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ON THE MONOTONE SYMMETRIC TRAVELLING SALESMAN PROBLEM: HYPOHAMILTONIAN/HYPOTRACEABLE GRAPHS AND FACETS*

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A graph G is called hypohamiltonian (hypotraceable) if it does not contain a hamiltonian cycle (chain) but if every vertex-deleted subgraph $G - v$ contains a hamiltonian cycle (chain). It is shown that certain classes of these graphs induce facets of the monotone symmetric travelling salesman polytope, i.e. the convex hull of the incidence vectors of all tours and subsets of tours. These results indicate that it is quite unlikely that an explicit complete characterization of this polytope can be obtained.

1. Introduction. The Travelling Salesman Problem (henceforth TSP) is one of the most intensively studied combinatorial optimization problems, and almost all techniques available to integer programmers have been tried to design workably efficient algorithms for its solution. Recent studies (cf. [7], [11]) indicate that a refined version of the old approach due to Dantzig et al. [4] using facial cutting planes seems to be very successful and promising for truly large problems. In this method the TSP is relaxed to a small linear program which is solved by standard or special (exploiting the particular structure) LP-techniques. If the LP-solution is a tour, the TSP is solved, otherwise facets of the travelling salesman polytope which are violated by the LP-optimum are generated to cut off this solution. This procedure is carried on until (1) a tour is reached or (2) no facial inequality can be determined that could slice off the LP-solution. If all facets of the travelling salesman polytope were known explicitly and good methods to find violated facets were available case (2) would never occur and the facial cutting plane approach would always converge (theoretically).

Tremendously large classes of facets for the travelling salesman problem have been characterized (cf. [8], [9]) but our knowledge here is still incomplete, even for small dimensions. The known facets however are "combinatorially pleasant", and, although their number is exponential, it seems possible—as in the case of matchings (cf. [6])—to implement efficient cutting plane algorithms based on these. One might hope that the missing facets will be found through intensive research efforts, that these facets are also "nice," and that the full linear system describing the travelling salesman polytope can be used in an algorithm of the type described above.

On the other hand the TSP is known to be NP-hard which makes one suppose that there is a trap in the hopeful thoughts above. As the matching problem shows, an exponential number of facets does not necessarily make a problem hard, hence one might suspect that the TSP has facets which are "combinatorially unpleasant." Some results indicating this will be established in this paper.

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2. Notation. All graphs $G = [V, E]$ we consider are undirected, have no loops and no multiple edges. $|V|$ is called the *order* of G.

The edges $e \in E$ are two-element subsets of V and are denoted by $e = \{i, j\}$. The complete graph of order n is denoted by $K_n = [V, E_n]$. A graph $G' = [W, F]$ is called a *subgraph* of G if $W \subset V$ and $F \subset E$. We use the following abbreviations:

$$G - v := [V - \{v\}, E - \{e : v \in e\}] \quad \text{for } v \in V,$$
$$G - e := [V, E - \{e\}] \quad \text{for } e \in E,$$
$$G + e := [V, E \cup \{e\}] \quad \text{for } e \in E_n - E.$$

$N(v)$ is the set of *neighbors* of v, i.e. $N(v) = \{w \in V : \exists e \in E \text{ such that } e = \{v, w\}\}$. The *degree* of $v \in V$, denoted by $d(v)$, is the number of neighbors of v. A *chain* of length $k - 1$ is a set of edges $P = \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}\}$, for short denoted by $P = [i_1, i_2, \ldots, i_k]$. If $\{i_k, i_1\} \in E$ then $C := P \cup \{\{i_k, i_1\}\}$ is called a *cycle* of length k denoted by $C = \langle i_1, i_2, \ldots, i_k \rangle$. We require that $i_p \neq i_q$, $1 \leq p < q \leq k$, for all cycles and chains. Chains of length $|V| - 1$ and cycles of length $|V|$ are called *hamiltonian*. Hamiltonian cycles are also called *tours*. A graph G is called *hamiltonian* if it contains a hamiltonian cycle; G is called *traceable* if it contains a hamiltonian chain. The symmetric travelling salesman problem can be described as follows: Given a complete graph $K_n = [V, E_n]$ and "distances" $c_e \in \mathbb{R}$ for all $e \in E_n$, find the shortest tour in K_n. In a natural way a polytope having (0, 1)-vertices can be associated to the TSP. Let $\tilde{T}_n$ be the set of all edge-sets which are tours or subsets of tours in K_n. To each edge $e \in E_n$ we associate a variable x_e, and to each $T \in \tilde{T}_n$ we associate an incidence vector x^T, i.e. a vector such that $x_e^T = 1$ if $e \in T$ and $x_e^T = 0$ otherwise. As $|E_n| = n(n-1)/2 =: m$ we have $x^T \in \mathbb{R}^m$. The convex hull $\tilde{Q}_T^n$ of all incidence vectors of tours and subsets of tours is called the *monotone symmetric travelling salesman polytope*, i.e.

$$\tilde{Q}_T^n := \operatorname{conv}\{x^T \in \mathbb{R}^m : T \in \tilde{T}_n\}.$$

It should be clear that every (symmetric) travelling salesman problem can be solved as a linear maximization problem over $\tilde{Q}_T^n$.

If $E \subset E_n$ then $x(E) := \sum_{e \in E} x_e$ is used to abbreviate summation.

$\tilde{Q}_T^n$ contains the zero-vector and all unit-vectors hence it is fully dimensional, $\dim \tilde{Q}_T^n = m$. An inequality $ax \leq a_0$ is called *valid* with respect to $\tilde{Q}_T^n$ if $\tilde{Q}_T^n \subset \{x \in \mathbb{R}^m : ax \leq a_0\}$; a valid inequality $ax \leq a_0$ is called *maximal* if the inequality obtained by increasing any component a_e of the vector a by any $\epsilon > 0$ is not valid; a valid inequality $ax \leq a_0$ is called a *facet* of $\tilde{Q}_T^n$ if $\dim(\tilde{Q}_T^n \cap \{x \in \mathbb{R}^n : ax = a_0\}) = \dim \tilde{Q}_T^n - 1 = m - 1$. A facet is obviously a maximal inequality.

As $\tilde{Q}_T^n$ is fully dimensional there exists a unique (up to a constant factor) finite non-redundant system of linear inequalities $Ax \leq b$ such that $\tilde{Q}_T^n = \{x \in \mathbb{R}^m : Ax \leq b\}$. This system is given by the set of all facets of $\tilde{Q}_T^n$. Clearly, the effectiveness of an LP-relaxation cutting plane approach to the TSP depends heavily on the knowledge of facets of $\tilde{Q}_T^n$. It was shown in [8] that the trivial inequalities, the subtour-elimination constraints and all comb inequalities define facets of $\tilde{Q}_T^n$, which implies that the number of inequalities necessary to characterize $\tilde{Q}_T^n$ completely is tremendously large. The system of these inequalities, however, does not suffice to establish $\tilde{Q}_T^n$. Here we will characterize a new class of facets which—in contrast to the comb- and subtour-elimination constraints—does not seem to be easily dealt with.

3. Hypohamiltonian and hypotraceable graphs.

DEFINITION 3.1. Let $G = [V, E]$ be a graph.

(a) G is called *hypohamiltonian* if

(a_1) G is not hamiltonian and

(a_2) $G - v$ is hamiltonian for all $v \in V$.

(b) G is called *hypotraceable* if

(b_1) G is not traceable and

(b_2) $G - v$ is traceable for all $v \in V$. ∎

A hypohamiltonian (hypotraceable) graph $G = [V, E]$ is called *maximal* if $G + e$ is hamiltonian (traceable) for all $e \in E_n - E$.

For the history of these unusual graphs and their known properties see [3], [8], or [12]. To date no good or nearly good characterizations of hypohamiltonian or hypotraceable graphs are known, and seem to be hard to obtain. The articles dealing with those graphs (cf. [1], [3], [5], [10], [12], [13]) usually exhibit new classes of hypohamiltonian or hypotraceable graphs showing that for certain orders n such graphs indeed exist or that these graphs possess strange and unexpected properties. For our purposes we are interested in hypohamiltonian and hypotraceable graphs which own a certain technically useful property:

DEFINITION 3.2. Let $G = [V, E]$ be a hypohamiltonian (or hypotraceable) graph. A node $v \in V$ is said to have *property* Δ if for any two nodes $v_1, v_2 \in N(v)$ one of the following conditions is satisfied:

(a) $G - v_1$ contains a hamiltonian cycle (chain) which contains the edge $\{v, v_2\}$.

(b) $G - v_2$ contains a hamiltonian cycle (chain) which contains the edge $\{v, v_1\}$.

(c) There exists a node $v_3 \in N(v)$ such that both $G - v_1$ and $G - v_2$ contain a hamiltonian cycle (chain) which contains the edge $\{v, v_3\}$.

G has *property* Δ if every node $v \in V$ has property Δ. ∎

We are going to show that almost all of the known hypohamiltonian and hypotraceable graphs have property Δ.

LEMMA 3.3. *Let $G = [V, E]$ be a hypohamiltonian or hypotraceable graph, then every node $v \in V$ with $d(v) \leq 5$ has property Δ.*

PROOF. We show the assertion for $d(v) = 5$, the other cases follow similarly.

(a) Let G be hypohamiltonian and $v \in V$, and let $N(v) := \{v_1, v_2, v_3, v_4, v_5\}$. W.l.o.g. we show that one of the conditions (a), (b), (c) of 3.1 is satisfied for each of the pairs $v_1, v_i \in N(v)$, $i = 2, 3, 4, 5$.

(a_1) $G - v_1$ contains a hamiltonian cycle C; we may assume w.l.o.g. that C contains the chain $[v_2, v, v_3]$. Therefore condition (a) of 3.1 is satisfied for the pairs $v_1, v_2 \in N(v)$ and $v_1, v_3 \in N(v)$.

(a_2) Suppose $G - v_1$ (or $G - v_4$) contains a hamiltonian cycle which contains the edge $\{v, v_4\}$ (or $\{v, v_1\}$), then condition (a) is fulfilled for $v_1, v_4 \in N(v)$. If this is not the case then each hamiltonian cycle in $G - v_4$ contains a chain $[v_i, v, v_j]$ where $i \neq 1 \neq j$. One of the nodes v_i, v_j is necessarily one of the nodes v_2, v_3, say v_3. Thus $v_3 \in N(v)$ is a node such that $G - v_1$ and $G - v_4$ contain a hamiltonian cycle which contains the edge $\{v, v_3\}$; this means that condition (c) of 3.1 is satisfied.

(a_3) Substituting v_5 for v_4 in part (a_2) we obtain the assertion for the pair $v_1, v_5 \in N(v)$.

This proves the hypohamiltonian case.

(b) Let G be hypotraceable and $v \in V$. If $v_1 \in N(v)$ then none of the hamiltonian chains K in $G - v_1$ ends in v, otherwise $K \cup \{\{v, v_1\}\}$ would be a hamiltonian chain in G. Therefore any hamiltonian chain in $G - v_1$ contains a chain $[v_i, v, v_j]$ where $v_i, v_j \in N(v)$. The proof of the hypotraceable case is now completely analogous to part (a). ∎

As most of the known hypohamiltonian and hypotraceable graphs are "sparse," i.e. have comparatively few edges and small node degrees, Lemma 3.3 is widely applicable.

EXAMPLE 3.4. All nodes of the following hypohamiltonian graphs have node degree at most five, hence these graphs have property Δ by Lemma 3.3:

(a) The class of hypohamiltonian graphs constructed by Chvátal [3] from so called Flip-Flops.

(b) The class of cubic hypohamiltonian graphs given by Bondy [1].

(c) All hypohamiltonian graphs that can be constructed by Thomassen's method [12] from cubic hypohamiltonian graphs.

Almost all nodes of the following hypohamiltonian graphs have node degree at most five, for the others property Δ can be checked easily:

(d) All but three nodes of the hypohamiltonian graphs $G_3(m)$, $m \geqslant 2$, and $G_3(m, k)$, $m \geqslant 2$, $k \geqslant 2$ found by Doyen and von Diest [5] have degree at most five.

(e) Only one node in the class of hypohamiltonian graphs given by Lindgren [10] has degree more than five, if $|V| \geqslant 22$.

(f) The hypohamiltonian graphs $G_5(m, k)$, $m \geqslant 2$ or $k \geqslant 2$, of Doyen and van Diest [5] have five nodes with degree more than five if $m + k \geqslant 5$. We are going to show that these five nodes also have property Δ.

The node set of $G_5(m,k)$-graphs, where $m \geqslant 2$ or $k \geqslant 2$, is $V = \{a_1, a_2, \ldots, a_{5m}, u_1, u_2, \ldots, u_5, b_1, b_2, \ldots, b_{5k}\}$, and the edge set E is given by the cycle $\langle a_1, a_2, \ldots, a_{5m}\rangle$, the cycle $\langle b_1, b_2, \ldots, b_{5k}\rangle$ and the edges $\{u_h, a_{h+5i}\}$, $h = 1, \ldots, 5$, $i = 0, 1, \ldots, m-1$; $\{u_h, b_{h+5j}\}$, $h = 1, \ldots, 5$, $j = 0, 1, \ldots, k$. The nodes u_h, $h = 1, \ldots, 5$ have degree $m + k$. Because of symmetry it is sufficient to prove that u_1 has property Δ. Figure 3.1 shows that $G_5(m, k) - a_1$ contains a hamiltonian cycle which contains the edges $\{u_1, a_6\}$, $\{u_1, b_1\}$.

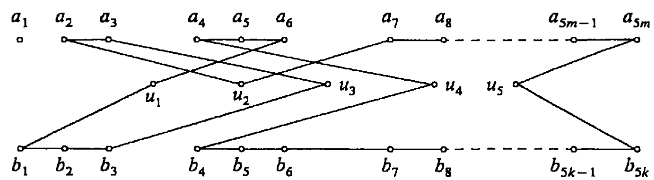

FIGURE 3.1

Thus for $a_1, a_6 \in N(u_1)$ and $a_1, b_1 \in N(u_1)$ condition (a) of Definition 3.1 is satisfied. Obviously the construction in Figure 3.1 can be modified such that for any pair $a_{1+5j}, a_{1+5i} \in N(u_1)$ and any pair $a_{1+5}, b_{1+5j} \in N(u_1)$ condition (a) is satisfied. By symmetry (invert the drawing) it is clear that condition (a) also holds for any $b_{1+5i}, b_{1+5j} \in N(u_1)$. These are all possible cases, therefore u_1 has property Δ. ∎

EXAMPLE 3.5. All hypotraceable graphs that can be constructed by Thomassen's method (cf. [12]) from cubic hypohamiltonian graphs only have nodes with degree at most four, and the hypotraceable graphs given by Thomassen in [14, Theorem 3.2] are cubic, hence by Lemma 3.3 they all have property Δ. ∎

The above examples which cover most of the known hypohamiltonian and hypotraceable graphs show that the technical property Δ is not out of the ordinary but owned by many of these graphs.

4. Facets of $\tilde{Q}_T^n$.

V. Chvátal [2] seems to be the first to have observed a relationship between hypohamiltonian graphs and the travelling salesman problem. He has shown that the Petersen graph, which is the smallest hypohamiltonian graph, defines a facet of a polytope closely related to $\tilde{Q}_T^n$.

In the following we consider all graphs $G = [V, E]$ of order n as subgraphs of the complete graph K_m, i.e. $V \subset \{1, 2, \ldots, m\}$, $E \subset E_m$.

LEMMA 4.1. *Let $G = [V, E]$ be a graph of order n.*

(a) *G is nonhamiltonian if and only if $x(E) \leqslant n - 1$ is valid with respect to $\tilde{Q}_T^m$ for all $m \geqslant n$.*

(b) *G is nontraceable if and only if $x(E) \leqslant n - 2$ is valid with respect to $\tilde{Q}_T^m$ for all $m \geqslant n$.*

PROOF. Clear. ∎

A graph $G = [V, E]$ of order n is *maximal nonhamiltonian (maximal nontraceable)* if $G + e$ is hamiltonian (traceable) for all $e \in E_n - E$. The following observation is obvious:

LEMMA 4.2. *Let $G = [V, E]$ be a graph of order n.*

(a) *G is maximal nonhamiltonian if and only if for all $f \in E_n - E$ and for all $\alpha > 0$ the inequality $x(E) + \alpha x_f \leqslant n - 1$ is not valid with respect to $\tilde{Q}_T^n$.*

(b) *G is maximal nontraceable if and only if for all $m \geqslant n$, for all $f \in E_m - E$ and for all $\alpha > 0$ the inequality $x(E) + \alpha x_f \leqslant n - 2$ is not valid with respect to $\tilde{Q}_T^m$.* ∎

Lemma 4.1 and Lemma 4.2 imply for hypohamiltonian and hypotraceable graphs:

PROPOSITION 4.3. *Let $G = [V, E]$ be a graph of order n.*

(a) *If G is maximal hypohamiltonian then $x(E) \leqslant n - 1$ is a maximal valid inequality with respect to $\tilde{Q}_T^n$.*

(b) *If G is maximal hypotraceable then $x(E) \leqslant n - 2$ is a maximal valid inequality with respect to $\tilde{Q}_T^m$ for all $m \geqslant n$.*

PROOF. Since hypohamiltonian resp. hypotraceable graphs are nonhamiltonian resp. nontraceable the corresponding inequalities are valid by Lemma 4.1. As maximal hypohamiltonian resp. maximal hypotraceable graphs are maximal nonhamiltonian resp. maximal hypotraceable none of the zero-coefficients can be increased by Lemma 4.2.

(a) Let $e = \{v, w\} \in E$, and $ax := x(E) \leqslant n - 1$. Since G is hypohamiltonian, $G - w$ contains a hamiltonian cycle C which contains an edge $\{u, v\}$ for some $u \in V$. Let $K := (C - \{u, v\}) \cup \{e\}$ then K is a hamiltonian chain in G, thus $x^K \in \tilde{Q}_T^n$ and $ax^K = n - 1$, i.e. the coefficient $a_e = 1$ cannot be increased without violating the validity of $ax \leqslant n - 1$.

(b) Let $e = \{v, w\} \in E$, and $ax := x(E) \leqslant n - 2$. Since G is hypotraceable $G - w$ contains a hamiltonian chain V which contains an edge $\{u, v\}$ for some $u \in V$. Let $K := (C - \{u, v\}) \cup \{e\}$ then K consists of two node-disjoint paths containing all nodes of G, therefore $x^K \in \tilde{Q}_T^m$ and $ax^K = n - 2$, i.e. a_e cannot be increased. ∎

Some of the hypohamiltonian and hypotraceable graphs described in the Examples 3.4 to 3.5 are maximal (like the Petersen graph) and some are not. But any nonmaximal graph can be completed (possibly in various ways) to a maximal graph, thus giving rise to many maximal inequalities. The following theorems show that any completion of hypohamiltonian and hypotraceable graphs with property Δ does not only create a maximal inequality but a facet of $\tilde{Q}_T^n$, hence an inequality which is essential for a complete and nonredundant linear characterization of the monotone symmetric travelling salesman polytope.

THEOREM 4.4. *Let $G = [V, E]$ be a hypohamiltonian graph of order n having property Δ. Let $G' = [V, E']$ be any maximal hypohamiltonian graph with $E \subset E'$ then the following holds:*

(a) *$x(E') \leqslant n - 1$ is a facet of $\tilde{Q}_T^n$.*

(b) *$x(E') \leqslant n - 1$ is not a facet of $\tilde{Q}_T^m$ for $m > n$.*

PROOF. (a) 1. Let $ax := x(E')$, $H_a := \{x \in \tilde{Q}_T^n : ax = n - 1\}$ and $H := \{T \in \tilde{T}_n : ax^T = n - 1\}$. We wish to show that $ax \leqslant n - 1$ is a facet of $\tilde{Q}_T^n$. Thus we may let $bx \leqslant b_0$ be a valid inequality such that $H_a \subset H_b := \{x \in \tilde{Q}_T^n : bx = b_0\}$. This implies

$b \neq 0$. As $\tilde{Q}_T^n$ is fully dimensional facets are unique up to a multiple constant i.e. $ax \leq n-1$ is a facet of $\tilde{Q}_T^n$ if and only if $b = \pi a$, $\pi \in \mathbb{R} - \{0\}$. Hence, we have to show that $b_e = \pi a_e$ for all $e \in E_n$.

2. We first show that $b_e = b_f$ for any two $e, f \in E$ which are adjacent. Let v be any node in V and v_1, v_2 be any two neighbors of v in G. By assumption G has property Δ hence one of the following three cases must occur:

Case 1: $G - v_1$ contains a hamiltonian cycle C which contains the edge $\{v, v_2\}$. Let $w \in V - \{v, v_1, v_2\}$ be any node such that $\{w, v_1\} \in E$. Such a node exists as in hypohamiltonian graphs all nodes obviously have degree at least three. The cycle C can be given as $C = \langle v, v_2, \ldots, w, w_2, \ldots \rangle$ where w_2 is some neighbor of w. The nodes v and w are not neighbors in C, i.e., $\{v, w\} \notin C$, $v \neq w_2$, otherwise C could be extended to a hamiltonian cycle in G, a contradiction. We define the following two hamiltonian chains in G:

$$K := (C - \{\{w, w_2\}\}) \cup \{\{w, v_1\}\}, \qquad L := (K - \{\{v, v_2\}\}) \cup \{\{v, v_1\}\}.$$

By construction $K, L \in H$, hence $x^K, x^L \in H_a \subset H_b$. Therefore

$$0 = b_0 - b_0 = bx^K - bx^L = b_{vv_2} - b_{vv_1}.$$

Case 2: $G - v_2$ contains a hamiltonian cycle which contains the edge $\{v, v_1\}$. Exchanging the roles of v_1 and v_2 in Case 1 proves this part.

Case 3: There is a neighbor v_3 of v in G such that $G - v_1$ and $G - v_2$ contain a hamiltonian cycle which contains $\{v, v_3\}$. Using the construction in Case 1 we can prove that $b_{vv_1} = b_{vv_3}$ and also $b_{vv_2} = b_{vv_3}$, i.e. we get $b_{vv_1} = b_{vv_2}$.

The Cases 1–3 prove that $b_e = b_f$ for all $e, f \in E$, where $e \cap f \neq \emptyset$.

3. Now let $e = \{u, v\}$, $f = \{y, z\}$ be any two edges in E. As a hypohamiltonian graph is connected there is a chain $[v, v_1, \ldots, v_k, y]$ in G connecting v and y. Using the result from 2 we conclude from $b_{uv} = b_{vv_1} = b_{v_1v_2} = \cdots = b_{v_ky} = b_{yz}$ that $b_e = b_f$ for all $e, f \in E$. Hence $b_e = \pi$ for all $e \in E$.

4. Let $f = \{u, v\} \in E' - E$. $G - v$ contains a hamiltonian cycle C which contains an edge $e = \{u, w\} \in E$ for some $w \in V$. Using the same construction as in Case 1 of 2, we can show that $b_f = b_e = \pi$, hence $b_e = \pi$ for all $e \in E'$.

5. Let $f \in E_n - E'$. As G' is maximal, $G + f$ contains a hamiltonian cycle C which contains f and $n - 1$ edges of E'. This implies that $C \in H$ and $K := C - \{f\} \in H$. Therefore $x^C, x^K \in H_b$, i.e. $0 = b_0 - b_0 = bx^C - bx^K = b_f$, hence $b_f = 0$ for all $f \in E_n - E'$.

Summing up 1 to 5, we have shown that $b = \pi a$ which proves (a).

(b) If $m > n$ then $E' \subseteq E_n = \{\{i, j\} : \{i, j\} \in V\}$ and $E' \neq E_n$. As $x(E_n) \leq n - 1$ is a valid inequality for $\tilde{Q}_T^m$—in fact it is a facet, cf. [8]—and as $x(E') \leq x(E_n)$ the hypohamiltonian inequality $x(E') \leq n - 1$ is not even a maximal inequality for $\tilde{Q}_T^m$. ∎

Previously known results on polytopes $\tilde{Q}_T^n$, cf. [8], suggested that for each successive n, $\tilde{Q}_T^n$ inherits via trivial lifting (adding zero components) all facets of $\tilde{Q}_T^{n-1}$ and in addition gains certain "new" facets. However, the above theorem shows that travelling salesman polytopes display a class of facets which is *not* trivially liftable. Moreover, the applicability of Theorem 4.4 is not restricted to a few isolated values of n since hypohamiltonian graphs of order n are known to exist for $n = 10, 13, 15, 16, 18$ and larger, cf. [13], and Examples 3.4 and 3.5 show that for every such n there are hypohamiltonian graphs having property Δ.

We will now show that the situation for hypotraceable graphs is still different.

THEOREM 4.5 *Let $G = [V, E]$ be a hypotraceable graph of order n having property Δ and $G' = [V, E']$ be any maximal hypotraceable graph with $E \subset E'$ then*

$$x(E') \leq n - 2 \text{ is a facet of } \tilde{Q}_T^m \text{ for all } m > n.$$

PROOF. 1. We use the same notation and considerations as in the proof of 4.4 part (a)1, we only substitute $n - 1$ by $n - 2$ and $\tilde{Q}_T^n$ by $\tilde{Q}_T^m$.

2. Let $v, v_1, v_2 \in V$ such that $\{v, v_1\}, \{v, v_2\} \in E$. G has property Δ hence the following cases must occur:

Case 1: $G - v_1$ contains a hamiltonian chain C containing $\{v, v_2\}$. Define $K := (C - \{\{v, v_2\}\}) \cup \{\{v, v_1\}\}$. By construction $C, K \in H$, hence $0 = b_0 - b_0 = bx^K - bx^C = b_{vv_1} - b_{vv_2}$.

Case 2: $G - v_2$ contains a hamiltonian chain C containing $\{v, v_1\}$, like Case 1.

Case 3: There is a neighbor v_3 of v in G such that both $G - v_1$ and $G - v_2$ contain a hamiltonian chain containing the edge $\{v, v_3\}$. Case 1 shows that $b_{vv_1} = b_{vv_3}$, and $b_{vv_2} = b_{vv_3}$, hence $b_{vv_1} = b_{vv_2}$.

3. Substituting "chain" for "cycle" the same arguments as in parts (a) 3, 4, and 5 of the proof of Theorem 4.4 show that

$$b_e = \pi \quad \text{for all } e \in E' \quad \text{and}$$
$$b_e = 0 \quad \text{for all } e \in E_n - E',\ E_n = \{\{i, j\} : i, j \in V\}.$$

4. It is easily seen that

$$b_e = 0 \quad \text{for all } e \in E_m \text{ such that } |e \cap V| \leq 1 \text{ holds.}$$

1 to 4 show that $b = \pi a$, which proves the theorem. ∎

Hence those maximal hypotraceable graphs obtained by completing hypotraceable graphs having property Δ constitute another class of complicated facets, this time, however, trivial lifting is possible; i.e. a facial hypotraceable inequality for $\tilde{Q}_T^n$ will also be facial for $\tilde{Q}_T^m$ for $m > n$.

Furthermore, the classes of facets found in Theorems 4.4 and 4.5 are not at all small. Each graph of the classes of hypotraceable and hypohamiltonian graphs given in Examples 3.4 to 3.5 gives rise to at least one (possibly many, if the graph is not maximal) set of facetial inequalities, because each labeling of the nodes such that the corresponding hypohamiltonian and hypotraceable inequalities are not identical defines a different facet of the travelling salesman polytope.

5. Conclusions. We have shown that certain hypohamiltonian and hypotraceable graphs induce facets of the monotone symmetric travelling salesman polytope, and that the number of such facets is not at all small. Thus in order to characterize $\tilde{Q}_T^n$ completely we have to characterize all those hypohamiltonian and hypotraceable graphs which induce facets. Although the complexity status of the decision problem "Is a given graph hypohamiltonian (hypotraceable)?" is presently unknown, there is a high probability that this problem is hard, i.e. at least NP- or $\overline{NP}$-complete. Therefore —in our opinion—it appears rather unlikely that an explicit, complete, and nonredundant characterization of the travelling salesman polytope can ever be obtained.

These results also imply the following: If such a graph $G = [V, E]$ is given and if we define "distances" $c_e = 1$ if $e \in E$, $c_e = 0$ if $e \in E_n - E$, then a facial cutting plane approach based on the LP max cx, $x \in \tilde{Q}_T^n$ will fail unless the facet induced by this graph is explicitly or implicitly in our cutting plane list, i.e. in order to insure convergence we have to have a procedure that generates this cutting plane. Again—in our opinion—it seems unlikely that an efficient method for this task can be designed. Although these negative results might indicate that the development of facial cutting plane algorithms for the travelling salesman problem is hopeless, recent computational studies (cf. [7], [11]) using only certain sets of "combinatorially pleasant" facets show the contrary. The reason for this seems to be the fact that facets used there (subtour-elimination-constraints, comb-inequalities) are comparatively large while the strange facets exhibited above contain only a relatively small number of tours. This might explain why many cases require only a few of the simple facets to prove optimality of a tour.

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ON THE OPTIMALITY OF (σ, S) POLICIES*

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A standard periodic review, stochastic, dynamic multiproduct inventory model is considered in which the ordering cost consists of linear portions for each product and a setup cost. This setup cost is incurred if an order is placed for any number of products. There is no separate setup cost incurred for each product ordered. This basic problem has been studied by Johnson (1967) and Wheeler (1968) among others. We extend their results by providing general conditions for the existence of an optimal policy and by further characterizing the optimal policy. In particular, we show that there exists an optimal (σ, S) policy. Such a policy does not order when the initial stock level is in σ and orders up to the vector level S otherwise (provided that such an order is feasible). Furthermore, the set σ is an upper layer (equivalently, an increasing set) with respect to a certain partial ordering. Such a policy reduces to an (s, S) policy in the single product case, in which case our conditions are very similar to those of Veinott (1966) and amount to a special case of the model of Schäl (1976). Our analysis is based in part on a generalization of quasi-convex functions (in the terminology of Porteus (1971)). Finite and infinite horizon results are given.

1. Introduction and model description: Finite horizon. We consider a standard periodic review, stochastic, dynamic m-product inventory model, $m \in \mathbb{N}^1$, in which the m-dimensional *demands* $D_1, \ldots, D_N$ in periods $1, \ldots, N$ are independent random variables with known distribution functions. Assume $D_i \in \vartheta_i$ for given subsets ϑ_i of $\mathbb{R}^m_+$,[1] $i = 1, \ldots, N$. This basic problem has been studied by Johnson (1967) and Wheeler (1968), among others.

Let $x_i = (x_i^1, \ldots, x_i^m)'$[2] denote the *stock on hand prior to placing any order* in period i. For convenience we assume delivery of an order is immediate. Let $y_i = (y_i^1, \ldots, y_i^m)'$ denote the *stock on hand after ordering* (and delivery) in period i. Thus $y_i - x_i$ is the vector of order quantities in period i. We require $y_i \geqslant x_i$ and $y_i \in Y_i$ for a specified subset Y_i of $\mathbb{R}^m$.[1] (If $a = (a^j)'$ and $b = (b^j)'$ are m-vectors of extended real numbers, we write $a \geqslant b$ if $a^j \geqslant b^j$ for all $j \in \{1, \ldots, m\}$. If, in addition, $a \neq b$, we write $a > b$.) Besides $Y_i = \mathbb{R}^m$ and $Y_i = \mathbb{R}^m_+$ one could think (more practically) of $Y_i = \{z \in \mathbb{R}^m \mid \sum_{j=1}^m (z^j)^+ \leqslant M_i\}$[3] for a given $M_i \in \mathbb{R}^1_+$ or $Y_i = \{z \in \mathbb{R}^m \mid z \leqslant b_i\}$ for a given $b_i \in \mathbb{R}^m_+$. In general we could think of Y_i as a specified closed and bounded subset of $\mathbb{R}^m$.

Because of the simple notation we consider the case $Y_i = \{z \in \mathbb{R}^m \mid z \leqslant b_i\}$ for a given $b_i \in \mathbb{R}^m_+$. b_i may be interpreted as the *capacity* of the given inventory model in period i, i.e., b_i^j denotes the maximum amount of units of product j in period i which may be stored. *At the end of period i, the amount of stock on hand* is assumed to be a specified Borel function $v_i(y_i, D_i)$ of y_i and D_i. Thus $x_{i+1} = v_i(y_i, D_i)$. Several interest-

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[1] We use $\mathbb{N}$ to denote the set of the positive natural numbers and $\mathbb{R}^m$, $\mathbb{R}^m_+$, $\bar{\mathbb{R}}^m$ to denote the set of column m-tuples of real, nonnegative real, extended real numbers, respectively.

[2] Given any row $(z^1, \ldots, z^m)$ we mean by $(z^1, \ldots, z^m)'$ the corresponding column and vice versa. For any function $F: \mathbb{R}^m \to \mathbb{R}^m$ we mean by $F^j(x)$ the jth component of $F(x)$.

[3] For a real number Z we write $Z^{\pm}$ for $\max(\pm Z, 0)$.

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