take

$$P_1' = (u_1, u_k, P, u_{j_0}, P', u_p)$$
$$P_1'' = (u_1, P, u_{i_0}, P'', u_p).$$

(c_2) $i_0 \geq r + s$:

Let

$$P_1' = (u_1, P, u_{i_0}, P'', u_p),$$
$$P_1'' = (u_1, u_k, P, u_{j_0}, P', u_p).$$

These pairs of paths satisfy Corollary 7. The contradiction follows as above.

Thus in Case (ii) we have constructed a cycle K of length $\geq m$ containing the path Q, which contradicts the assumption that G does not contain such a cycle, and we are done. □

Theorem 8 has some immediate Corollaries and also includes some of the classical theorems on graphs containing cycles of a certain minimum length.

Corollary 9. *Let $d_1, \ldots, d_n$ be the degree sequence of a graph $G = (V, E)$. Let $n \geq 3$, $q \geq 2$ and let the following condition be satisfied:*

$$d_k \leq k \leq q - 1 \Longrightarrow d_{n-k} \geq n - k. \tag{3}$$

Furthermore, let G be 2-connected if $q - 1 < n - d_{n-1} - 1$ holds and $d_k > k$ holds for all $1 \leq k \leq q - 1$. Then G contains a cycle of length at least $\min\{n, 2q\}$.

Proof. Take $r = 0$ in Theorem 8. □

One of the well-known theorems implied by Theorem 8 is the following due to Pósa [7], which generalizes results of Dirac [5].

Corollary 10 (Pósa [7]). *Let $d_1, \ldots, d_n$ be the degree sequence of a 2-connected graph G. Let $q \geq 2$, $n \geq 2q$. If*

$$d_k > k \quad \text{for all } k = 1, \ldots, q - 1, \tag{4}$$

then G contains a cycle of length at least $2q$.

Proof. Immediate from Corollary 9. □

For bipartite graphs a simple trick yields:

Corollary 11. *Let $G = (V, W, E)$ be a bipartite graph with degree sequences $d(v_1) \leq \cdots \leq d(v_n)$ and $d(w_1) \leq \cdots \leq d(w_m)$, $n \leq m$. If*

$$d(w_k) \leq k \leq n - 1 \Longrightarrow d(v_{n-k}) \geq m - k + 1, \tag{5}$$

then G contains a cycle of length $2n$.

Proof. Construct $G^* = (V \cup W, E^*)$ by adding all edges to E which have both endpoints in V. Clearly G^* contains a cycle of length $2n$ if and only if G does. If G satisfies (5) then G^* satisfies (3). As (5) implies that $d(w_1) \geq 2$ and V defines a clique in G^*, G^* is 2-connected. □

Standard theorems giving sufficient conditions for a graph to be hamiltonian can also be derived from Theorem 8.

Corollary 12 (Berge, [1, p. 204]). *Let $G = (V, E)$ be a graph with degree sequence $d_1, \ldots, d_n$. Let r be an integer, $0 \leq r \leq n - 3$. If for every k with $r < k < \frac{1}{2}(n + r)$ the following condition holds:*

$$d_{k-r} \leq k \Longrightarrow d_{n-k} \geq n - k + r, \tag{6}$$

then for each subset Q of edges, $|Q| = r$, that forms a path there is a hamiltonian cycle in G that contains Q.

Proof. Clearly (6) is equivalent to (2) if $m = n$. We have to prove that (6) implies $(r+2)$-connectedness.

If there is a k with $r < k < \frac{1}{2}(n+r)$ such that $d_{k-r} \leq k$, then by the arguments of the proof of Theorem 8, Section (1) (a) $(r+2)$-connectedness is assured.

If $d_{k-r} > k$ for all $r < k < \frac{1}{2}(n+r)$, we have $d_q \geq q + r$, where $q := \left\lceil \frac{n-r}{2} \right\rceil$. Furthermore $2q \geq n - r$ and $q \leq n - r - 1$ (as $r \leq n-3$), thus $q + r \leq d_q \leq d_{n-r-1}$. This implies

$$q = 2q - q \geq n - (r+q) > n - (q+r) - 1 \geq n - d_{n-r-1} - 1.$$

Thus condition (1) of Proposition 1 is satisfied and G is $(r+2)$-connected. □

Actually Berge proved a stronger theorem saying that Q only has to be a set of edges of cardinality r such that the connected components of Q are paths.

Corollary 13 (Chvátal [4]). *If the degree sequence $d_1, \ldots, d_n$ of a graph G, $n \geq 3$, satisfies*

$$d_k \leq k < \tfrac{1}{2} n \implies d_{n-k} \geq n - k, \tag{7}$$

then G contains a hamiltonian cycle.

Proof. Take $r = 0$ in Corollary 12. □

Furthermore, Chvátal showed that this theorem is best possible in the sense that if there is a degree sequence of a graph not satisfying (7) then there exists a non-hamiltonian graph having a degree sequence which majorizes the given one. This proves that Theorem 8 is also best possible in this special case. Moreover Chvátal (see [4]) showed that most of the classical results on hamiltonian graphs are contained in his theorem, and therefore are also implied by Theorem 8.

A trivial consequence of Corollary 13 which however is not too "workable" is

Corollary 14. *Let G' be an induced subgraph of a graph G having $m \leq n$ vertices. If the degree sequence $d'_1, \ldots, d'_m$ of G' satisfies (7) then G contains a cycle of length m.* □

4. Some examples

(a) We first show that the number m implied by Theorem 8 giving the minimum length of a cycle containing a given path cannot be increased, i.e. we give an example of a graph G with a path Q of length r such that the longest cycle containing Q has length m.

Consider a graph with two disjoint vertex sets A and B. A is a clique of q

vertices, and B consists of p isolated vertices. Each vertex of A is linked to each vertex of B by an edge. Suppose that $1 < q - r$ and $p \geqslant q - r + 1$. The degree sequence of G is

$$\underbrace{q, q, \ldots, q,}_{p \text{ times}} \underbrace{n-1, \ldots, n-1}_{q \text{ times}}.$$

Hence we have

$$d_i > i + r \quad \text{for } i < q - r,$$

$$d_{q-r} = (q-r) + r = q,$$

$$d_{n-(q-r)-r} = d_{n-q} = q < q + 1 = (2q - r) + 1 - (q - r) \leqslant n - (q - r).$$

By Theorem 8 for each path Q of length r there is a cycle of length $2q - r$ containing Q.

If we choose a path Q of length r such that all vertices of Q are contained in A it is obvious that no longer cycle containing Q exists.

(b) We give an example showing that the assumption of $(r+2)$-connectedness in Theorem 8 under the specified conditions is necessary.

Consider the graph G consisting of three vertex sets A, B, C. A and B have k vertices and are complete, C has $r+1$ vertices and is complete. Each vertex of C is joined to each vertex of $A \cup B$ by an edge. Hence G is $(r+1)$-connected but not $(r+2)$-connected. Take a path Q of length r in C. Clearly the maximal length of a cycle containing Q is $k + r + 1$. The degree sequence of this graph is

$$\underbrace{k+r, \ldots, k+r,}_{2k \text{ times}} \underbrace{n-1, \ldots, n-1}_{r+1 \text{ times}}.$$

We have $d_i > i + r$ for $0 < i \leqslant k - 1$, therefore Theorem 8 would imply the existence of a cycle of length at least $2k + r$ containing Q.

(c) We give an example showing that Corollary 14 is not stronger than Corollary 9.

Consider a graph consisting of two disjoint cliques A, B, each having m vertices. Link A and B by two disjoint edges. Obviously this graph is hamiltonian. The degree sequence is

$$\underbrace{m-1, \ldots\ldots, m-1,}_{2m-4 \text{ times}} m, m, m, m.$$

Corollary 9 implies that there exists a cycle of length $\geqslant 2m - 2$, but Corollary 14 does not imply a cycle of length $\geqslant 2m - 2$.

(c_1) Delete 2 vertices of A, both must necessarily be distinct from the two vertices linking A to B. The degree sequence is

$$\underbrace{m-3,\ldots,m-3}_{m-4 \text{ times}}, m-2, m-2, \underbrace{m-1,\ldots,m-1}_{m-4 \text{ times}}, m, m$$

which does not satisfy (7).

(c_2) Delete one vertex of A and one of B, again both must be distinct from the vertices linking A to B. The degree sequence is

$$\underbrace{m-2,\ldots\ldots,m-2}_{2\,(m-3) \text{ times}}, m-1, m-1, m-1, m-1$$

which also does not satisfy (7).

It is clear that Corollary 9 does not imply Corollary 14.

(d) Bondy proved (see [3]) the following

Theorem (Bondy). *Let G be a 2-connected graph with degree sequence $d_1,\ldots,d_n$. If*

$$d_j \leq j,\ d_k \leq k\ (j \neq k) \Longrightarrow d_j + d_k \geq c, \tag{8}$$

then G has a cycle of length at least min (c, n). □

Chvátal showed that in the case $c = n$ his theorem (Corollary 13) implies Bondy's theorem, thus in the hamiltonian case Corollary 9 is stronger than the theorem of Bondy. In general this is obviously not true, nor is the converse as the following example shows: The graph has three vertex sets A, B, C. $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$, $|C| = m$. The edges are the following: $\{a_1, b_1\}$, $\{a_1, b_2\}$, $\{a_2, b_1\}$, $\{a_2, b_3\}$, $\{a_3, b_2\}$, $\{a_3, b_3\}$, $\{a_3, b_4\}$, and all edges having both endpoints in $B \cup C$. The degree sequence is

$$2, 2, 3, \underbrace{n-4,\ldots,n-4}_{m \text{ times}}, n-3, n-2, n-2, n-2.$$

$d_2 \leq 2$ and $d_3 \leq 3$. By Pósa's theorem there is a cycle of length ≥ 4, by Bondy's theorem there exists a cycle of length ≥ 5. As $d_{n-2} \geq n-2$ and $d_{n-3} \geq n-3$ and $d_i > i$, $4 \leq i < \frac{1}{2}n$, G is hamiltonian by Corollary 9.

(e) In [8] Woodall stated the following (to my knowledge unsettled)

Conjecture. *Let $d_1,\ldots,d_n$ be the degree sequence of a 2-connected graph G, $m \leq n-3$, and let the following condition be satisfied:*

$$\begin{cases} d_{k+m} > k & \text{for } 1 \leq k < \frac{1}{2}(n-m-1), \\ d_{k+m+1} > k & \text{if } k = \frac{1}{2}(n-m-1). \end{cases} \tag{9}$$

Then G contains a cycle of length at least $n - m$. □

Obviously Corollary 9 does not imply Woodall's Conjecture, but surprisingly nor does the Conjecture imply Corollary 9, although in most cases Woodall's Conjecture — if true — would be "better" than Corollary 9.

We give an example: Let n and m be both odd (or even), $j = \frac{1}{2}(n - m - 2)$ and $j^2 \geq \frac{1}{2}(n + m)$ (which is a solvable condition).

Consider the following graph consisting of three vertex sets A, B, $\{v\}$. B has $j + 1$ elements and is complete, v is linked to all elements of B by an edge. A consists of $j + m$ isolated vertices, each element of A is linked to exactly j vertices of B such that each element of B is linked to at least $m + 1$ vertices of A. This is possible as $(j + m)j = jm + j^2 \geq jm + \frac{1}{2}(n + m) = jm + j + m + 1 = (m + 1)(j + 1)$. The degree sequence of this graph is

$$\underbrace{j, \ldots\ldots, j}_{j+m \text{ times}}, j + 1, \underbrace{m_1, \ldots\ldots, m_{j+1}}_{j+1 \text{ times}}$$

where $m_i \geq n - j$ for $i = 1, \ldots, j + 1$. We have

$$d_{k+m} > k \quad \text{for } 1 \leq k \leq j - 1,$$
$$d_{j+m} = j \quad \text{and } j < \tfrac{1}{2}(n - m - 1)$$

Thus Woodall's Conjecture does not imply a cycle of length $\geq n - m$. On the other hand

$$d_k > k \quad \text{for } 1 \leq k \leq j - 1,$$

$$d_j = j \quad \text{and } d_{n-j} = m_1 \geq n - j.$$

Hence by Corollary 9 there exists a cycle of length $\geq 2(j + 1) = n - m$.

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