ON INTERSECTIONS OF LONGEST CYCLES

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ABSTRACT Let G be a 2-connected graph, let $k \in \{2, 3, 4, 5\}$, and suppose that G has at least $k+1$ nodes. We prove in this paper that whenever two different longest cycles of G meet in k nodes, then these nodes form an articulation set of G. We give some applications of this result, e.g. a lower bound on the circumference of vertex-transitive graphs, and conjecture possible generalizations.

1. Introduction and main results

A considerable amount of very successful research effort has been spent on the following type of problem: Given a graph (or digraph) with some properties (e.g. assumptions on the connectivity or degree sequence) and given a certain number of nodes or edges or paths, determine a lower (or upper) bound on the length of a longest cycle (or path) of the graph containing (not containing) the given nodes, edges, or paths; see for instance [**3–6**, **8**, **9**] and many others.

To our knowledge almost nothing has been done to determine the intersection patterns of longest cycles. Such results are of interest, for instance, in designing recursive algorithms (e.g. in combinatorial optimization) in which in every step a longest cycle is shrunk or deleted and one wants to know a bound on the length of the longest cycle in the resulting graph. The case where two longest cycles meet in two nodes has been analysed in [**7**] (to obtain a polynomial algorithm for the max-cut problem in graphs without long odd cycles).

In this section we state the main results of the paper. The (non-standard) notation can be found in Section 2, and the proofs in the subsequent sections. In this paper we only consider *2-connected graphs* (the essential case) with at least four nodes. The circumference of such graphs is at least four and every two longest cycles meet in at least two nodes. The result of [**7**] can be stated as follows:

THEOREM 1.1 *Let C and D be two different longest cycles of a graph G meeting in exactly two nodes, say in u and v. Let $C = P_1 \cup P_2$ and $D = Q_1 \cup Q_2$, where P_i, Q_i, $i = 1, 2$ are $[u, v]$-segments of C (respectively*

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D). Then the following hold:

(a) *$\{u, v\}$ is an articulation set of G.*
(b) *$|P_1| = |P_2| = |Q_1| = |Q_2|$ and therefore the circumference is even.*
(c) *The truncated paths $\bar{P}_1$, $\bar{P}_2$, $\bar{Q}_1$, $\bar{Q}_2$ obtained by removing the end-nodes u, v of P_1, P_2, Q_1, Q_2 are in different components of $G-\{u, v\}$.*
(d) *Every longest cycle contains u and v.*
(e) *For every longest cycle B, the two paths of $B-\{u, v\}$ in $G-\{u, v\}$ belong to different components of $G-\{u, v\}$.* □

This result is generalized here to the following.

THEOREM 1.2 *Let $k \in \{3, 4, 5\}$, and let $G = [V, E]$ be a graph with at least $k+1$ nodes. Suppose that C and D are two different longest cycles meeting in a set W of exactly k nodes. Then the following hold:*

(a) *W is an articulation set of G.*
(b) *In the case $k = 3$, the paths obtained by removing W from C and D are in different components of $G-W$.*

The proof of Theorem 1.2 will be given in Sections 3 and 4; conjectures about further generalizations in Section 5. We first show that not all cases of Theorem 1.1 are generalizable; this is, for instance, obvious for (b) of Theorem 1.1. Moreover, we discuss some consequences of Theorem 1.2.

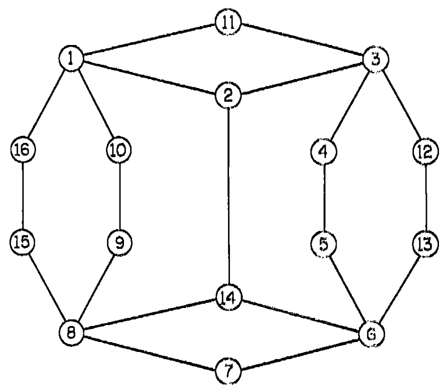

FIG. 1.

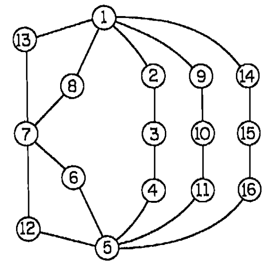

FIG. 2.

Theorem 1.2(b) does not hold for $k = 4$. Consider the graph G of Fig. 1. The circumference of G is 10. The cycle C through the nodes $1, 2, \ldots, 10$ and the cycle D through 1, 11, 3, 12, 13, 6, 14, 8, 15, 16 meet in $W = \{1, 3, 6, 8\}$. $G-W$, however, has only seven components, since node 2 of $C-W$ and node 14 of $D-W$ form one component of $G-W$. If Theorem 1.2(b) were to hold for $k = 4$, $G-W$ should have eight components.

Theorem 1.1(d) cannot be generalized to the case $k = 3$. Consider the graph G of Fig. 2. The circumference of G is 8, and the cycles C_1 through the nodes $1, 2, \ldots, 8$, C_2 through the nodes 1, 9, 10, 11, 5, 12, 7, 13, and C_3 through the nodes $1, 2, \ldots, 5$, 16, 15, 14 are longest cycles of G. C_1 and C_2 meet in $\{1, 5, 7\}$, while C_1 and C_3 meet in $\{1, 5\}$ only. Thus, if two longest cycles meet in three nodes, then not all longest cycles necessarily contain these three nodes.

To prove Theorem 1.2 we want to assume that the circumference is at least $k+1$. This can be done because of the following observations. A 2-connected graph with at least four nodes has a cycle of length at least four. So this assumption is automatically fulfilled for $k = 3$. The 2-connected

those shown in Fig. 3. There may be additional parallel edges, but no "diagonals". Theorem 1.2 is obviously true for these graphs. The 2-connected graphs with at least six nodes and circumference five look like those shown in Fig. 4. The graph G' in Fig. 4 is a graph on four nodes containing a path of length three from u to v, and further parallel edges may exist. For the graphs shown in Fig. 4, Theorem 1.2 clearly holds as well.

We may therefore assume in the sequel that G has circumference at least $k+1$, for $2 \leq k \leq 5$.

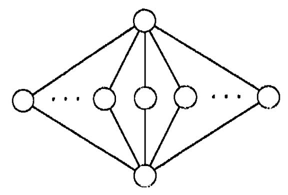

FIG. 3. The 2-connected graphs with circumference 4.

We shall now give an application of Theorem 1.2.

Babai [2] proved that every connected vertex-transitive graph with at least four nodes contains a cycle of length greater than $(3n)^{1/2}$. If a vertex-transitive graph is k-connected, $k \in \{4, 5, 6\}$, then Theorem 1.2 implies that every two different longest cycles meet in at least k nodes. Following the (nice and simple) proof of Babai [2], we can conclude

THEOREM 1.3 *If G is a k-connected vertex-transitive graph, $k \in \{4, 5, 6\}$, then G contains a cycle of length greater than $(kn)^{1/2}$.* □

Note that vertex-transitive graphs are regular. By a result of Mader [10] and Watkins [12], the connectivity of a connected d-regular graph is at least $\frac{2}{3}(d+1)$. Thus, if a connected vertex-transitive graph G is d-regular we get

G is Hamiltonian if $d = 2$ (obvious);

$$G \text{ has circumference at least } \begin{cases} (3n)^{1/2} & \text{if } d = 3, \\ (4n)^{1/2} & \text{if } d = 4, 5, \\ (5n)^{1/2} & \text{if } d = 6, \\ (6n)^{1/2} & \text{if } d \geq 7. \end{cases}$$

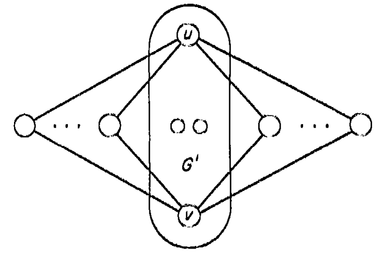

FIG. 4. The 2-connected graphs with circumference 5.

Although a number of people believe that all but four connected vertex-transitive graphs are Hamiltonian (see, for example, Alspach [1]), it seems that the lower bounds on the circumference given above are the best ones known at present.

The proofs of Theorem 1.2 for the various cases do not use the high symmetry of vertex-transitive graphs, and it seems to me that the constant $k^{1/2}$ in the bound can be considerably improved, but no improvement on the factor $n^{1/2}$ (the important one) can be obtained this way. For instance, if G has at least six nodes and is 3-regular, then one can obtain a lower bound of $2n^{1/2}$ for the circumference of G (see Alspach [1]).

2. Notation and some lemmas

Graphs are denoted by $G = [V, E]$ where V is the node set and E the edge set. Our graphs may have loops or multiple edges.

A path P with end-nodes u and v is called a $[u, v]$*-path*. The nodes of P different from u and v are called *internal nodes* of P. The path resulting from P by removing the two end-nodes of P is called the *truncation of* P and is denoted by $\bar{P}$. A truncated path $\bar{P}$ may be empty (if P is an edge), consist of one node (if P has two edges), or may be a path in the usual sense. A path P is called *internally disjoint from* a path or cycle C if none of the internal nodes of P is a node of C.

If P and Q are paths which are internally disjoint from each other and have at least one end-node in common, then $P \cup Q$ denotes the *concatenation* of P and Q which is either a path or a cycle. If C and D are paths or cycles and R is a path with one end-node contained in C, one end-node contained in D, and which is internally disjoint from C and D, then R is called a $[C, D]$*-path*. If we fix an orientation of a cycle C and have two nodes u, v of C, then we can speak of the $[u, v]$*-segment* of C which is the path from u to v consistent with the orientation.

If C and D are two cycles which both contain the nodes u and v and if P is a $[u, v]$-segment of C and Q a $[u, v]$-segment of D, then P and D are called *parallel* if P is internally disjoint from D and Q internally disjoint from C. We say that P and Q are *parallel paths on the cycles* C and D.

The length of a path or cycle C is the number of its edges and is denoted by $|C|$. The length of a longest cycle in a graph G is called the *circumference* of G.

An *articulation set* is a set of nodes whose removal results in a disconnected graph or the graph with one node.

The following simple lemmas help to reduce considerably the number of cases to be considered in the proof of our theorem.

LEMMA 2.1 *Let P and Q be two parallel paths on two longest cycles of a graph G. Then P and Q have the same length.*

PROOF Let C and D be two longest cycles of G and let P and Q be parallel paths on C and D. Then C and D can be written as concatenations of paths as follows: $C = P \cup P'$ and $D = Q \cup Q'$. By assumption $P \cup Q'$ and $Q \cup P'$ are cycles, and since C and D have maximum length we get $|P \cup Q'| \leq |Q \cup Q'|$ implies $|P| \leq |Q|$, and $|Q \cup P'| \leq |P \cup P'|$ implies $|Q| \leq |P|$, which proves the claim. □

LEMMA 2.2 *Let C and D be two longest cycles of G having a common node u. Let P be a segment on C and Q be a segment on D such that P and Q have u as one end-node and such that P is internally disjoint from D and Q is internally disjoint from C. Then G contains no $[\bar{P}, \bar{Q}]$ path internally disjoint from C and D.*

PROOF Suppose that R is a path in G connecting an internal node of P, say x, to an internal node of Q, say y, such that R is internally disjoint from C and D. We may assume that the situation is as depicted in Fig. 5, where $C = P \cup P_1 = P' \cup P'' \cup P_1$, $D = Q \cup Q_1 = Q' \cup Q'' \cup Q_1$ and P, P_1 are $[u, v]$-segments of C, while Q, Q_1 are $[u, w]$-segments of D.

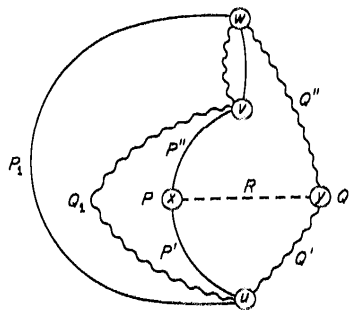

FIG. 5.

Clearly, $D_1 := Q_1 \cup P' \cup R \cup Q''$ and $C_1 := P_1 \cup Q' \cup R \cup P''$ are cycles of G, and hence $|D_1| \leq |D|$ implies $|P' \cup R| \leq |Q'|$ and $|C_1| \leq |C|$ implies $|Q' \cup R| \leq |P'|$. Therefore we have $|Q'| < |Q' \cup R| \leq |P'| < |P' \cup R| \leq |Q'|$, which is a contradiction. □

LEMMA 2.3 *Let C and D be two longest cycles of a graph G which are concatenations of three paths, i.e. $C = P_1 \cup P_2 \cup P_3$, $D = Q_1 \cup Q_2 \cup Q_3$. Suppose that P_1 and Q_1 are parallel, and P_2 and Q_2 are internally disjoint respectively from D and C. Then G contains no $[\bar{P}_1, \bar{P}_2]$-, $[\bar{P}_1, \bar{Q}_2]$-, $[\bar{Q}_1, \bar{Q}_2]$-, and no $[\bar{Q}_1, \bar{P}_2]$-paths internally disjoint from C and D.*

PROOF We show that G contains no $[\bar{P}_1, \bar{Q}_2]$-path internally disjoint from C and D. The other cases follow by symmetry.

Suppose that R is a path linking a node x of $\bar{P}_1$ to a node y of $\bar{Q}_2$ which is internally disjoint from C and D. We may assume that the situation is as depicted in Fig. 6, where $C = P_1' \cup P_1'' \cup P_2 \cup P_3$ (P_3 is a $[z, u]$-segment of C) and $D = Q_1 \cup Q_2' \cup Q_2'' \cup Q_3$ (Q_3 is a $[w, u]$-segment of D). We see that $C_1 := P_1' \cup R \cup Q_2' \cup P_2 \cup P_3$ and $D_1 := Q_1 \cup P_1'' \cup R \cup Q_2'' \cup Q_3$ are cycles and that $|C_1| + |D_1| = |C| + |D| + 2|R|$, which contradicts the fact that C and D have maximum length. □

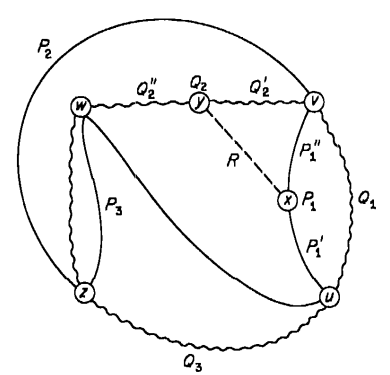

FIG. 6.

LEMMA 2.4 *Let P and Q be two parallel paths on two longest cycles C and D of a graph G. Let S be a segment on one of the cycles disjoint from the other. If there is a $[\bar{P}, S]$-path (respectively a $[\bar{Q}, S]$-path) internally disjoint from D, then there is no $[\bar{Q}, S]$-path (respectively no $[\bar{P}, S]$-path) internally disjoint from C and D.*

PROOF Suppose that S is a segment on D and suppose that there are a

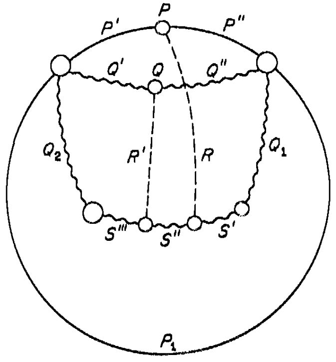

FIG. 7.

$[\bar{P}, S]$-path R and a $[\bar{Q}, S]$-path R'. Then, up to isomorphic rearrangements, the situation can be depicted as shown in Fig. 7. Here $S = S' \cup S'' \cup S'''$, $Q = Q' \cup Q''$, $P = P' \cup P''$, $C = P \cup P_1$, $D = Q \cup Q_1 \cup S \cup Q_2$. Clearly, $C_1 := P_1 \cup P'' \cup R \cup S'' \cup R' \cup Q'$ and $C_2 = P_1 \cup Q'' \cup R' \cup S'' \cup R \cup P'$ are cycles which are not longer than C, but $|C_1| + |C_2| = 2|C| + 2(|R| + |R'| + |S''|)$, a contradiction. □

3. The cases $k=3, 4$

Note that the cases (a), (b) and (c) of Theorem 1.1 follow immediately from Lemmas 2.1, 2.2 and 2.3. The cases (d) and (e) need different arguments and are not proved here.

PROOF OF THEOREM 1.2(a), (b) FOR $k = 3$ Let the node set W in which the two longest cycles C and D intersect be $\{u, v, w\}$. Then C and D can be written as concatenations of three paths, say $C := P_1 \cup P_2 \cup P_3$ and $D := Q_2 \cup Q_3$, such that P_1, Q_1 are parallel $[u, v]$-segments, P_2, Q_2 are parallel $[v, w]$-segments and P_3, Q_3 are parallel $[u, v]$-segments. Since $|V| \geq 4$ and G is 2-connected, C and D have length at least 4, so at least one of the paths P_i, $i = 1, 2, 3$, has length at least 2, say P_1. By Lemma 2.1 Q_1 has the same length as P_1, and by Lemma 2.2, there is no $[\bar{P}_1, \bar{Q}_1]$-path in G disjoint from C and D.

Each of the paths P_2, P_3, Q_2, Q_3 satisfies the assumptions of Lemma 2.3 with respect to the parallel paths P_1 and Q_1. Therefore there are no $[\bar{P}_1, \bar{P}_2]$-, $[\bar{P}_1, \bar{P}_3]$-, $[\bar{P}_1, \bar{Q}_2]$-, $[\bar{P}_1, \bar{Q}_3]$-paths internally disjoint from C and D. This implies that the component of $G-W$ containing $\bar{P}_1$ does not contain any of the (non-empty) truncated paths $\bar{P}_2$, $\bar{P}_3$, $\bar{Q}_1$, $\bar{Q}_2$, $\bar{Q}_3$. Thus we have shown that $G-W$ has at least two components (the ones containing $\bar{P}_1$ and $\bar{Q}_1$) and that each of the paths $C-W$ and $D-W$ belongs to a different component of $G-W$, which finishes the proof. □

PROOF OF THEOREM 1.2(a) FOR $k=4$ Suppose the two longest cycles C and D meet in $\{u_1, u_2, u_3, u_4\}$. Let $C := P_1 \cup P_2 \cup P_3 \cup P_4$, where P_i is a $[u_i, u_{i+1}]$-segment of C, $i = 1, 2, 3$, and P_4 a $[u_4, u_1]$-segment. There are two possible intersection patterns for $D = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$, namely, each of the four segments of D is parallel to a segment of C, or exactly two segments of C are parallel to segments of D.

Case 1: Each segment of D is parallel to a segment of C, say P_i is parallel to Q_i, $i = 1, 2, 3, 4$ By assumption one of the paths P_i has length at least two, say P_1. Using Lemmas 2.1, 2.2 and 2.3 we obtain, just as in the proof above for the case $k = 3$, that there can be no $[\bar{P}_1, \bar{Q}_1]$-, $[\bar{P}_1, \bar{Q}_2]$-, $[\bar{P}_1, \bar{Q}_4]$-, $[\bar{P}_1, \bar{P}_2]$-, $[\bar{P}_1, \bar{P}_4]$-paths in G internally disjoint from C and D.

If the truncated path $\bar{P}_3$ (and hence $\bar{Q}_3$) is empty, then $\bar{Q}_1$ and $\bar{P}_1$ belong to different components of $G-W$. If $\bar{P}_3$ (and hence $\bar{Q}_3$) is non-empty, then by Lemma 2.2 there is no $[\bar{P}_3, \bar{Q}_3]$-path internally disjoint from C and D. By Lemma 2.2 there is a $[\bar{P}_1, \bar{P}_3]$-path (or a $[\bar{P}_1, \bar{Q}_3]$-path) internally disjoint from C and D, there can be no $[\bar{Q}_1, \bar{P}_3]$-path (or $[\bar{Q}_1, \bar{Q}_3]$-path) internally disjoint from C and D. Hence $\bar{P}_1$ and $\bar{Q}_1$ belong to different components of $G-W$ in this case too, i.e. W is an articulation set.

Case 2: Two segments of C are parallel to segments of D, say P_1, Q_1 and P_3, Q_3 are parallel One of the paths P_i has length at least two. There are two subcases to consider:

Case 2.1: One of the paths P_i parallel to a path Q_i has length at least two, say P_1 The proof of subcase 2.1 is word for word the same as the proof of Case 1.

Case 2.2: One of the paths P_2 or P_4 has length at least two, say P_2 By Lemma 2.2 there is no $[\bar{P}_2, \bar{Q}_i]$-path internally disjoint from C and D for $i = 1, \ldots, 4$. By Lemma 2.3 there is no such $[\bar{P}_2, \bar{P}_1]$-path and no such $[\bar{P}_2, \bar{P}_3]$-path. The only remaining possibility is that there is a $[\bar{P}_2, \bar{P}_4]$-path R internally disjoint from C and D. We may assume that the situation is as depicted in Fig. 8.

Then $D_1 := Q_1 \cup P_2' \cup R \cup P_4' \cup Q_3 \cup Q_4$ and $D_2 := Q_1 \cup Q_2 \cup Q_3 \cup P_2'' \cup R \cup P_4''$ are cycles which by assumption are not longer than D. How-

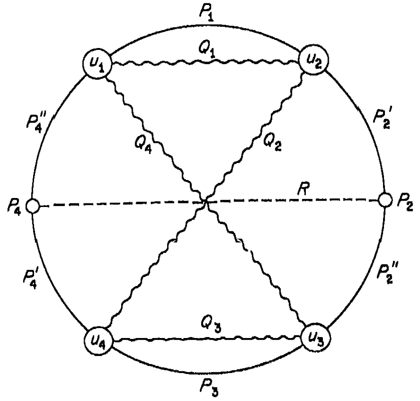

FIG. 8.

ever, $|D_1|+|D_2|=|D|+|Q_1|+|P_2|+|Q_3|+|P_4|+2|R|=2|D|+2|R|$, since $|P_2|+|P_4|=|Q_2|+|Q_4|$, which is a contradiction. So there is no $[\bar{P}_2, \bar{P}_4]$-path internally disjoint from C and D either. This shows that the component containing $\bar{P}_2$ contains none of the other truncated paths and thus that W is an articulation set. □

REMARK 3.1 If $C=P_1\cup P_2\cup P_3\cup P_4$ and $D=Q_1\cup Q_2\cup Q_3\cup Q_4$ are longest cycles as above which meet in exactly four nodes u_1, u_2, u_3, u_4 and if the truncated paths $\bar{P}_i$, $\bar{Q}_i$, $i=1,\ldots,4$ are non-empty, then one can conclude from the proof above that $G-\{u_1, u_2, u_3, u_4\}$ has at least seven components. □

The graph G of Fig. 1 has two longest cycles meeting in four nodes W and where all truncated paths are non-empty. By Remark 3.1, $G-W$ has the minimum possible number of components. Moreover, one can show that there is no graph with these properties with fewer nodes than the graph of Fig. 1.

4. The case $k=5$

We shall now prove Theorem 1.2(a) for $k=5$. Let $W=\{u_1,\ldots,u_5\}$ be the set of nodes in which the longest cycles C and D intersect, and assume that $C=P_1\cup P_2\cup P_3\cup P_4\cup P_5$, where P_i is the $[u_i, u_{i+1}]$-segment of C, $i=1,\ldots,4$ and P_5 the $[u_5, u_1]$-segment. Let $D=Q_1\cup Q_2\cup Q_3\cup Q_4\cup Q_5$. By enumerating the possible ways in which C and D can intersect, one can see that (up to renumbering) there are four cases to consider, which we depict graphically.

Case 1 All segments are parallel (Fig. 9).

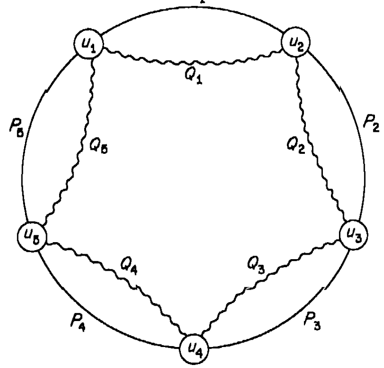

FIG. 9.

Case 2 Three segments are parallel (Fig. 10).

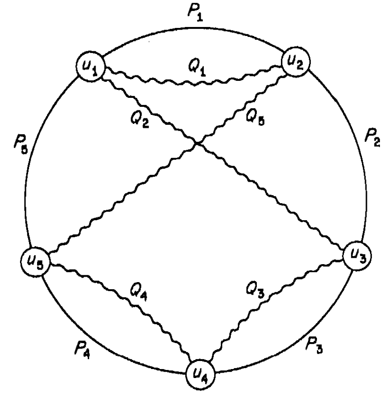

FIG. 10.

Case 3 Two segments are parallel (Fig. 11).

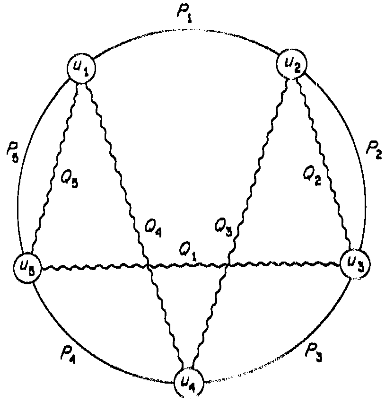

FIG. 11.

Case 4 No segments are parallel (Fig. 12).

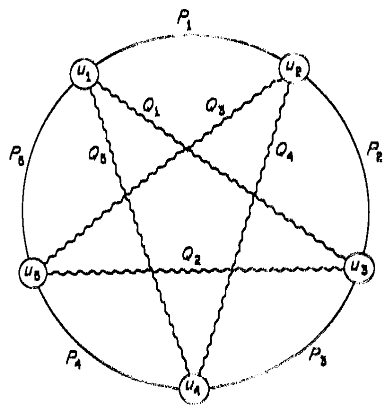

FIG. 12.

We have to discuss each of these cases separately. By going into detail one can also give lower bounds on the number of components of $G-W$ as in Remark 3.1, but we restrict ourselves to the proof that W is an articulation set. We assume in all cases that W is not an articulation set and derive a contradiction. As shown in Section 1, we can assume that the circumference of G is at least 6.

PROOF FOR CASE 1 Since the circumference of G is at least 6, one of the paths P_i has length at least 2. By symmetry we may assume that $|P_1| \geq 2$.

By Lemmas 2.2 and 2.3, G contains no $[\bar{P}_1, X]$-path internally disjoint from C and D for all $X \in \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_5, \bar{P}_2, \bar{P}_5\}$. Suppose there is a path to one of the other truncated paths. By symmetry we may assume that there is a $[\bar{P}_1, \bar{Q}_3]$-path R in G internally disjoint from C and D. The end-nodes of R split P_1 into P_1' and P_1'' and Q_3 into Q_3' and Q_3''. (Here and in the sequel we assume that the splitting is "clockwise", i.e. the segment on C following P_5 is P_1', then P_1'', then P_2, and on D we go from Q_2 to Q_3' to Q_3'' to Q_4.) Then $C_1 := R \cup Q_3' \cup P_3 \cup P_4 \cup P_5 \cup Q_1 \cup P''$ is a cycle in G. $|C_1| \leq |C|$ implies $|R \cup Q_3' \cup P_1''| \leq |P_2|$.

Therefore $|P_2| \geq 3$, and if $G-W$ is connected then there must be a path linking $\bar{P}_2$ to one of the other truncated paths. As before, $\bar{P}_2$ can only be linked to one of the paths $\bar{P}_4$, $\bar{P}_5$, $\bar{Q}_4$, $\bar{Q}_5$. By symmetry we may assume that G contains a $[\bar{P}_2, \bar{Q}_5]$-path R' internally disjoint from C and D. R' splits P_2 into P_2' and P_2'' and Q_5 into Q_5' and Q_5''.

Set

$$C_2 := P_1'' \cup P_2' \cup R' \cup Q_5'' \cup P_5 \cup P_4 \cup P_3 \cup Q_3' \cup R$$

and

$$C_3 := P_1' \cup Q_1 \cup Q_2 \cup P_2'' \cup R' \cup Q_5' \cup P_4 \cup Q_3'' \cup R,$$

then C_2 and C_3 are cycles satisfying $|C_1|+|C_2| = 2(|C|+|R|+|R'|)$, which contradicts the maximality of $|C|$.

PROOF FOR CASE 2 One of the paths P_i has length at least two. There are three subcases to consider.

Subcase 2.1 $|P_1| \geq 2$.
Subcase 2.2 $|P_2| \geq 2$ or $|P_5| \geq 2$, say $|P_2| \geq 2$.
Subcase 2.3 $|P_3| \geq 2$ or $|P_4| \geq 2$, say $|P_3| \geq 2$.

PROOF OF SUBCASE 2.1, $|P_1| \geq 2$ By Lemmas 2.2 and 2.3 G contains no $[\bar{P}_1, X]$-path internally disjoint from C and D for all $X \in \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_5, \bar{P}_2, \bar{P}_5\}$. We now show that there are no such $[\bar{P}_1, X]$-paths for $X \in \{\bar{P}_3, \bar{P}_4, \bar{Q}_3, \bar{Q}_4\}$ as well. By symmetry it is sufficient to prove that

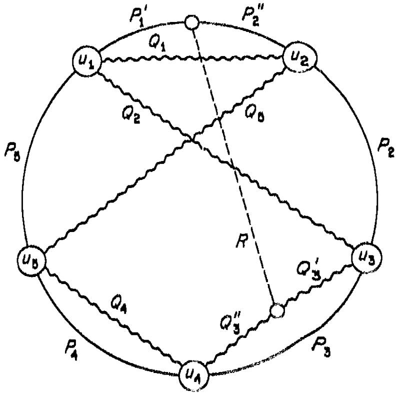

FIG. 13.

there is no $[\bar{P}_1, \bar{Q}_3]$-path internally disjoint from C and D. Suppose R is such a path.

R splits P_1 into $P_1' \cup P_1''$ and Q_3 into $Q_3' \cup Q_3''$ (see Fig. 13). Then we can construct the following cycles:

$$C_1 := R \cup P_1'' \cup Q_1 \cup P_5 \cup P_4 \cup P_3 \cup Q_3',$$
$$D_1 := R \cup P_1' \cup Q_2 \cup P_2 \cup Q_5 \cup Q_4 \cup Q_3'',$$

which satisfy $|C_1|+|D_1| = |C|+|D|+2|R|$, a contradiction.

PROOF OF SUBCASE 2.2, $|P_2| \geq 2$ By Lemmas 2.2 and 2.3 there is no $[\bar{P}_2, X]$-path internally disjoint from C and D for all $X \in \{\bar{P}_1, \bar{P}_3, \bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \bar{Q}_5\}$. So there may be such paths for $X \in \{\bar{Q}_4, \bar{P}_4, \bar{P}_5\}$. The cases $X = \bar{Q}_4$ and $X = \bar{P}_4$ are symmetric, so we have to consider two further subcases.

(a) Suppose there is a $[\bar{P}_2, \bar{P}_5]$-path R internally disjoint from C and D. R splits P_2 into $P_2' \cup P_2''$ and P_5 into $P_5' \cup P_5''$. Consider the cycles

$$C_1 := P_1 \cup P_2' \cup R \cup P_5' \cup P_4 \cup P_3 \cup Q_2,$$
$$D_1 := Q_1 \cup Q_5 \cup Q_4 \cup Q_3 \cup P_2'' \cup R \cup P_5'',$$

then $|C_1|+|D_1| = |C|+|D|+2|R|$, a contradiction.

(b) Suppose that there is a $[\bar{P}_2, \bar{Q}_4]$-path R internally disjoint from C and D and that R splits P_2 into $P_2' \cup P_2''$ and Q_4 into $Q_4' \cup Q_4''$. (Such a path in fact may exist!) Since $|P_4| = |Q_4| \geq 2$ and we assume that $G - W$ is connected, there must be a $[\bar{P}_4, X]$-path internally disjoint from C and D for some of the truncated paths X. By Lemmas 2.2 and 2.3 there can be no such path for $X \in \{\bar{P}_3, \bar{P}_5, \bar{Q}_3, \bar{Q}_4, \bar{Q}_5\}$, by Lemma 2.4 no such path for $X = \bar{P}_2$, and by Subcase 2.1 no such path for $X \in \{\bar{P}_1, \bar{Q}_1\}$. The only possibility remaining is a $[\bar{P}_4, \bar{Q}_2]$-path R' internally disjoint from C and D splitting P_4 into $P_4' \cup P_4''$ and Q_2 into $Q_2' \cup Q_2''$ (see Fig. 14). The cycles

$$C_1 := P_1 \cup P_2' \cup R \cup Q_4' \cup P_3 \cup Q_2' \cup R' \cup P_4'' \cup P_5,$$
$$D_1 := Q_1 \cup Q_5 \cup Q_4'' \cup R \cup P_2'' \cup Q_3 \cup P_4' \cup R' \cup Q_2''$$

satisfy $|C_1|+|D_1| = |C|+|D|+2(|R|+|R'|)$, a contradiction. This finishes the proof of Subcase 2.2.

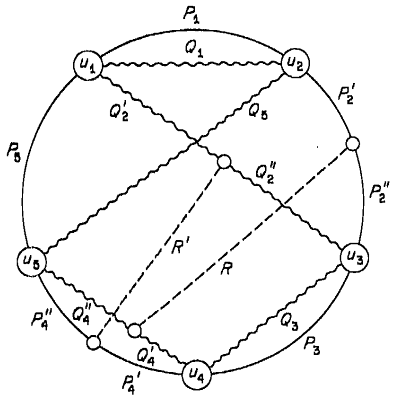

FIG. 14.

PROOF OF SUBCASE 2.3, $|P_3| \geq 2$ By Lemmas 2.2 and 2.3 there can be no $[\bar{P}_3, X]$-path internally disjoint from C and D for $X \in \{\bar{Q}_2, \bar{Q}_3, \bar{Q}_4, \bar{P}_2, \bar{P}_4\}$. Moreover, by Subcase 2.1 there can be no such path for $X \in \{\bar{Q}_1, \bar{P}_1\}$. What remains is the possibility of a $[\bar{P}_3, \bar{P}_5]$-or a $[\bar{P}_3, \bar{Q}_5]$-path internally disjoint from C and D. Note that these two cases are symmetric, and in fact the existence of a $[\bar{P}_3, \bar{P}_5]$-path is equivalent to the existence of a $[\bar{P}_2, \bar{Q}_4]$-path which has been discussed in Subcase 2.2(b). This proves Subcase 2.3.

PROOF OF CASE 3 As in Case 2, we also have to discuss the three subcases $|P_1| \geq 2$, $|P_2| \geq 2$ and $|P_3| \geq 2$. The proof runs along the same lines as that of Case 2.

PROOF OF CASE 4 One can show that there is no $[\bar{P}_i, \bar{Q}_j]$-path internally disjoint from C and D for all $i, j \in \{1, \ldots, 5\}$ at all. Namely, suppose that $|P_1| \geq 2$, then by Lemmas 2.2 and 2.3 there are no $[\bar{P}_1, X]$-paths internally disjoint from C and D for all $X \in \{\bar{Q}_1, \bar{Q}_3, \bar{Q}_4, \bar{Q}_5\}$. Suppose that there is such a $[\bar{P}_1, \bar{Q}_2]$-path R splitting P_1 into $P_1' \cup P_1''$ and Q_2 into $Q_2' \cup Q_2''$, then one of the cycles $C_1 := R \cup P_1'' \cup Q_4 \cup P_3 \cup Q_1 \cup P_5 \cup Q_2''$ and $D_1 := R \cup P_1' \cup Q_5 \cup P_4 \cup Q_3 \cup P_2 \cup Q_2'$ must be longer than C, a contradiction. This shows that $G - W$ is disconnected. This finishes the proof of the case $k = 5$. □

5. Some Conjectures

One can of course ask what is the largest number k for which Theorem 1.2(a) holds. In fact, $k = 5$ is best possible. Consider the graph G with $n \geq 8$ nodes shown in Fig. 15. G has circumference 6 and contains two different cycles of length 6 meeting in the six nodes $1, 2, \ldots, 6$ such that the removal of these six nodes leaves a connected graph.

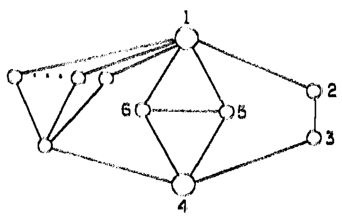

FIG. 15.

It seems, however, sensible to add to Theorem 1.2 the condition that the circumference of G should be at least $k + 1$ (for $k \leq 5$ this slightly weakens the theorem but is the only interesting case). Then the graph of Fig. 15 is no longer a counterexample. Indeed I believe that this version of Theorem 1.2(a) is also true for $k = 6$ and 7, but I have resisted any inclination to try the enumerative proof technique used above to solve these cases.

For $k = 8$ the Petersen graph P gives a counterexample. If u and v are two adjacent nodes of P, then $P - u$ (respectively $P - v$) contains a Hamiltonian cycle C (respectively D). The intersection of C and D is the eight nodes of P different from u and v. These eight nodes do not form an articulation set by construction.

I do not know of an infinite number of graphs in which two longest cycles meet in a set of eight nodes which is not an articulation set, but I know of such a class for $k = 9$. Consider Fig. 16. The graph shown therein has $n \geq 12$ nodes and circumference 10. The cycles C through the nodes $1, 2, \ldots, 10$ and D through 2, 3, 4, 11, 8, 9, 10, 5, 6, 7 are longest cycles of length 10 and meet in the set $W = \{2, 3, \ldots, 10\}$. $G - W$ consists of one component.

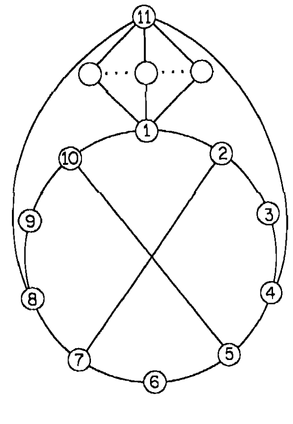

FIG. 16.

It should be clear that connectivity plays a role in the problem we consider. So one can state the questions above in more generality.

For every integer $k \geq 2$ let $f(k)$ denote the largest integer with the following property: If G is a k-connected (non-Hamiltonian) graph and if two longest cycles of G meet in at most $f(k)$ nodes then these nodes form an articulation set.

Let $f'(k)$ be defined as $f(k)$ above, making the additional assumption that the k-connected graph G has circumference larger than $f'(k)$. Clearly, we have $f'(k) \geq f(k)$.

Theorem 1.2(a) and the graph of Fig. 15 show that $f(2) = 5$. The Petersen graph shows that $f'(3) \leq 7$, and we conjecture that $f'(2) = f'(3) = 7$. It seems to be a very hard problem to determine the functions f and f'. I believe however that the following is true:

CONJECTURE 5.1 $f(k) \geq k$. □

In other words, if G is k-connected and two longest cycles meet in k nodes, then these nodes form an articulation set. Conjecture 5.1 – if true – would imply

CONJECTURE 5.2 *In a k-connected graph two longest cycles meet in at least k nodes.* □

I have been told by R. Häggkvist and A. Bondy that Conjecture 5.2 was first conjectured in 1979 by Scott Smith (then a high-school student), and that the truth of the conjecture probably has been verified up to $k = 10$ (up to $k = 6$ it follows from Theorem 1.2(a)). The general case is still unsettled.

Another interesting problem coming up in certain applications is to determine the intersection pattern of two longest odd cycles or a longest odd and longest even cycle. This will be investigated in a forthcoming paper.

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