2. $\mathfrak{N}_i^2 = \{N_{ji} = (j, i) \in K\}$, $Z_2 = \emptyset \Rightarrow \bar{Z}_2 = \tilde{K}_{0i}$ ergibt $\mathsf{E}(L_i) \geqq \max_{\mathfrak{N}_i} \{\mathsf{E}(L_j) + \mathsf{E}(X_{\varphi^{-1}(j,i)})\}$; diese Ungleichung bildet die Grundlage für die PERT-Schätzung

$$(\hat{\mathsf{E}}^{\mathrm{PERT}}(L_0) = 0, \qquad \hat{\mathsf{E}}^{\mathrm{PERT}}(L_i) = \max_{\mathfrak{N}_i^2} \{\hat{\mathsf{E}}^{\mathrm{PERT}}(L_j) + \hat{\mathsf{E}}^{\mathrm{PERT}}(X_{\varphi^{-1}(j,i)})\}, \qquad i \in P - \{0\}).$$

3. $\mathfrak{N}_i^3 = \{N_{ji} = (j, i) \in K\}$, $Z_3 = \hat{K}_i \, (= \bigcup_{\mathfrak{N}_i^3} K_{ji})$ ergibt

$$\mathsf{E}(L_i) \geqq \mathsf{E}_{\hat{K}_i} \max_{\mathfrak{N}_i^3} \{\mathsf{E}_{\bar{\hat{K}}_i}(L_j) + X_{\varphi^{-1}(j,i)}\},$$

diese Ungleichung bildet die Grundlage für die FULKERSON-Schätzung

$$(F(L_0) = 0, \qquad F(L_i) = \mathsf{E}_{\hat{K}_i} \max_{\mathfrak{N}_i^3} \{F(L_j) + X_{\varphi^{-1}(j,i)}\}, \qquad i \in P \setminus \{0\}).$$

Während, wie man sich leicht überzeugt, in den Fällen 1.—3. die Bedingungen a)—b) automatisch erfüllt waren, setzen wir jetzt

4. $\mathfrak{N}_i^4 = \{N_{ji} = W_{ji} \mid W_{ji}$ Weg von j nach i, wobei a)—b) erfüllt ist$\}$,

$Z_4 = \bigcup_{\mathfrak{N}_i^4} K(W_{ji})$ ergibt $\mathsf{E}(L_i) \geqq \mathsf{E}_{Z_4} \max_{\mathfrak{N}_i^4} \{\mathsf{E}_{\bar{Z}_4}(L_j) + L(W_{ji})\}$,

diese Ungleichung bildet die Grundlage für die ROBILLARD-TRAHAN Schätzung

$$(RT(L_0) = 0, \qquad RT(L_i) = \mathsf{E}_{Z_4} \max_{\mathfrak{N}_i^4} \{RT(L_j) + L(W_{ji})\}, \qquad i \in P \setminus \{0\}).$$

Man kann zeigen

$$\mathsf{E}(L_i) \geqq RT(L_i) \geqq F(L_i) \geqq \hat{\mathsf{E}}^{\mathrm{PERT}}(L_i). \tag{4}$$

Bei einer Verbesserung der wegen ihrer einfachen Berechnungsmöglichkeit wohl am häufigsten benutzten Schätzer nach der PERT-Methode hat man sich also Kenntnis über Verteilungen und entsprechende Erwartungswerte von kritischen Weglängen für Subnetzpläne zu verschaffen.

Inwieweit sich Monotoniebeziehungen der in (4) beschriebenen Art auch auf andere Schätzmöglichkeiten (bezüglich geeignet zu wählender Systeme $\mathfrak{N}_i$ und der Inklusionsbeziehung bzgl. Teilmengen von $\tilde{K}_{0i}$) übertragen lassen, wird noch untersucht.

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ZAMM 58, T 469 – T 471 (1978

M. GRÖTSCHEL

Hypo-Hamiltonian Facets of the Symmetric Travelling Salesman Polytope

The monotone symmetric travelling salesman polytope $\tilde{Q}_T^n$ is defined as the convex hull of all incidence vectors of hamiltonian cycles (tours) and subsets of tours in a complete undirected graph with n nodes. It is known that $\tilde{Q}_T^n$ has tremendously large classes of facets which are in a certain sense combinatorially pleasant (see [5]). Here we will show that $\tilde{Q}_T^n$ also has combinatorially unpleasant facets. Two of these classes of complicated facets are identified. They are induced by certain hypo-HAMILTONian and hypotraceable graphs, which are graphs possessing some strange properties.

Given n cities and distances $c_{ij} \in \mathbb{R}$ between each pair i, j of cities, the problem of finding the shortest roundtrip visiting each city exactly once is called the *travelling salesman problem*. If $c_{ij} = c_{ji}$ the travelling salesman problem is called *symmetric*. The symmetric travelling salesman problem (henceforth TSP) can be adequately modeled in graph theoretical terms. Each city i is identified with a node i, each link between two cities i, j is represented by an edge $\{i, j\}$ carrying a "weight" c_{ij} which represents the distance. The problem of finding the shortest roundtrip is now the problem of finding the shortest HAMILTONian cycle (also called *tour*) in the complete undirected graph $K_n = [V_n, E_n]$, where $V_n = \{1, \dots, n\}$, $E_n = \{\{i, j\} : i, j \in V_n, i \neq j\}$.

The formulation of the TSP as an integer or linear program has been shown to be of great value for designing efficient algorithms, cf. [6], and for solving large problems, cf. [3], [5]. This can be done as follows:

For each edge $e \in E_n$ we define a variable x_e. For each subset F of the edge set E_n an *incidence vector* $x^F = (x_e^F)_{e \in E_n}$ is defined in the following way: $x_e^F = 1$ if $e \in F$, $x_e^F = 0$ if $e \notin F$. Let $\tilde{T}_n$ be the set of tours and subsets of tours in the graph $K_n = [V_n, E_n]$. Then the (*monotone symmetric*) *travelling salesman polytope* is defined to be

$$\tilde{Q}_T^n := \operatorname{conv} \{x^T \in \mathbb{R}^{|E_n|} : T \in \tilde{T}_n\} .$$

It should be clear that the travelling salesman problem can (theoretically) be solved as a linear program over $\tilde{Q}_T^n$, cf. [5]. Results from polyhedron theory imply that there exists a system of linear inequalities $Ax \leqq b$, $x \geqq 0$, such that $\tilde{Q}_T^n = \{x \in \mathbb{R}^{|E_n|} : Ax \leqq b, x \geqq 0\}$. However a system characterizing $\tilde{Q}_T^n$ completely and non-redundantly is not known explicitely although tremendously large classes of facets of $\tilde{Q}_T^n$ have been found already, cf. [5]. (A linear inequality $ax \leqq b$ is called a *facet* of $\tilde{Q}_T^n$ if it is *valid* with respect to $\tilde{Q}_T^n$, that is $\tilde{Q}_T^n \subset \{x : ax \leqq b\}$ and if $\dim (\tilde{Q}_T^n \cap \cap \{x : ax = b\}) = \dim \tilde{Q}_T^n - 1$, i.e. the facets are those inequalities which are necessary to characterize a polytope.) All these classes contain combinatorially pleasant facets, i.e. inequalities which can be characterized completely by few parameters. Here we shall show that $\tilde{Q}_T^n$ also has facets which cannot be characterized by simple parameters and which are not easy to determine. This implies that it is most unlikely that an explicit complete characterization of $\tilde{Q}_T^n$ can ever be given.

In what follows all graphs we are dealing with are undirected, have no loops and no multiple edges. If $G = [V, E]$ is a graph we use the following abbreviations: If $v \in V$ then $G - v := [V - \{v\}, E - \{e \in E : v \in e\}]$, if $e \in E$ then $G - e := [V, E - \{e\}]$, if $e \in E_n - E$ then $G + e := [V, E \cup \{e\}]$.

Definition 1: Let $G = [V, E]$ be a graph.

(a) G is called HAMILTON*ian* if G contains a HAMILTONian cycle.

(b) G is called *hypo*-HAMILTON*ian* if

(b_1) G is not HAMILTONian and

(b_2) $G - v$ *is* HAMILTONian for all $v \in V$.

(c) G is called *traceable* if G contains a HAMILTONian chain.

(d) G is called *hypotraceable* if

(d_1) G is not traceable and

(d_2) $G - v$ is traceable for all $v \in V$.

Although it is not obvious that such strange graphs as hypo-HAMILTONian or hypotraceable graphs exist, infinite classes of these graphs can be constructed; for hypo-HAMILTONian graphs see [1], [2], [4], [7], [8], for hypotraceable graphs see [7], [8]. Prominent hypo-HAMILTONian graphs are the PETERSEN-graph and the COXETER-graph (cf. [1]), the smallest known hypotraceable graph has 34 nodes and was constructed by THOMASSEN [7] using four PETERSEN-graphs.

Considering hypo-HAMILTONian and hypotraceable graphs $G = [V, E]$ with $|V| = n$ as subgraphs of the complete graph $K_m = [V_m, E_m]$, $m \geqq n$, we are able to relate these graphs to the TSP in the following way:

Proposition 2:

(a) *If $G = [V, E]$ is a hypo-*HAMILTON*ian graph then the hypo-*HAMILTON*ian inequality*

$$\sum_{e \in E} x_e \leqq n - 1$$

is a valid inequality with respect to $\tilde{Q}_T^m$ for $m \geqq n$.

(b) *If $G = [V, E]$ is a hypotraceable graph then the hypotraceable inequality*

$$\sum_{e \in E} x_e \leqq n - 2$$

is a valid inequality with respect to $\tilde{Q}_T^m$ for $m \geqq n$.

Proposition 2 gives rise to the conjecture that some or all of these inequalities are facets of $\tilde{Q}_T^m$. Some of these however can immediately be excluded from the candidate list.

Definition 3: A hypo-HAMILTONian (hypotraceable) graph $G = [V, E]$, $|V| = n$, is called *maximal* if $G + e$ is HAMILTONian (traceable) for all $e \in E_n - E$.

Proposition 4: *If $G = [V, E]$, $|V| = n$, is a hypo-*HAMILTON*ian (hypotraceable) graph* which is not maximal *then the corresponding hypo-*HAMILTON*ian (hypotraceable) inequality is not a facet of $\tilde{Q}_T^m$ for all $m \geqq n$.*

In order to prove the desired result we need a further

Definition 5: A node $v \in V$ in a hypo-HAMILTONian (hypotraceable) graph $G = [V, E]$ *has property Δ* if for each pair v_1, v_2 of neighbours of v one of the following properties is satisfied:

(a) $G - v_1$ contains a HAMILTONian cycle (chain) which contains the edge $\{v, v_2\}$.

(b) $G - v_2$ contains a HAMILTONian cycle (chain) which contains the edge $\{v, v_1\}$.

(c) There is a neighbour v_3 of v such that $G - v_1$ and $G - v_2$ contain a HAMILTONian cycle (chain) which contains $\{v, v_3\}$.

G has *property Δ* if every node $v \in V$ has property Δ.

Although this property Δ seems strange, almost all known hypo-HAMILTONian and hypotraceable graphs have property Δ (see [5]). This is due to the fact that these graphs in general are "thin" (have only few edges and small node-degrees) and that the following result holds:

Proposition 6: *Every node in a hypo-*HAMILTON*ian (hypotraceable) graph which is contained in at most five edges has property Δ.*

Combining all these concepts we can now show

Theorem 7: *Let $G = [V, E]$, $|V| = n$, be a hypo-*HAMILTON*ian graph having property Δ, then for all maximal hypo-*HAMILTON*ian graphs $G' = [V, E']$ with $E \subset E'$ the following is true:*

(a) $\sum_{e \in E'} x_e \leqq n - 1$ *is a facet of* $\tilde{Q}_T^n$.

(b) $\sum_{e \in E'} x_e \leqq n - 1$ *is not a facet of* $\tilde{Q}_T^m$ *for all* $m > n$.

Theorem 8: *Let $G = [V, E]$, $|V| = n$, be a hypotraceable graph having property Δ, then for all maximal hypotraceable graphs $G' = [V, E']$ with $E \subset E'$ the following is true:*

$$\sum_{e \in E'} x_e \leqq n - 2 \quad \text{is a facet of } \tilde{Q}_T^m \text{ for all } m \geqq n .$$

Theorems 7 and 8 imply that in order to characterize $\tilde{Q}_T^n$ completely all hypo-HAMILTONian and hypotraceable graphs have to be characterized completely. However this is a problem which is at least as hard as the travelling salesman problem itself as it can be shown that the decision problem: "Given a graph $G = [V, E]$, is G hypo-HAMILTONian or hypotraceable?" contains an NP-complete subproblem in the sense of complexity-theory, hence is contained in the same class of problems as the TSP.

For the proofs of the above results see [5] where further investigations of the facial structure of $\tilde{Q}_T^n$ and related polytopes as well as computational studies exploiting these results have been reported.

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L. GRYGAROVÁ / F. NOŽIČKA

Über asymptotische Berührung von konvexen Mengen

Für ein allgemeines konvexes Optimierungsproblem

$$\min_{x \in M} \{F(x)\} \tag{1}$$

mit konvexer Zielfunktion $F(x)$ und mit abgeschlossener konvexer Restriktionenmenge $M \neq \emptyset$ tritt nur einer der folgenden Fälle auf:

a) Die Funktion $F(x)$ ist nach unten unbeschränkt über M und somit

$$M_{\text{opt}} = \{\tilde{x} \in M \mid FF(\tilde{x}) = \min_{x \in M} \{F(x)\}\} = \emptyset , \qquad \inf_{x \in M} \{F(x)\} = -\infty .$$

b) Es gibt eine Zahl ν mit $\nu = \inf_{x \in M} \{F(x)\}$, wobei zugleich $M_{\text{opt}} = \emptyset$ ist.

c) Das Problem (1) ist lösbar, d. h. $M_{\text{opt}} \neq \emptyset$.

Vom geometrischen Standpunkt aus entspricht der Fall c) einer Punktberührung von zwei speziellen abgeschlossenen konvexen Mengen M_1 und M_2 im E^n. Dagegen führt der Fall b) zu dem geometrischen Begriff einer asymptotischen Berührung von zwei abgeschlossenen konvexen Mengen M_1 und M_2 im E^n.

Eine nähere Untersuchung des Begriffs der asymptotischen Berührung von zwei abgeschlossenen konvexen Mengen im E^n führt zu interessanten Ergebnissen, die eine eindeutige geometrische Charaktersierung eines solchen Begriffs liefern. Die entsprechenden, hier aufgestellten, notwendigen und hinreichenden Bedingungen für eine solche asymptotische Berührung liefern dann unmittelbar ähnliche Bedingungen für das Auftreten des obigen Falles b).