# On Minimal Strong Blocks

Martin Grötschel
*RHEINISCHE FRIEDRICH—WILHELMS—UNIVERSITÄT BONN*

## ABSTRACT

It is shown that strong blocks, i.e., digraphs that are strongly connected and have no cutnodes, have an ear-decomposition. This result is used to prove that the number $q$ of arcs of minimal strong blocks is bounded by $p \leq q \leq 2p-3$ and that minimal strong blocks contain at least two nodes with indegree and outdegree equal to one.

## 1. NOTATION

The purpose of this paper is to characterize (minimal) strong blocks constructively and to use this characterization for the derivation of certain properties of these digraphs.

We adopt the graph-theoretical notation of Harary [4, Chap. 16], using the terms node instead of point and edge instead of line, but briefly recall some definitions. The graph $G$ of a digraph $D=(V, A)$ has the same node set as $D$ and contains an edge $uv$ if $(u, v) \in A$ or $(v, u) \in A$.

A connected digraph is called *strong* if every arc is contained in a cycle. A digraph is called a *block* if its graph is 2-(node)-connected. A strong block with the property that the digraph obtained by removing any of the arcs is not a strong block, is called *minimal*. A node with both indegree and outdegree equal to one is called a *carrier*. If $D=(V, A)$ is a digraph with node set $V$ and arc set $A$ and $F \subset A$, then $V(F)$ is the set of nodes of $V$ that are contained in at least one arc of $F$. The number of nodes of a digraph is denoted by $p$, the number of its arcs is denoted by $q$.

## 2. STRONG BLOCKS

Many classes of graphs and digraphs (e.g., connected, 2-connected graphs, strong digraphs) have a constructive characterization called

Journal of Graph Theory, Vol. 3 (1979) 213–219

0364-9024/79/0003-0213$01.00

*ear-decomposition* (c.f., Ref. 6) of the following kind:

> *Take a "basic" graph or digraph (e.g., tree, cycle) and add "ears" (e.g., edges, paths) repeatedly in a well defined way.*

If the bases and ears are appropriately defined, then one can often show that a graph belongs to a certain class of graphs if and only if it can be constructed (or decomposed) in the above way. Such a characterization also holds for strong blocks.

**Theorem 2.1.** Let $D=(V,A)$, $A\neq\varnothing$, be a digraph, let $F=\varnothing$, and let the following algorithm be given:

(2.1) Take any cycle $C$ in $D$, and add the arcs of $C$ to $F$.

(2.2) Take any path $P=v_1, v_2, \ldots, v_k$ in $D$ with the following properties: $v_1$, $v_k\in V(F)$, $v_1\neq v_k$, $v_i\in V-V(F)$ for $i=2,\ldots,k-1$. Add the arcs of $P$ to $F$.

(2.3) Repeat (2.2) until no path $P$ with the required properties exists.

Then the algorithm terminates with $A=F$ if and only if $G$ is a strong block.

***Proof.*** The above algorithm clearly creates a strong block, thus, if a digraph can be constructed this way, it is a strong block.

Let $D=(V,A)$ be a strong block, and suppose the algorithm generates a cycle (a cycle always exists) and a sequence of paths and terminates although $A\neq F$.

If there were an arc $(u,v)\in A-F$ with $u$, $v\in V(F)$, then this arc would be a path with the desired properties, thus under the above assumption no such arc exists and $V-V(F)\neq\varnothing$.

Since $D$ is strong, every node in $V-V(F)$ can be reached from the nodes in $V(F)$ by a path, hence there is a node $v\in V(F)$ contained in arcs $(u,v)\in F$, $(v,w)\in A-F$, i.e., $w\in V-V(F)$. The 2-connectedness of $D$ implies the existence of a semicycle $Z$ containing $(u,v)$ and $(v,w)$. Let $S$ denote the semipath in $Z$ from $v$ to $v'$ where $v'(\neq v)$ is the first among the nodes in $V(F)$ that is reached by traversing $Z$ from $v$ in the direction of $(v,w)$. Call the arcs in $S$ that have the same orientation as $(v,w)$ *positively oriented* and call the others *negatively oriented.*

If all arcs in $S$ are positively oriented, then $S$ is a path satisfying the requirements of (2.2), a contradiction. Since $D$ is strong, every negatively oriented arc $x\in S$ is contained in a cycle $C_x$. Substituting the path $C_x-\{x\}$ for each negatively oriented arc $x$ in the path $S$ we obtain a walk $W$ from $v$ to $v'$. $W$ contains a path $T$ from $v$ to $v'$ and $T$ clearly contains a path that satisfies all properties required in (2.2), contradicting the assumption. ∎

**Corollary 2.2.** Strong blocks have an ear-decomposition where the basis graphs are cycles and the ears are paths with distinct endnodes.

## 3. MINIMAL STRONG BLOCKS

Theorem 2.1 also renders possible a simple but useful characterization of minimal strong blocks.

**Proposition 3.1.** A strong block $D$ is minimal iff there is no ear-decomposition of $D$ in which a path of length one is added in one of the steps (2.2).

***Proof.*** If $D$ is not minimal, then there is an arc $x \in A$ such that $D-x$ is a strong block. Any ear-decomposition of $D-x$ and the addition of $x$ to $D-x$ gives an ear-decomposition of $D$ where the path added in the last step (2.2) has length one.

Let $D$ be a strong block for which in an ear-decomposition in the $k$th execution of step (2.2) a path of length 1, i.e., an arc $x=(u, v) \in A-F$, is added to the previously constructed arc set $F$. The digraph $D-x$ can be constructed by the ear-decomposition where only the $k$th execution of step (2.2) is left out and all other steps are performed as in the ear-decomposition of $D$. Therefore $D-x$ is a strong block and $D$ is not minimal. ∎

**Corollary 3.2.** Every minimal strong block $D$ contains at least one carrier.

***Proof.*** Take any ear-decomposition of $D$. The path added in the last execution of (2.2) has length 2 or more by Proposition 3.1. This path and hence $D$ contains at least one carrier. ∎

Other properties of minimal strong blocks now follow.

**Proposition 3.3.** If $D=(V, A)$ is a minimal strong block, then the following holds:

(a) $D$ contains no parallel arcs.
(b) If $p \geq 3$, then $(u, v) \in A$ implies $(v, u) \notin A$.
(c) No cycle in $D$ has a chord.
(d) No graph of any 4-node induced subdigraph of $F$ is isomorphic to $K_4$.

***Proof.*** The first three parts (a), (b), (c) are obvious.

Suppose that the graph of the subdigraph $H$ induced by the nodes $W=\{1, 2, 3, 4\}$ is isomorphic to $K_4$, thus $\mathrm{id}_H(v)+\mathrm{od}_H(v)=3$ for all $v \in W$. We consider several cases.

**Case 1.** $od_H(1)=3$. Since $D$ is strong, there exists a cycle $C$ in $D$ containing the arc $(1,2)$. The nodes 3 and 4 cannot be contained in $C$ because of (c). Suppose $(3,2)\in A$, then $(1,2)$ is a chord of a cycle. Similarly, $(1,3)$ is contained in a cycle not containing the nodes 2 and 4. If $(2,3)\in A$, then $(1,3)$ is the chord of a cycle. Therefore the graph of $H$ does not contain the edge 23, a contradiction.

**Case 2.** $id_H(1)=3$ is contradicted analogously to Case 1.

**Case 3.** $od_H(1)=2$, $id_H(1)=1$. Without loss of generality let $(1,2)$, $(1,3)$, $(4,1)\in A$. Since $od_H(4)=3$ leads to a contradiction, so by symmetry we may assume that $(4,2)$, $(3,4)\in A$. As $id_H(2)=3$ is impossible, we must have $(2,3)\in A$. But then $(1,3)$ is the chord of the cycle $\langle 1,2,3,4\rangle$, a contradiction.

**Case 4.** $id_H(1)=2$, $od_H(1)=1$ is contradicted as in Case 3. ∎

Exploiting the ear-decomposition characterization of minimal strong blocks given in Proposition 3.1 we get:

**Theorem 3.4.** Let $D=(V,A)$ with $p\geq 3$ be a minimal strong block. Then

$$p\leq q\leq 2p-3$$

and these bounds are best possible.

***Proof.*** Let $C$ be a cycle in $D$ of length $k_0$; Proposition 3.3(b) implies $k_0\geq 3$. We reconstruct $D$ with the algorithm given in Theorem 2.1 starting with $C$ in step (2.1) and adding paths in step (2.2).

Let $m$ be the number of times step (2.2) was carried out, and let $k_i$, $i=1,\ldots,m$, be the length of the paths that were added. By Proposition 3.1 every path added in step (2.2) has length at least 2, hence contains at least one new node not contained in the previously constructed digraph $(V(F),F)$. This implies $m\leq p-3$. Therefore we obtain:

$$q=\sum_{i=0}^{m} k_i, \qquad p=k_0+\sum_{i=1}^{m}(k_i-1)=\sum_{i=0}^{m} k_i-m=q-m$$

which gives $q\leq 2p-3$.

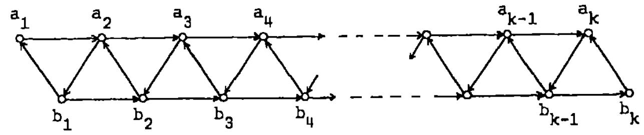
A digraph with less than $p$ arcs is neither strong nor 2-connected. A cycle is a minimal strong block with $p$ arcs. Example 3.5 now shows that the upper bound is also best possible. ∎

**Example 3.5.** We give two classes of minimal strong blocks of all orders $p\geq 3$ that have $2p-3$ arcs.

(a) Let $V:=\{a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_j\}$ where $k\geq 2$, $j\in\{k-1,k\}$.

Let

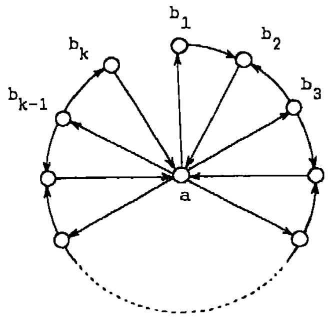
$$\begin{aligned} A := &\{(a_i, a_{i+1}) : i = 1, \dots, k-1\} \\ &\bigcup \{(b_i, b_{i+1}) : i = 1, \dots, j-1\} \\ &\bigcup \{(a_i, b_{i-1}) : i = 2, \dots, k\} \\ &\bigcup \{(b_i, a_i) : i = 1, \dots, j\} \end{aligned}$$




Clearly the removal of any of the arcs $(a_i, a_{i+1})$ or $(b_i, b_{i+1})$ destroys 2-connectedness and the removal of any of the other arcs destroys strong connectedness.

(b) Let $V := \{a, b_1, b_2, \dots, b_k\}$, $k \geqslant 2$ and let

$$\begin{aligned} A := &\{(a, b_i) : i = 1, \dots, k \text{ and } i \text{ odd}\} \\ &\bigcup \{(b_i, a) : i = 1, \dots, k \text{ and } i \text{ even}\} \\ &\bigcup \{(b_i, b_{i+1}) : i = 1, \dots, k-1 \text{ and } i \text{ odd}\} \\ &\bigcup \{(b_{i+1}, b_i) : i = 2, \dots, k-1 \text{ and } i \text{ even}\} \end{aligned}$$




The removal of any of the arcs $(a, b_i)$, $(b_j, a)$ destroys strong connectedness, the removal of any of the other arcs, 2-connectedness. The node $a \in V$ has the maximal possible degree $p-1$, c.f., Proposition 3.3 (a) and (b).

Minimally 2-connected graphs are treated extensively in Refs. 5 and 7 while minimally strongly connected digraphs were characterized in Refs. 1, 2, and 5. Plummer [7] proved that any minimally 2-connected graph has at least two nodes of degree two, it is shown in Ref. 1 (p. 31) that every minimal strong digraph contains at least two carriers, and Geller [2] proved that the same holds for minimally strongly connected blocks. Thus it seems likely that Corollary 3.2 can be sharpened. We proceed to do so. The following properties of digraphs are easily verified:

**Lemma 3.6.** Let $D=(V, A)$, $p \geqslant 2$, be a digraph.

(a) If $D$ is strong, then every block of $D$ is strong.
(b) If $D$ is a minimal strong block and if for some $x \in A$, $D-x$ is strong, then every block of $D-x$ is a minimal strong block.
(c) Every arc is contained in at most one block.

We can now prove

**Theorem 3.7.** Every minimal strong block contains at least two carriers.

***Proof.*** Let $D=(V, A)$ be a minimal strong block. If $D$ is minimally strongly connected, the assertion follows from Ref. 1 (p. 31). Therefore we may assume that $p \geqslant 3$ and that $D$ contains an arc $x$ such that $D-x$ is strongly but not 2-connected.

Let $C$ be a cycle in $D$ that contains $x$. We carry out an ear-decomposition of $D$ starting with $C$ in step (2.1). $D-x$ contains $s$ blocks, where $s \geqslant 2$, and all are minimal strong blocks by Lemma 3.6(b). (Note that none of these minimal strong blocks has a single edge (i.e., $K_2$) as its underlying undirected graph.) In every execution of step (2.2) in the ear-decomposition of $D$ a path $P$ is added to the previously constructed digraph $(V(F), F)$ where only the endpoints $v_1$, $v_k$ of $P$ are in $V(F)$. As $(V(F), F)$ is 2-connected there is a semipath $S$ from $v_k$ to $v_1$ with arcs in $F-\{x\}$. Thus the union of $S$ and $P$ is a semicycle, which proves that all arcs in $P$ are contained in the same block of $D-x$. By Lemma 3.6(c) different blocks have no arc in common, therefore an ear-decomposition of $D$ starting with $C$ reconstructs $D$ "blockwise."

To each of the $s$ blocks of $D-x$ a "last" path is added in some execution of step (2.2). By Proposition 3.1 each of these "last" paths contains at least one carrier, hence each of the $s \geqslant 2$ blocks of $D-x$ contains at least one carrier, which proves the theorem. ∎

The proof of Theorem 3.7 shows that the statement can be formulated in a sharper way:

**Corollary 3.8.** Let $D=(V,A)$ be a minimal strong block, and let $S:=\{x\in A: D-x \text{ is strong}\}$. Let

$$s:=\begin{cases}\max\limits_{x\in S}\{r: D-x \text{ contains } r \text{ blocks}\}, & \text{if } S\neq\varnothing\\ 2, & \text{if } S=\varnothing\end{cases}$$

then $D$ contains at least $s$ carriers.

Theorem 3.7 has a nice application in polyhedral combinatorics since it makes a short proof of the facial characterization of the branching polytope possible which uses elementary linear algebra and the above result only, c.f. Ref. 3.

## ACKNOWLEDGMENTS

The author is indebted to L. Lovász for pointing out the ear-decomposition technique of Ref. 6 to him. This characterization of strong blocks shortened former proofs of Theorems 3.4 and 3.7 considerably. The author is also grateful to the referees for calling Refs. 2 and 5 to his attention. This research was supported by the Sonderforschungsbereich 21(DFG), Institut für Ökonometrie und Operations Research, Universität Bonn.

## References

[1] C. Berge, *Graphs and Hypergraphs*. North Holland, Amsterdam (1973).

[2] D. P. Geller, Minimally strong digraphs. *Proc. Edinburgh Math. Soc.* 17 (1970) 15–22.

[3] M. Grötschel, Strong blocks and the optimum branching problem. Report No. 7786-OR, Institut für Ökonometrie und Operations Research, Universität Bonn (1977).

[4] F. Harary, *Graph Theory*. Addison-Wesley, Reading, Mass. (1969).

[5] S. Hedetniemi, Characterizations and constructions of minimally 2-connected graphs and minimally strong digraphs. *Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing*. R. C. Mullin et al., Eds. Utilitas Mathematica, Winnipeg (1971).

[6] L. Lovász and M. D. Plummer, On minimal elementary bipartite graphs. *J. Combinatorial Theory, Ser. B* 23 (1977) 127–138.

[7] M. D. Plummer, On minimal blocks. *Trans. Amer. Math. Soc.* 134 (1968) 555–567.