The Monotone 2-Matching Polytope on a Complete Graph

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Abstract: *Based on J. Edmonds's [1] complete linear characterization of the convex hull of all incidence vectors of 2-matchings in a graph we show which of the given inequalities are essential i.e. define facets of the polyhedron under consideration in the case of a complete graph.*

1. Introduction and Notation

One of the main streamlines of current research in integer programming deals with the problem of finding inequalities in order to state integer programs as linear programs i.e. to replace the discreteness conditions by linear inequalities (see [1], [2], [3], [4], [5], [6], [7]). The results of these efforts have been extremely helpful in order to devise good (i.e. polynomial) algorithms for some combinatorial optimization problems and to prove their correctness, furthermore, they gave rise to a number of improved cutting-plane and branch-and-bound algorithms for hard integer programming problems.
In this note we consider the 2-matching problem the associated polyhedron of which was completely characterized by J. Edmonds [1]. Assuming that the underlying graph is complete we show which of the inequalities found are facets of this polyhedron.

A graph $G = [V,E]$ consists of a finite, nonempty set of nodes V and a finite set of edges E. Each edge $e \in E$ is a two-element subset of V. Throughout this paper n is the number of nodes $(n = |V|)$ and m is the number of edges $(m = |E|)$. A graph $G = [V,E]$ is called complete if E is the set of all two-element subsets of V. The complete graph on n nodes is denoted by $K_n = [V,E]$.

A subset M of edges in a graph $G = [V,E]$ is called a 2-matching if each node $v \in V$ is contained in at most two edges of M.
If $v_1, v_2, \ldots, v_s$ are nodes such that $\{v_i, v_{i+1}\} \in E$, $i=1,\ldots,s-1$, then we call the set of edges $\{\{v_i, v_{i+1}\} | i=1,\ldots,s-1\}$ a chain and denote it by $[v_1, v_2, \ldots, v_s]$. If additionally $\{v_1, v_s\} \in E$ then $[v_1, v_2, \ldots, v_s] \cup \{\{v_1, v_s\}\}$ is called a cycle and is denoted by $\langle v_1, v_2, \ldots, v_s \rangle$. The number of edges of a chain (cycle) is called the length of the chain (cycle). A cycle is called

<u>hamiltonian</u> if it has length n. <u>$\omega(W)$</u>$:=\{e\varepsilon E \mid |e \cap W| = 1\}$, we write <u>$\omega(v)$</u> instead of $\omega(\{v\})$; <u>$E(W)$</u>$:=\{e\varepsilon E \mid e \subset W\}$; <u>$N(v)$</u>$:=\{w\varepsilon V \mid \{v,w\} \varepsilon E\}$.

If $G = [V,E]$ is a graph we associate to each edge $e \varepsilon E$ a component x_e of a vector $x \varepsilon \mathbb{R}^m$; for $F \subset E$ the vector $\underline{x}^F \varepsilon \mathbb{R}$ with

$$x_e^F = \begin{cases} 1 & \text{if } e\varepsilon F \\ 0 & \text{if } e \notin F \end{cases}$$

is called the <u>incidence vector</u> of F.

An inequality $ax \le a_o$ is called <u>valid</u> with respect to a polytope $P \subset \mathbb{R}^m$ if $P \subset \{x\varepsilon\mathbb{R}^m \mid ax \le a_o\}$. A subset $F \subset P$ is called a <u>face</u> of the polytope P if there exist valid inequalities $a^i x \le a_o^i$ $i=1,\ldots,k$ such that $F = P \cap \{x\varepsilon\mathbb{R}^m \mid a^i x = a_o^i \text{ for } i=1,\ldots,k\}$. A face $F \subset P$ is called a <u>facet</u> of P if dim F = dim P - 1, in other words F is a facet of P if F contains dim P affinely independent vectors.

For every polytope P there exists a system of inequalities $Ax \le b$ such that $P = \{x\varepsilon\mathbb{R}^m \mid Ax \le b\}$. If the polytope P is fully-dimensional, a minimal system of inequalities to describe P can be found as follows: Characterize all facets F of P and for each facet F find one valid inequality $a^F x \le a_o^F$ such that $F = P \cap \{x \mid a^F x = a_o^F\}$. The system of these inequalities satisfies $P = \{x\varepsilon\mathbb{R}^m \mid a^F x \le a_o^F \text{ for all facets } F\}$ and is non-redundant i.e. if we drop any of these inequalities then the resulting system of inequalities does not describe P. If P is fully-dimensional then for each facet F of P there exists a unique (up to a constant factor) valid inequality $ax \le a_o$ such that $F = P \cap \{x \mid ax=a_o\}$. Using standard terminology we call this inequality a facet.

2. *The monotone 2-matching polytope*

The 2-matching problem can be stated as follows:

> *Given a graph $G = [V,E]$ and weights $c_e \varepsilon \mathbb{R}$ for all $e \varepsilon E$.*
> *Find a 2-matching M in G such that $\sum_{e\varepsilon M} c_e$ is maximal.*

Via incidence vectors of 2-matchings a 0-1-polytope can be associated to this problem:

$$\underline{\tilde{Q}_{2M}(G)} := \text{conv}\{x^M \in \mathbb{R}^m \mid x^M \text{ is the incidence vector of a 2-matching } M \text{ in } G\}.$$

As every subset of a 2-matching is a 2-matching this polytope has the monotonicity property, that is if x is a vertex of $\tilde{Q}_{2M}(G)$ and y is a 0-1 vector such that $y \leq x$ then y is a vertex of $\tilde{Q}_{2M}(G)$. Every vertex of $\tilde{Q}_{2M}(G)$ is in one-to-one correspondence to a 2-matching in G and vice versa. Hence the 2-matching problem can be solved via the linear program

$$\max \; cx$$
$$x \in \tilde{Q}_{2M}(G).$$

The polytope $\tilde{Q}_{2M}(G)$ is called the monotone 2-matching polytope on G. If G is the complete graph K_n we denote the monotone 2-matching polytope by $\underline{\tilde{Q}^n_{2M}}$.

Although theoretical results imply that for every polytope there exists a system of inequalities which completely and non-redundantly describes this polytope , such a system is in general not known explicitly for polytopes (e.g. associated to combinatorial optimization problems) which are defined as the convex hull of a finite set of points. However for the monotone 2-matching polytope on a graph G a complete system of inequalities is available.

As all vertices of $\tilde{Q}_{2M}(G)$ are 0-1 vectors it is obvious that the inequalities

$$(1) \quad x_e \geq 0 \qquad \forall e \in E,$$

$$(2) \quad x_e \leq 1 \qquad \forall e \in E$$

are valid with respect to $\tilde{Q}_{2M}(G)$. Given a 2-matching M in G then by definition every node is contained in at most two edges of M, hence the inequalities

$$(3) \quad \sum_{e \in \omega(v)} x_e \leq 2 \qquad \forall v \in V$$

are valid with respect to $\tilde{Q}_{2M}(G)$. The polytope defined by the inequalities (1), (2), (3) contains $\tilde{Q}_{2M}(G)$ but in general it is not equal to $\tilde{Q}_{2M}(G)$. A new system of inequalities valid with respect to $\tilde{Q}_{2M}(G)$ was defined by J. Edmonds [1] .

Proposition 1

If $G = [V,E]$ *is a graph and* $W \subset V, Z \subset \omega(W)$, $|Z|$ *odd then*

$$(4) \quad \sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \leq |W| + \frac{|Z|-1}{2}$$

is a valid inequality with respect to $\tilde{Q}_{2M}(G)$. □

The inequalities (4) defined in Proposition 1 are called 2-matching constraints. A deep result of J. Edmonds is

Theorem 2 [1]

Given a graph $G = [V,E]$. *A complete linear characterization of the monotone 2-matching polytope* $\tilde{Q}_{2M}(G)$ *is given by the following inequalities*

$$(1) \quad x_e \geq 0 \qquad \forall e \in E$$

$$(2) \quad x_e \leq 1 \qquad \forall e \in E$$

$$(3) \quad \sum_{e \in \omega(v)} x_e \leq 2 \quad \forall v \in V$$

$$(4) \quad \sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \leq |W| + \frac{|Z|-1}{2} \quad \forall \; W \subset V, \; Z \subset \omega(W), \; |Z| \text{ odd.}$$ □

Which of these inequalities are facets of $\tilde{Q}_{2M}(G)$ and which are inessential will be shown in the following for the case of a complete graph K_n.

3. *Trivial facets and redundant 2-matching inequalities*

In order to decide whether a valid inequality is a facet or not it is necessary to know the dimension of the polytope under consideration. The empty set and all sets containing only one edge are 2-matchings hence

$\tilde{Q}_{2M}(G)$ contains the null-vector and all unit-vectors, i.e. $\tilde{Q}_{2M}(G)$ contains $m+1$ affinely independent vectors. This implies

Proposition 3

$$\dim \ \tilde{Q}_{2M}(G) = |E| = m \qquad \square$$

It is easy to see which of the inequalities (1), (2), (3) are facets of $\tilde{Q}_{2M}(G)$.

Proposition 4

Let $G = [V,E]$ be a graph, then the inequalities

$$(1) \quad x_e \geq 0 \qquad \forall e \in E$$

$$(2) \quad x_e \leq 1 \qquad \forall e \in E$$

are facets of $\tilde{Q}_{2M}(G)$.

Proof

Let $e \in E$.

a) The nullvector and the incidence vectors of the 2-matchings $\{f\}$ for all $f \in E-\{e\}$ satisfy $x_e \geq 0$ with equality. Hence $\tilde{Q}_{2M}(G) \cap \{x|x_e = 0\}$ contains m affinely independent vectors, which proves that $x_e \geq 0$ is a facet of $\tilde{Q}_{2M}(G)$.

b) The incidence vectors of $\{e\}$ and $\{e,f\}$ for all $f \in E-\{e\}$ are 2-matchings, they satisfy $x_e \leq 1$ with equality, and they are affinely independent. $\square$

Proposition 5

Let $G = [V,E]$ be a graph and $v \in V$.

$$(3) \quad \sum_{e \in \omega(v)} x_e \leq 2$$

is a facet of $\tilde{Q}_{2M}(G)$ if and only if $|\omega(v)| \geq 3$.

Proof

a) If $|\omega(v)| \leq 1$ then (3) is obviously not even a supporting hyperplane, if $\omega(v) = \{e,f\}$ then every 2-matching the incidence vector of which satisfies (3) with equality must contain e and f. Hence $\tilde{Q}_{2M}(G) \cap \{x \mid \sum_{e\varepsilon\omega(v)} x_e = 2\}$ is contained in the two hyperplanes $x_e = 1$, $x_f = 1$ therefore it cannot be a facet.

b) If $|\omega(v)| \geq 3$ take three edges $e_1, e_2, e_3 \in \omega(v)$. The incidence vectors of the following m 2-matchings satisfy inequality (3) with equality and are linearly (hence affinely) independent:
$\{e_1,e_2\}$, $\{e_2,e_3\}$, $\{e_1,e_3\}$; $\{e_1,e\}$ for all $e\varepsilon\omega(v) - \{e_1,e_2,e_3\}$; and $\{e_1,e_2,f\}$ for all $f\varepsilon E\text{-}\omega(v)$, thus (3) is a facet of $\tilde{Q}_{2M}(G)$. □

It may occur that for two different nodes $v,w\varepsilon V$ the inequalities (3) define the same facet of $\tilde{Q}_{2M}(G)$ namely if v is the only neighbour of w and vice versa. As $(|\omega(v)|=)|\omega(w)| \geq 3$ has to be satisfied this only happens if we allow parallel edges in a graph.

If P is a fully-dimensional polytope and $ax \leq a_o$ is valid with respect to P and if there exist other valid inequalities $bx \leq b_o$ and $cx \leq c_o$ such that either $a \leq b$, $a_o \geq b_o$ (in this case we call $ax \leq a_o$ <u>dominated</u>) or $a \leq b + c$, $a_o \geq b_o + c_o$ then obviously $ax \leq a_o$ is redundant and is not a facet of P. We are going to check which of the 2-matching constraints are redundant with respect to $\tilde{Q}_{2M}(G)$.

The inequalities (2) and (3) (in some cases) are formally contained in the 2-matching constraints. In order to have a precise terminology we exclude them from the 2-matching constraints.

Lemma 6

A 2-matching constraint is redundant with respect to $\tilde{Q}_{2M}(G)$ if $|W| \leq 2$.

Proof

a) Let $|W| = 1$. If $|Z| = 1$ then the 2-matching inequality is an inequality of type (2). If $|Z| \geq 3$ it is dominated by (or equal to) an inequality of type (3).

b) Let $|W|=\{v,w\}$. Let $|Z|=1$, e.g. $Z=\{\{u,v\}\}$. If $\{v,w\}\in E$ then the 2-matching constraint is $x_{vw}+x_{uv}\le 2$, hence it is redundant as $x_{uv}\le 1$ and $x_{vw}\le 1$ are facets by Proposition 4.
If $\{v,w\}\notin E$ the assertion is obvious.

If $|Z|=3$ then there is a vertex in W which is contained in at least two of the edges in Z. Let $Z=\{f,g,h\}$ and $v\in f$, $v\in g$ then

$$\sum_{e\in E(W)} x_e + \sum_{e\in Z} x_e \le \sum_{e\in\omega(v)} x_e + x_h \le 2+1 = |W| + \frac{|Z|-1}{2} = 3.$$

This implies the redundancy of the 2-matching constraint.
If $|Z|\ge 5$ then we have

$$\sum_{e\in E(W)} x_e + \sum_{e\in Z} x_e \le \sum_{e\in\omega(v)} x_e + \sum_{e\in\omega(w)} x_e \le 2+2 \le |W| + \frac{|Z|-1}{2},$$

hence the 2-matching constraint is redundant. ▯

<u>Lemma 7</u>

A 2-matching constraint is redundant with respect to $\tilde{Q}_{2M}(G)$ if $|W|=3$ and $|Z|=1$.

<u>Proof</u>

Let $W=\{u,v,w\}$, $Z=\{\{u,z\}\}$. If $\{v,w\}\in E$ then

$$\sum_{e\in E(W)} x_e + x_{uz} \le \sum_{e\in\omega(u)} x_e + x_{vw} \le 2+1 = |W| + \frac{|Z|-1}{2}.$$

If $\{v,w\}\notin E$ the redundancy of the 2-matching constraint is obvious. ▯

<u>Lemma 8</u>

A 2-matching constraint is redundant with respect to $\tilde{Q}_{2M}(G)$ if there are two edges $e,f\in Z$ such that $e\cap f\neq\emptyset$.

<u>Proof</u>

Let $Z=\{e_1,e_2,\dots,e_k\}$, and $e_{k-1}\cap e_k\neq\phi$. Let $v\in e_{k-1}\cap e_k$, and without loss of generality $v\notin e_i$, $i=1,\dots,k-2$.

a) $v \in W$. Defining

$W' := W - \{v\}$, $Z' := Z - \{e_{k-1}, e_k\}$ we obtain

$$\sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \le \Big(\sum_{e \in E(W')} x_e + \sum_{e \in Z'} x_e \Big) + \sum_{e \in \omega(v)} x_e$$
$$\le |W| + \frac{|Z'| - 1}{2} + 2$$
$$= |W| + \frac{|Z| - 1}{2} ,$$

hence the 2-matching constraint is redundant.

b) $v \notin W$. We define $W' := W \cup \{v\}$, $Z' := Z - \{e_{k-1}, e_k\}$. This gives

$$\sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \le \sum_{e \in E(W')} x_e + \sum_{e \in Z'} x_e$$
$$\le |W'| + \frac{|Z'| - 1}{2}$$
$$= |W| + \frac{|Z| - 1}{2}$$

therefore the 2-matching constraint under consideration is dominated. □

4. *The non-trivial facets of* $\tilde{Q}^n_{2M}$

We now restrict ourselves to the monotone 2-matching polytope $\tilde{Q}^n_{2M}$ on the complete graph $K_n = [V,E]$. In order to find all those 2-matching constraints that define facets of $\tilde{Q}^n_{2M}$ we know from section 3 that we need to consider only those 2-matching constraints $\sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \le |W| + \frac{|Z| - 1}{2}$ such that $e \cap f = \emptyset \ \forall e,f \in Z$ and $|Z| \ge 3$, or $|Z| = 1$ and $|W| \ge 4$.
It turns out that in the case of the complete graph this is exactly the class of facet-defining 2-matching constraints.

<u>Theorem 9</u>

Let $n \ge 4$, $W \subset V$, $Z \subset \omega(W)$, $|Z|$ *odd.*
The 2-matching constraint $\sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \le |W| + \frac{|Z| - 1}{2}$
defines a facet of $\tilde{Q}^n_{2M}$ *if and only if one of the following conditions holds:*

(i) $|Z| \geq 3$ *and* $e \cap f = \emptyset$ $\forall e,f \in Z$;

(ii) $|Z| = 1$ *and* $|W| \geq 4$.

Proof

"only if": Follows immediately from Lemmas 6, 7 and 8.

"if":

1) Let $ax \leq a_o$ be a 2-matching constraint satisfying (i) or (ii), and let $H_a := \tilde{Q}^n_{2M} \cap \{x \in \mathbb{R}^m \,|ax = a_o\}$ be the face of $\tilde{Q}^n_{2M}$ defined by $ax \leq a_o$.

We show that any hyperplane $bx = b_o$ for which $H_a \subset \{x \in \mathbb{R}^m \,|bx = b_o\}$ holds, necessarily satisfies $b = \alpha a$. This implies $b_o = \alpha a_o$, and hence that the affine space spanned by H_a is $\{x \in \mathbb{R}^m \,|ax = a_o\}$. Therefore $\dim H_a = m - 1 = \dim \tilde{Q}^n_{2M} - 1$ which proves that $ax \leq a_o$ is a facet of $\tilde{Q}^n_{2M}$.

Without loss of generality let $W = \{1,2,\ldots,s\}$,
$Z = \{\{1,s+1\}, \{2,s+2\},\ldots, \{k, s+k\}\}$, where $k \geq 1$ is odd.

Let $bx = b_o$ be a hyperplane such that $H_a \subset \{x \in \mathbb{R}^m \,|bx = b_o\} =: H_b$, and let

$$(1) \quad \alpha := b_{1,s+1} \quad .$$

2) Suppose $ax := \sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \leq |W| + \frac{|Z|-1}{2} =: a_o$ satisfies condition (i).

Let M be the 2-matching consisting of the chain $[k, k+1,\ldots,s,1]$, the edges $\{i,i+1\}$ for $1 \leq i \leq k-2$, i odd, and the edges $Z - \{\{1,s+1\}\}$; let $M' := (M - \{\{1,2\}\}) \cup \{\{1,s+1\}\}$. The 2-matchings M and M' contain $s + \frac{k-1}{2}$ edges of $E(W) \cup Z$, hence their incidence vectors x^M, $x^{M'}$ satisfy $ax \leq a_o$ with equality. From $x^M, x^{M'} \in H_a \subset H_b$ follows

$$0 = b_o - b_o = bx^M - bx^{M'} = b_{12} - b_{1,s+1} = b_{12} - \alpha \text{ , hence } b_{12} = \alpha \,.$$

This construction can be carried out in a similar way replacing the node 2 by any of the nodes $3,\ldots,k$; we obtain analogously

$$(2) \quad b_{1i} = \alpha \qquad \forall i \in \{2,\ldots,k\}.$$

Knowing b_{1i} we conclude (using the same construction the other way around) that

$$(3) \quad b_{i,s+i} = \alpha \qquad \forall\, i \varepsilon \{1,\dots,k\} \text{ holds.}$$

This information and 2-matchings defined as above lead to

$$(4) \quad b_{ij} = \alpha \qquad 1 \leq i < j \leq k.$$

If $s>k$ we define a 2-matching M" as follows:
$M'' := (M' - \{\{1,s\}\}) \cup \{\{1,2\}\}$, obviously $x^{M''} \varepsilon H_a$ hence

$$0 = bx^{M'} - bx^{M''} = b_{1s} - b_{12} = b_{1s} - \alpha.$$

We repeat this construction in a similar way firstly for all $j\varepsilon\{k+1,\dots,s\}$ and secondly for all $i\varepsilon\{2,\dots,k\}$ and for all $j\varepsilon\{k+1,\dots,s\}$ and obtain

$$(5) \quad b_{ij} = \alpha \qquad \forall\, i\varepsilon \{1,\dots,k\},\ \forall\, j\varepsilon\{k+1,\dots,s\}.$$

If $s>k+1$ we define $M''' := (M' - \{\{s-1,s\}\}) \cup \{\{2,s-1\}\}$, again we have $x^{M'''} \varepsilon H_a \subset H_b$, and therefore

$$0 = bx^{M'} - bx^{M'''} = b_{s-1,s} - b_{2,s-1} = b_{s-1,s} - \alpha .$$

This construction yields in the same manner

$$(6) \quad b_{ij} = \alpha \qquad k+1 \leq i < j \leq s .$$

Summarizing the partial results (1) - (6), we conclude

$$(7) \quad b_e = \alpha\, a_e \qquad \forall\, e \,\varepsilon\, E(W) \cup Z.$$

If $e \notin E(W) \cup Z$ it is a trivial exercise to find a 2-matching N, such that $N' := N \cup \{e\}$ is a 2-matching and $x^N \varepsilon H_a$. $e \notin E(W) \cup Z$ implies $x^{N'} \varepsilon H_a$ and thus $0 = bx^{N'} - bx^N = b_e$. Therefore

$$(8)\quad b_e = 0 = \alpha\, a_e \qquad \forall\, e \notin E(W) \cup Z.$$

Hence we have shown $b = \alpha\, a$ which proves the case (i).

3) Suppose condition (ii) is satisfied: $k := |Z| = 1$, and $s := |W| \geq 4$. Every hamiltonian cycle $M := \langle 1, i_2, \ldots, i_s \rangle$ in the induced subgraph $[W, E(W)]$ is a 2-matching and satisfies $x^M \in H_a$. The same is true for every chain $N := [1, s+1, i_2, \ldots, i_s]$. Hence

$$0 = b_o - b_o = bx^M - bx^N = b_{1,i_s} - b_{1,s+1} = b_{1,i_s} - \alpha .$$

This yields

$$(9)\quad b_{1j} = \alpha \qquad \forall\, j \in \{2, \ldots, s\}.$$

Let $M_o = \langle i_2, i_3, \ldots, i_s \rangle$ be a hamiltonian cycle in $[W-\{1\}, E(W-\{1\})]$ (which does exist because of $|W| \geq 4$). $M' := M_o \cup \{\{1, s+1\}\}$ is a 2-matching containing $|W|$ edges, therefore $x^{M'} \in H_a$. Likewise $x^{M''} \in H_a$ for $M'' := (M' - \{\{i_2, i_3\}\}) \cup \{\{1, i_2\}\}$. Thus we have

$$0 = bx^{M'} - bx^{M''} = b_{i_2,i_3} - b_{1,i_2} = b_{i_2,i_3} - \alpha,$$

and similarly

$$(10)\quad b_{ij} = \alpha \qquad 1 \leq i < j \leq k.$$

(9) and (10) yield

$$(11)\quad b_e = \alpha = \alpha\, a_e \qquad \forall\, e \in E(W) \cup Z.$$

As in the case 2) it is trivial to show that

$$b_e = 0 = \alpha\, a_e \qquad \forall\, e \notin E(W) \cup Z.$$

This completes the proof of (ii) and therefore the whole proof. □

Edmonds's complete characterization of the monotone 2-matching polytope (see Theorem 2), Proposition 4 and 5, and Theorem 9 yield

Theorem 10

Let $K_n = [V,E]$ *be the complete graph on* n *nodes,* $n \geq 4$*, and let* $\tilde{Q}^n_{2M}$ *be the convex hull of all incidence vectors of 2-matchings in* K_n*. Then the following system of inequalities is a complete and non-redundant linear characterization of* $\tilde{Q}^n_{2M}$:

$$(1) \quad x_e \geq 0 \qquad \forall\, e \in E,$$

$$(2) \quad x_e \leq 1 \qquad \forall\, e \in E,$$

$$(3) \quad \sum_{e \in \omega(v)} x_e \leq 2 \qquad \forall\, v \in V,$$

$$(4) \quad \sum_{e \in E(W)} x_e + \sum_{e \in Z} x_e \leq |W| + \frac{|Z|-1}{2} \quad \forall\, W \subset V,\ Z \subset \omega(W),\ |Z| \text{ odd}$$

such that (i) $|Z| \geq 3$ *and* $e \cap f = \emptyset\ \forall\, e,f \in Z$
or (ii) $|Z| = 1$ *and* $|W| \geq 4$ *holds.* □

Theorem 10 gives a complete characterization of the monotone 2-matching polytope on an arbitrary graph, however in general it does not provide a non-redundant characterization (although it is less redundant than the characterization given in Theorem 2). The question which of the 2-matching constraints are redundant with respect to the polytope $\tilde{Q}_{2M}(G)$ is still open.

W. Pulleyblank [7] considered the polytope

$$Q'_{2M}(G) := \text{conv}\{x \in \mathbb{R}^m \mid x_e \in \{0,1,2\},\ \sum_{e \in \omega(v)} x_e \leq 2 \quad \forall v \in V\}$$

instead of $\tilde{Q}_{2M}(G)$ and elegantly solved this problem in the more general setting of b-matchings. His results however do not carry over to the case allowing only 0-1 variables; as for the resulting problem some more graph theoretical considerations have to be pursued.

As the 2-matching problem is very closely related to the symmetric travelling salesman problem it is an interesting question which of the facets of the monotone 2-matching polytope $\tilde{Q}^n_{2M}$ carry over to the monotone travelling salesman polytope $\tilde{Q}^n_T$, hence what kind of structure does $\tilde{Q}^n_T$ "inherit" from $\tilde{Q}^n_{2M}$. This problem was completely solved in [4]. It turns out that almost all facets of $\tilde{Q}^n_{2M}$ are also facets of $\tilde{Q}^n_T$.

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Eingegangen am: 29. 4. 77