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Polyhedral Combinatorics

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The subject *polyhedral combinatorics* has grown so explosively in the recent twenty years that it is virtually impossible to write an (annotated) bibliography of this field aiming at a high degree of completeness. This is rather easy for the period before 1970 when the field had not yet completely emerged as a subject in its own right. There are classics like

L.R. Ford, D.R. Fulkerson (1962). *Flows in Networks,* Princeton University Press, Princeton, NJ,

which is a landmark book in network flow theory, like

G.B. Dantzig, D.R. Fulkerson, S.M. Johnson (1954). Solution of a large-scale traveling-salesman problem. *Oper. Res. 2,* 393-410,

which describes the basic techniques of cutting plane generation and separation within a linear programming framework (although not totally explicit and aware of all relations to polyhedral theory, but many of the fundamental ideas can be found there), like

A.J. Hoffman, J.B. Kruskal (1956). Integral boundary points of convex polyhedra. [Kuhn & Tucker 1956] (see below), 223-246,

which introduces totally unimodular matrices (called matrices with the unimodular property there) and recognizes them as an important class of matrices in polyhedral combinatorics with many applications, or like

J. Edmonds (1965). Maximum matching and a polyhedron with 0,1-vertices. *J. Res. Nat. Bur. Standards Sect. B 69,* 125-130,

J. Edmonds (1965). Paths, trees, and flowers. *Canad. J. Math. 17,* 449-467,

where the first complete linear description of a polytope associated with a combinatorial optimization problem (matching) is given which is nontrivial (in the sense that it needs more inequalities, in fact an exponential number, for the description than are necessary for the obvious integer programming formulation) and where a polynomial time algorithm based on linear programming techniques is presented utilizing this description, showing for the first time that structure (nice characterization of the inequalities) matters more than number (of inequalities) for the design of good algorithms. In particular this and the subsequent work of Edmonds (and coauthors), for instance

J. Edmonds (1970). Submodular functions, matroids, and certain polyhedra. R. Guy *et al.* (eds.). *Combinatorial Structures and their Applications,* Gordon and Breach, New York, 69-87,

J. Edmonds, E.L. Johnson (1970). Matching: a well-solved class of integer linear programs. *Ibidem,* 89-92,

J. Edmonds, R. Giles (1977). A min-max relation for submodular functions on graphs. *Ann. Discrete Math. 1,* 185-204,

has influenced the development of polyhedral combinatorics to a great extent.

It seems that even the work on special subjects like polyhedral aspects of the traveling salesman problem or stable sets in graphs or on submodular flows has grown so rapidly that it may take almost a book to survey these topics in detail. Fortunately, a number of very good and up to date survey papers on various aspects of polyhedral combinatorics have been written recently which treat the existing literature in depth. For those who wish to get acquainted with polyhedral combinatorics we recommend reading (some of) those papers we will describe in the sequel. Most of these papers have a large number of references which make it easy to get to the present research literature and approach the frontiers of this science.

Before starting our *survey of surveys* we would like to mention a project carried out at the Institut für Operations Research, Universität Bonn, Nassestrasse 2, D-5300 Bonn, West Germany. The Institute collects and compiles all papers and books on *Integer Programming and Related Areas.* Classified bibliographies are published in the Lecture Notes in Economics and Mathematical Systems of Springer Verlag about every three or four years containing almost everything published in this area during the covered time period. The first three books of this series are the following:

C. Kastning (ed.) (1976). *Integer Programming and Related Areas: a Classified Bibliography,* Lecture Notes in Economics and Mathematical Systems 128, Springer, Berlin,

D. Hausmann (ed.) (1978). *Idem 1976-1978,* Ibidem 160, Springer, Berlin,

R. von Randow (ed.) (1982). *Idem 1978-1981,* Ibidem 197, Springer, Berlin.

The papers and books are classified according to a subject list. Papers and books on aspects of polyhedral combinatorics can be found for instance under the following subjects:

- adjacency on integer polyhedra;
- blocking, antiblocking, integer rounding;
- cutting planes;
- duality in integer programming;
- facets of integer polyhedra;
- group theoretic approach (corner polyhedra);
- integer polyhedra.

These bibliographies are valuable sources of literature on combinatorial optimization and make the search for existing papers on a subject quite easy.

We now give an overview on papers which survey various aspects of polyhedral combinatorics. Many of these papers have been prepared for tutorial purposes and outline the subject they treat in a way that is also suitable for the beginner.

Polyhedral combinatorics deals with the application of polyhedral theory and linear algebra to combinatorial problems. There are only very few books on polyhedral theory (compared to the vast amount of linear algebra books), and moreover, most of these books do not treat all those polyhedral topics which are of interest in polyhedral combinatorics. The same is also true for most of the books on linear programming. Two books - although dealing with more general subjects - which are good sources for the type of polyhedral theory we need are:

J. Stoer, C. Witzgall (1970). *Convexity and Optimization in Finite Dimensions I,* Grundlehren der mathematischen Wissenschaften 163, Springer, Berlin,

R.T. Rockafellar (1970). *Convex Analysis,* Princeton University Press, Princeton, NJ.

A comprehensive survey of the part of polyhedral theory used in mathematical programming is

A. Bachem, M. Grötschel (1982). New aspects of polyhedral theory. B. Korte (ed.). *Modern Applied Mathematics: Optimization and Operations Research,* North-Holland, Amsterdam, 51-106.

A classic collection of papers on various aspects of polyhedral theory and combinatorics is

H.W. Kuhn, A.W. Tucker (eds.) (1956). *Linear Inequalities and Related Systems,* Princeton University Press, Princeton, NJ,

where many of the results of the algebraic theory of polyhedra (which today are mostly considered as basic knowledge) appear for the first time. Most of the papers in this book are still worth reading, not only for historical interest.

Many of the survey papers mentioned in the sequel also give a short overview over the particular part of polyhedral theory necessary for the paper. In the near future the following book will appear that contains all those areas of linear algebra and polyhedral theory (theoretical as well as algorithmical aspects) which are of interest in combinatorial optimization:

A. Schrijver (to appear). *Polyhedral Combinatorics,* Wiley, Chichester.

At present, the results of polyhedral combinatorics are scattered over the literature, and only a few books have a chapter on special topics of it. The forthcoming book by Schrijver will be an exceptionally comprehensive treatment of polyhedral combinatorics and cover almost all parts of the theory known to date. We highly recommend this book (parts of which are circulating in the scientific community) to students and researchers. The broadest and most up to date survey paper covering the whole field is

W.R. Pulleyblank (1983). Polyhedral combinatorics. A. Bachem, M. Grötschel, B. Korte (eds.). *Mathematical Programming: the State of the Art - Bonn 1982,* Springer, Berlin, 312-345.

The paper starts with basic polyhedral theory, deals with faces, facets, dimension and adjacency, various representations, dual integrality, discusses prominent examples (e.g. matching polyhedra, traveling salesman polytopes) and points out algorithmic features (good algorithms, $\mathcal{P}$- and $\mathcal{NP}$-problems, separation, cutting planes). The main subject of

A. Schrijver (1983). Min-max results in combinatorial optimization. *Ibidem,* 439-500

are min-max relations, but a large part consists of applications of such relations to polyhedral combinatorics. Particular attention is given to complete descriptions of polyhedra associated with combinatorial optimization problems. Especially worth mentioning is the description of refinement techniques that yield far reaching generalizations from a few fundamental results. Two papers dealing with a special topic are:

M. Grötschel, M.W. Padberg (1985A). Polyhedral theory. E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys (eds.). *The Traveling Salesman Problem,* Wiley, Chichester,

M. Grötschel, M.W. Padberg (1985B). Polyhedral algorithms. *Ibidem.*

This is a series of two papers on polyhedral aspects of the traveling salesman problem. The papers contain almost everything about this subject that is known to date. Broad coverage is given to the use of polyhedral methods in the design of practical LP-based cutting plane algorithms. All polyhedral and algorithmical aspects are treated in general terms and are exemplified by the traveling salesman problem. A somewhat older paper - but still of interest - dealing with the same kind of questions as [Grötschel & Padberg 1985A,B] for another combinatorial optimization problem is

E. Balas, M.W. Padberg (1976). Set partitioning: a survey. *SIAM Rev. 18,* 710-760.

[Grötschel & Padberg 1985B] and [Balas & Padberg 1976] in particular deal with the algorithmic implications of polyhedral combinatorics for special $\mathcal{NP}$-hard problems, i.e. how knowledge of classes of facets for polytopes associated with a hard combinatorial optimization problem can be combined with LP-techniques and branch and bound to obtain practically efficient algorithms. A general description of this kind of approach is given in

M. Grötschel (1982). Approaches to hard combinatorial optimization problems. B. Korte (ed.). *Modern Applied Mathematics: Optimization and Operations Research,* North-Holland, Amsterdam, 437-515,

where the practical use - even of partial linear characterizations of combinatorial polytopes - is demonstrated. On the other hand, the paper

C.H. Papadimitriou (1984). Polytopes and complexity. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization,* Academic Press, New York, 295-305

shows that (unless $\mathcal{P} = \mathcal{NP}$) many polyhedral questions about $\mathcal{NP}$-hard problems turn out to be even harder than the original problems. So in a theoretical sense, polyhedral combinatorics does not seem to be a feasible line of attack at $\mathcal{NP}$-hard problems. The word theoretical should be underlined here, since the success of algorithms for $\mathcal{NP}$-hard problems based on results of polyhedral combinatorics shows the contrary empirically. Successful practical experiences with real-world applications of this kind are described in a number of papers, for instance in [Grötschel & Padberg 1985B] and

M. Grötschel, M. Jünger, G. Reinelt (to appear). A cutting plane algorithm for the linear ordering problem. *Oper. Res.*

Since the fundamental work of Gomory, see for instance

R.E. Gomory (1958). Outline of an algorithm for integer solutions to linear programs. *Bull. Amer. Math. Soc. 64,* 275-278,

cutting planes have been considered intensively in integer programming from a theoretical as well as a practical point of view. Illuminating articles about the geometrical background and basic techniques for deriving cutting planes are

V. Chvátal (1973). Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Math. 4,* 305-337,

and (considering a more general case)

A. Schrijver (1980). On cutting planes. *Ann. Discrete Math. 9,* 291-296.

An introductory article giving a good survey of existing cutting plane methods is

L.A. Wolsey (1979). Cutting plane methods. A.G. Holzmann (ed.). *Operations Research Support Methodology,* Dekker, New York, 441-466.

The field of polyhedral combinatorics has received a new and very stimulating impetus through the invention of the *ellipsoid method.* Here polyhedral results and cutting planes can be used in a novel way. In fact, it turns out that finding an optimum solution to a combinatorial optimization problem in polynomial time is equivalent to finding a cutting plane for the associated polytope in polynomial time. The latter problem is called *separation problem* and asks whether for a given point y and a polyhedron P the fact that y is in P or not can be proved and if $y \notin P$ a hyperplane separating y from P (a cutting plane) can be found. Generalizations of the ellipsoid method and their use together with polyhedral combinatorics for the polynomial-time solvability of combinatorial optimization problems are discussed in

M. Grötschel, L. Lovász, A. Schrijver (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica 1,* 169-197,

R. M. Karp, C.H. Papadimitriou (1982). On linear characterizations of combinatorial optimization problems. *SIAM J. Comput. 11,* 620-632,

M.W. Padberg, M.R. Rao (to appear). The Russian method for linear inequalities III: bounded integer programming. *Math. Programmming Stud.*

The geometrical flavor of this approach together with an outline of several applications is described in the paper

M. Grötschel, L. Lovász, A. Schrijver (1984). Geometric methods in combinatorial optimization. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization,* Academic Press, New York, 167-183,

which was prepared for an instructional series on this subject and avoids the ugly technical details which are necessary for a rigorous treatment of the subject. Such a treatment together with further generalizations and more applicatins will be contained in the forthcoming book

M. Grötschel, L. Lovász, A. Schrijver (to appear). *The Ellipsoid Method and Combinatorial Optimization,* Springer, Berlin.

The application of the ellipsoid method to a combinatorial optimization problem mainly rests on the fact that for the associated polyhedron a complete linear characterization is known. A widely used technique for obtaining such results exploits what is called *total dual integrality.* The importance of this concept for polyhedral combinatorics, algorithms and complexity is described in the instructional paper

J. Edmonds, R. Giles (1984). Total dual integrality of linear inequality systems. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization,* Academic Press, New York, 117-129.

Duality plays a central role in polyhedral combinatorics (and not only here). Two special geometric duality theories, called *blocking* and *anti-blocking theory,* have given many new insights into the relations between various combinatorial problems - in particular for packing and covering type problems. A seminal paper on this subject is

D.R. Fulkerson (1971). Blocking and anti-blocking pairs of polyhedra. *Math. Programming 1,* 168-194.

It turned out that the investigation of anti-blocking relations can be viewed as the study of certain problems on perfect graphs. A beautiful theory with many deep results has evolved here. The book

C. Berge, V. Chvátal (eds.) (1984). *Topics on Perfect Graphs, Ann. Discrete Math. 21*

contains most of the fundamental papers on this subject and a number of new results. So a very good overview over the historical development of this area up to very recent research can be gained from this book.

A subject that has to be mentioned here as well is the work of Seymour on general *min-max relations.* Seymour has extended a number of classic min-max relations like various versions of the max-flow min-cut theorem to quite general frameworks. He states his results usually in matroid language and often characterizes the matroids for which certain min-max relations hold via forbidden minor theorems. His work has quite a polyhedral flavor, since polyhedral results (e.g. LP-duality or complementary slackness) are used quite often and many of his theorems can be nicely interpreted in the language of polyhedral combinatorics. Two papers we would like to mention in this area are

P.D. Seymour (1977). The matroids with the max-flow min-cut property. *J. Comb. Theory Ser. B 23,* 189-222,

P.D. Seymour (1981). Matroids and multicommodity flows. *European J. Combin. 2,* 257-290.

These articles are not easy to read but worth the effort.

A subject that grew out of the study of matroid polytopes and polymatroids is the investigation of *submodular functions*; see [Edmonds 1970] above. These functions are in some sense the combinatorial analogues of convex functions and there are a number of interesting polyhedra, separation theorems and min-max relations associated with them. A survey of this topic is given in

L. Lovász (1983). Submodular functions and convexity. A. Bachem, M. Grötschel, B. Korte (eds.). *Mathematical Programming: the State of the Art - Bonn 1982,* Springer, Berlin, 235-257.

Another line of attack at combinatorial problems from a polyhedral point of view is based on Gomory's group problem and his theory of corner polyhedra. More generally, it is now often called the *subadditive approach* to integer programming. The theory of corner polyhedra can be found in

A. Bachem (1976). *Beiträge zur Theorie der Corner Polyeder,* Mathematical

Systems in Economics 29, Hain, Meisenheim am Glan,

and a survey of subadditive techniques in

E.L. Johnson (1979). On the group problem and a subadditive approach to integer programming. *Ann. Discrete Math. 5,* 97-112.

A recent research paper showing nice applications of this approach is

G. Gastou, E.L. Johnson (to appear). Binary group and Chinese postman polyhedra. *Math. Programming.*

Polyhedral combinatorics does of course have its special branches. Flourishing areas are for instance the hunt for *facets* of polyhedra associated with combinatorial problems (see the papers under the subject 'facets of integer polyhedra' in [Kastning 1976], [Hausmann 1978], [Von Randow 1982], or see [Pulleyblank 1983] and [Grötschel & Padberg 1985A]) and the hunt for *complete linear characterizations* of such polyhedra (see [Pulleyblank 1983] and [Schrijver 1983]). Another topic of interest is the study of *adjacency relations* on combinatorial polytopes. A good treatment of this subject is

D. Hausmann (1980). *Adjacency on Polytopes in Combinatorial Optimization,* Mathematical Systems in Economics 49, Hain, Meisenheim am Glan.

There are a number of *expository papers* that describe certain polyhedral or linear programming aspects of combinatorial problems. We want to mention a few of them. The paper

V. Chvátal (1975). Some linear programming aspects of combinatorics. D. Corneil, E. Mendelsohn (eds.). *Proc. Conf. Algebraic Aspects of Combinatorics, Toronto, 1975,* Congressus Numerantium 13, Utilitas Mathematica, Winnipeg, 2-30

describes basic examples and notions and is aimed at an audience with no previous knowledge of linear programming. The exposition is very nice and worth reading.

A.J. Hoffman (1982). Ordered sets and linear programming. I. Rival (ed.). *Ordered Sets,* Reidel, Dordrecht, 619-634

shows how linear programming ideas can be used to prove theorems about ordered sets (Dilworth's theorem and generalizations such as the theorems of Greene and Greene & Kleitman) and, the other way around, how concepts from partially ordered sets can be used in polyhedral theory (study of lattice polyhedra).

Various roles of *totally unimodular matrices* (these are matrices for which each square submatrix has determinant 1, 0 or -1) in polyhedral combinatorics are surveyed in

A.J. Hoffman (1960). Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. R. Bellman, M. Hall (eds.).

Combinatorial Analysis, American Mathematical Society, Providence, RI, 113-128,

A.J. Hoffman (1976). Total unimodularity and combinatorial theorems. *Linear Algebra Appl. 13*, 103-108,

A.J. Hoffman (1979). The role of unimodularity in applying linear inequalities to combinatorial theorems. *Ann. Discrete Math. 4*, 73-84.

In [Hoffman 1960] emphasis is laid on Hall's theorem, several variations and generalizations are stated and its relations to transportation and transshipment problems is explored. [Hoffman 1976] shows how total unimodularity can be used to give a common proof to a variety of transversal theorems, and [Hoffman 1979] states a number of interesting results about polytopes associated with combinatorial optimization problems.

We have already mentioned before that *duality* plays a central role in many aspects of combinatorial optimization. A large number of results of this type are surveyed in

L. Lovász (1977). Certain duality principles in integer programming. *Ann. Discrete Math. 1*, 363-374.

An overview of polytopes associated with combinatorial optimization problems can be found in

L. Lovász (1979). Graph theory and integer programming. *Ann. Discrete Math. 4*, 141-158.

Here also several examples are presented illuminating some of the proof techniques used in polyhedral combinatorics.

As mentioned above, there exist no monographs (yet) covering a large part of the field 'polyhedral combinatorics', but there are a number of proceedings volumes or paper collections that mainly contain articles on this subject. To mention some of these books which are worth looking at to get an idea about the present research topics we quote:

P.L. Hammer, E.L. Johnson, B.H. Korte, G.L. Nemhauser (eds.) (1977). *Studies in Integer Programming, Ann. Discrete Math. 1*,

M.L. Balinski, A.J. Hoffman (eds.) (1978). *Polyhedral Combinatorics, Math. Programming Stud. 8*,

P.L. Hammer, E.L. Johnson, B.H. Korte (eds.) (1979). *Discrete Optimization I, Ann. Discrete Math. 4*,

P.L. Hammer, E.L. Johnson, B.H. Korte (eds.) (1979). *Discrete Optimization II, Ann. Discrete Math. 5*,

M.W. Padberg (ed.) (1980). *Combinatorial Optimization, Math. Programming Stud. 12*,

A. Bachem, M. Grötschel, B. Korte (eds.) (1982). *Bonn Workshop on*

Combinatorial Optimization, Ann. Discrete Math. 16.

This survey of the literature on polyhedral combinatorics is by no means complete, but we think it gives a reasonable collection of papers and books which may be suitable for the beginner to start exploring the subject. We have mainly concentrated on very recent books and papers some of which have not appeared yet but are circulating as preprints. Most of these papers will however appear in the years 1984 and 1985.